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Research article

Orthogonal F-contractions on O-complete b-metric space

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Abstract: In this paper, we introduce the concepts of an orthogonal F-contractive type mapping, an orthogonal Kannan F-contractive type mapping and an orthogonal F-expanding type mapping. We prove some fixed point theorems for these mappings in orthogonal complete b-metric spaces. The obtained results generalize and extend some of the well known results in the literature. An example is presented to support our results.

Keywords: orthogonal set; orthogonal b-metric space; orthogonal continuous; orthogonal preserving; orthogonal F-contraction; fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Indeed, it is the first metric fixed point result, called the Banach contraction theory, is one of the most important outcomes of mathematical analysis. It is the most widely used fixed point result in many branches of mathematics and it is generalised in several different directions. One natural way to reinforce the Banach contraction concept is the replacement of the metric space by other generalised metric spaces. The concept of a b-metric space was introduced by Bakhtin [9] and Czerwik [11]. In the setting of b-metric spaces, which is a generalisation of the Banach contraction principle in metric spaces, they also defined the fixed point outcome. In 2015, the concepts of generalised F-Suzuki type contraction and F-Suzuki type contraction mappings were introduced by Alsulami, Karapinar and Piri [5] and the fixed point theorems on complete b-metric space were proved. Gornicki [16] introduced the concepts for F-expanding mapping in 2018 and proved several fixed point theorems and also introduced the concepts of F-contractions in complete b-metric spaces by Lukacs and Kajanto [22]. On the other hand, in many branches of mathematics, the notion of an orthogonal set has many applications and has several ways of orthogonality. The modern definition of orthogonality in metric spaces was introduced by Eshaghi Gordji, Ramezani, De la Sen and Cho [14] and demonstrated the fixed point result for contraction mappings in metric spaces fitted with the new orthogonality. In addition, they used these findings to assert the existence and uniqueness of the first-ordinary differential equation solution, while the Banach contraction mapping can not be applied to this problem. Eshaghi Gordji and Habibi [12] proved fixed point theory in generalized orthogonal metric space. Sawangsup, Sintunavarat and Cho [24] introduced the new concept of an orthogonal F-contraction mapping and proved the fixed point theorems in orthogonal-complete metric space. The orthogonal contractive type mappings have been studied by many authors and important results have been obtained in [2–4, 6–8, 13, 15, 18–20, 23, 25, 27, 28].

In this paper, we introduce the new concepts of an orthogonal F-contractive type mapping, an orthogonal Kannan F-contractive type mapping and an orthogonal F-expanding type mapping and prove the fixed point theorems in an orthogonal complete b-metric space.

2. Preliminaries

Throughout this paper, we denote by W, $\mathbb{R}^+$ and $\mathbb{N}$ a nonempty set, the set of positive real numbers and the set of positive integers, respectively.
The concept of a b-metric space was introduced by Bakhtin [9] and Czerwik [11] as follows.

Definition 2.1. [11] Let W be a nonempty set and $s \geq 1$. Suppose that a function $d : W \times W \to \mathbb{R}^+$ satisfies the following conditions for all $u, v, w \in W$:

(•) $d(u, v) = 0$ if and only if $u = v$;
(•) $d(u, v) = d(v, u)$;
(•) $d(u, v) \leq s[d(u, w) + d(w, v)]$.

Then (W, d, s) is called a b-metric space with the coefficient s.

Example 2.2. [11] Define a mapping $d : W \times W \to \mathbb{R}^+$ by $d(u, v) = |u - v|^2$ for all $u, v \in \mathbb{W}$. Then (W, d, s) is a b-metric space with the coefficient $s = 2$.

The following example shows that there exists a b-metric which is not a metric.

Example 2.3. [1] Let $W = \{0, 1, 2\}$ and let $d : W \times W \to [0, \infty)$ be defined by

$$d(0, 1) = 1, d(0, 2) = \frac{1}{2} \text{ and } d(1, 2) = 2,$$

with $d(u, u) = 0$ and $d(u, v) = d(v, u)$ for all $u, v \in W$. Notice that d is not a metric, since we have $d(1, 2) > d(1, 0) + d(0, 2)$. However, it is easy to see that d is a b-metric with $s \geq \frac{4}{3}$.

The following example shows that b-metric which is not a continuous mapping.

Example 2.4. [21] Let $(W, d, 2)$ be a complete b-metric space, where $W = [0, 2]$ and the function $d : W \times W \to [0, \infty)$ with $d(u, v) = (u - v)^2$. Let $H : W \to W$ be a mapping, defined by

$$Hu = \begin{cases} 1 & \text{if } u \in [0, 1] \\ \frac{u^2}{6} & \text{if } u \in (1, 2]. \end{cases}$$

Then H is discontinuous at $u = 1$.

The following example shows that b-metric which is a continuous mapping.

Example 2.5. [21] Let $(W, d, 2)$ be a complete b-metric space, where $W = [0, \infty)$ and the function $d : W \times W \to [0, \infty)$ with $d(u, v) = (u - v)^2$. Let $H : W \to W$ be a mapping, defined by

$$Hu = \begin{cases} \frac{-u}{2} & \text{if } u \in [-1, 0) \\ 2u & \text{if } u \geq 0. \end{cases}$$

Then H is a continuous mapping.

Then we recall the concept of a control function which was introduced by Wardowski [26].

Definition 2.6. [26] Let $\mathfrak{I}$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

(F_1) F is strictly increasing;
(F_2) for each sequence $\{\alpha_n\}$ of positive numbers, we have

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

In 2015, Cosentino, Jleli, Sarmet and Vetro [10] introduced the following condition in Definition 2.6 to obtain some fixed point results in b-metric spaces.

(F_4) Let $s \geq 1$ be a real number. For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau > 0$,
$\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ for all $n \in \mathbb{N}$.

In 2015, Alsulami, Karapinar and Piri [5] introduced the concepts of generalized F-Suzuki type contraction and F-Suzuki type contraction in complete b-metric space as follows.

Definition 2.7. [5] Let (W, d, s) be a b-metric space with constant $s \geq 1$. A mapping $H : W \to W$ is said to be a generalized F-Suzuki type contraction mapping if there are $F \in \mathfrak{I}$ and $\tau > 0$ such that for all $u, v \in W$ with $u \neq v$, $\frac{1}{2s} d(u, Hu) < d(u, v)$ implies

$$\tau + F(d(Hu, Hv)) \leq aF(d(u, v)) + bF(d(u, Hu)) + cF(d(v, Hv)),$$

where $c \in [0, 1)$ and $a, b \in [0, 1]$ are real numbers with $a + b + c = 1$ and H satisfies the conditions (F_1) and (F_2).

Definition 2.8. [5] Let (W, d, s) be a b-metric space with constant $s \geq 1$. A mapping $H : W \to W$ is said to be a F-Suzuki type contraction mapping if there are $F \in \mathfrak{I}$ and $\tau > 0$ such that for all $u, v \in W$ with $u \neq v$, $\frac{1}{2s} d(u, Hu) < d(u, v)$ implies

$$\tau + F(d(Hu, Hv)) \leq F(d(u, v)).$$

In a b-metric space (W, d, s), a mapping $H : W \to W$ is said to be a Picard operator [22] if it has a unique fixed point $v \in W$ and the Picard iteration $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = Hu_n, \ \ n = 0, 1, 2, ...$$

converges to v for any $u_0 \in W$.

Later, in 2019, Goswami, Haokip and Mishra [17] defined a new type of F-contarctive mapping with F satisfying conditions (F_1), (F_2), (F_3) and (F_4) and proved some fixed point theorems as follows.

Definition 2.9. [17] For a b-metric space (W, d, s), a mapping $H : W \to W$ is said to be an F-contractive type mapping if there exists $\tau > 0$ such that $d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{3}\left\{F(d(u, v)) + F(d(u, Hu)) + F(d(v, Hv))\right\},$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{3}\left\{F(d(u, v)) + F(d(u, Hv)) + F(d(v, Hu))\right\},$$

for all $u, v \in W$.

Theorem 2.10. *[17] Let (W, d, s) be a complete b-metric space and $H : W \to W$ be an F-contractive type mapping. Then H is a Picard operator.*

Definition 2.11. [17] Let (W, d, s) be a b-metric space. A mapping $H : W \to W$ is said to be a Kannan F-contractive type mapping if there exists $\tau > 0$ such that $d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{2}\left\{F(d(u, Hu)) + F(d(v, Hv))\right\},$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{2}\left\{F(d(u, Hv)) + F(d(v, Hu))\right\},$$

for all $u, v \in W$.

Theorem 2.12. *[17] Let (W, d, s) be a complete b-metric space and $H : W \to W$ be a Kannan F-contractive type mapping. Then H is a Picard operator.*

Theorem 2.13. *[17] Let (W, d, s) be a complete b-metric space and $H : W \to W$ be an asymptotically regular(briefly, A.R) mapping such that, for some $\tau > 0$, $d(u, Hu)d(v, Hv) \neq 0$ implies*

$$\tau + F(sd(Hu, Hv)) \leq F(d(u, Hu)) + F(d(v, Hv)),$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq F(d(u, Hv)) + F(d(v, Hu)),$$

for all $u, v \in W$. Then H has a fixed point of $z \in W$.

Definition 2.14. [17] A mapping $H : W \to W$ is said to be an F-expanding type mapping if there exists $\tau > 0$ such that $d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(u, v)) \leq \frac{1}{3} \{F(d(Hu, Hv)) + F(d(u, Hu)) + F(d(v, Hv))\},$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(u, v)) \leq \frac{1}{3} \{F(d(Hu, Hv)) + F(d(u, Hv)) + F(d(v, Hu))\},$$

for all $u, v \in W$.

Lemma 2.15. *[16] Let (W, d, s) be a complete b-metric space and $H : W \to W$ be surjective. Then there exists a mapping $H^* : W \to W$ such that $H \circ H^*$ is the identity mapping in W.*

Theorem 2.16. *[17] Let (W, d, s) be a complete b-metric space and $H : W \to W$ be surjective and an F-expanding type mapping. Then H has a unique fixed point $t \in W$.*

On the other hand, Eshaghi Gordji, Ramezani, De la Sen and Cho [14] introduced the concept of an orthogonal set (or O-set), some examples and some properties of the orthogonal sets as follows:

Definition 2.17. [14] Let $W \neq \phi$ and $\perp \subseteq W \times W$ be a binary relation. If $\perp$ satisfies the following condition:

$$\exists u_0 \in W : (\forall u \in W, u \perp u_0) \quad \text{or} \quad (\forall u \in W, u_0 \perp u),$$

then it is called an orthogonal set (briefly, O-set). We denote this O-set by $(W, \perp)$.

Example 2.18. [14] Let W be the set of all people in the world. Define the binary relation $\perp$ on W by $v \perp u$ if v can give blood to u. Accoding to Table 1, if u_0 is a person such that his(her) blood type is O-, then we have $u_0 \perp u$ for all $u \in W$. This means that $(W, \perp)$ is an O-set. In this O-set, u_0 (in Definition 2.17) is not unique. Note that, in this example, u_0 may be a person with blood type AB+. In this case, we have $u \perp u_0$ for all $u \in W$.

Table 1: Example of O-set.

Type	You can give blood to	You can receive blood from
A+	A+AB+	A+A-O+O-
O+	O-A+B+AB+	O+O-
B+	B+AB+	B+B-O+O-
AB+	AB+	Everyone
A-	A+A-AB+AB-	A-O-
O-	Everyone	O-
B-	B+B-AB+AB-	B-O-
AB-	AB+AB-	AB-B-O-A-

Example 2.19. [14] Let $W = \mathbb{Z}$. Define the binary relation $\perp$ on W by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(W, \perp)$ is an O-set.

Example 2.20. [14] Let (W, d) be a metric space and $H : W \to W$ be a Picard operator, that is, H has a unique fixed point $u^* \in W$ and $\lim_{n\to\infty} H^n(u) = u^*$ for all $v \in W$. We define the binary relation $\perp$ on W by $v \perp u$ if

$$\lim_{n\to\infty} d(u, H^n(v)) = 0.$$

Then $(W, \perp)$ is an O-set.

Example 2.21. [14] Let $W = [0, \infty)$ and define $u \perp v$ if $uv \in \{u, v\}$. Then, by setting $u_0 = 0$ or $u_0 = 1$, $(W, \perp)$ is an O-set.

Now, we give the concepts of an O-sequence, a $\perp$-continuous mapping, an O-complete orthogonal metric space, a $\perp$-preserving mapping and a weakly $\perp$-preserving mapping.

Definition 2.22. [14] Let $(W, \perp)$ be an O-set. A sequence $\{u_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, u_n \perp u_{n+1}) \quad or \quad (\forall n \in \mathbb{N}, u_{n+1} \perp u_n).$$

Definition 2.23. [14] A tripled $(W, \perp, d)$ is called an orthogonal metric space if $(W, \perp)$ is an O-set and (W, d) is a metric space.

Definition 2.24. [14] Let $(W, \perp, d)$ be an orthogonal metric space. Then a mapping $H : W \to W$ is said to be orthogonally continuous (or $\perp$-continuous) in $u \in W$ if for each O-sequence $\{u_n\}$ in W with $u_n \to u$ as $n \to \infty$, we have $H(u_n) \to H(u)$ as $n \to \infty$. Also, H is said to be $\perp$-continuous on W if H is $\perp$-continuous in each $u \in W$.

Remark 2.25. [14] Every continuous mapping is $\perp$-continuous and the converse is not true.

Definition 2.26. [14] Let $(W, \perp, d)$ be an orthogonal metric space. Then W is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

Remark 2.27. [14] Every complete metric space is O-complete and the converse is not true.

Definition 2.28. [14] Let $(W, \perp, d)$ be an orthogonal metric space. A mapping $H : W \to W$ is said to be $\perp$-preserving if $Hu \perp Hv$ whenever $u \perp v$. Also, $H : W \to W$ is said to be weakly $\perp$-preserving if $H(u) \perp H(v)$ or $H(v) \perp H(u)$ whenever $u \perp v$.

In this paper, we modify the concepts of F-contractive type mapping, Kannan F-contractive type mapping and F-expanding type mapping to an orthogonal sets and prove some fixed point theorems for F-contractive type mapping, Kannan F-contractive type mapping and F-expanding type mapping. Also, we give some example to illustrate our results.

3. Main results

In this section, inspired by the notions of F-contractive type mapping, Kannan F-contractive type mapping and F-expanding type mapping, defined Goswami, Haokip and Mishra [17], we introduce a new F-contractive type mapping, Kannan F-contractive type mapping and F-expanding type mapping and prove some fixed point theorems for these contraction mapping in an orthogonal b-metric space.

Definition 3.1. Let $(W, \perp, d, s)$ be an orthogonal b-metric space. A mapping $H : W \to W$ is said to be an orthogonal F-contractive type mapping (briefly, $H_\perp$-contractive) on $(W, \perp, s)$ if there are $F \in \Im$ and $\tau > 0$ such that for all $u, v \in W$ with $u \perp v$ and $[d(Hu, Hv) > 0, d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{3}\{F(d(u, v)) + F(d(u, Hu)) + F(d(v, Hv))\}], \tag{3.1}$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{3}\{F(d(u, v)) + F(d(u, Hv)) + F(d(v, Hu))\}. \tag{3.2}$$

Definition 3.2. Let $(W, \perp, d, s)$ be an orthogonal b-metric space. A mapping $H : W \to W$ is said to be an orthogonal Kannan F-contractive type mapping (briefly, Kannan $H_\perp$-contractive) on $(W, \perp, s)$ if there are $F \in \Im$ and $\tau > 0$ such that for all $u, v \in W$ with $u \perp v$ and $[d(Hu, Hv) > 0, d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{2}\{F(d(u, Hu)) + F(d(v, Hv))\}] \tag{3.3}$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq \frac{1}{2}\{F(d(u, Hv)) + F(d(v, Hu))\}. \tag{3.4}$$

Definition 3.3. Let $(W, \perp, d, s)$ be an orthogonal b-metric space. A map $H : W \to W$ is said to be an orthogonal F-expanding type mapping (briefly, $H_\perp$-expanding type mapping) on $(W, \perp, s)$, if there are $F \in \Im$ and $\tau > 0$ such that for all $u, v \in W$ with $u \perp v$ and $[d(Hu, Hv) > 0, d(u, Hu)d(v, Hv) \neq 0$ implies

$$\tau + F(sd(u, v)) \leq \frac{1}{3}\{F(d(Hu, Hv)) + F(d(u, Hu)) + F(d(v, Hv))\}] \tag{3.5}$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(u, v)) \leq \frac{1}{3}\{F(d(Hu, Hv)) + F(d(u, Hv)) + F(d(v, Hu))\}. \tag{3.6}$$

Now, we give the fixed point theorem for F-contractive type mapping in an O-complete b-metric space $(W, \perp, d, s)$.

Theorem 3.4. *Let $(W, \perp, d, s)$ be an O-complete orthogonal b-metric space, u_0 be an orthogonal element and $H : W \to W$ be a mapping. Suppose that there exist $F \in \Im$ and $\tau > 0$ such that the following conditions hold:*

(i) H is $\perp$-preserving;
(ii) H is an $H_\perp$-contractive type mapping;
(iii) H is $\perp$-continuous.

Then H is a Picard operator.

Proof. Since $(W, \perp)$ is an O-set,

$$\exists u_0 \in W : (\forall u \in W, u \perp u_0) \quad \text{or} \quad (\forall u \in W, u_0 \perp u).$$

It follows that $u_0 \perp Hu_0$ or $Hu_0 \perp u_0$. Let $u_n = Hu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_n = u_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that u_n is a fixed point of H. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $d(u_n, u_{n+1})$ by λ_n for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since H is $\perp$-preserving, we have

$$u_n \perp u_{n+1} \quad \text{or} \quad u_{n+1} \perp u_n$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{u_n\}$ is an O-sequence. Since H is $H_\perp$-contractive type mapping, we have

$$F(s\lambda_n) \leq \frac{1}{3} \{F(d(u_{n-1}, u_n)) + F(d(u_{n-1}, u_n)) + F(d(u_n, u_{n+1}))\} - \tau$$

or

$$F(s\lambda_n) \leq F(\lambda_{n-1}) - \frac{3}{2}\tau.$$

By the condition (F_4), we have

$$F(s^n\lambda_n) \leq F(s^{n-1}\lambda_{n-1}) - \frac{3}{2}\tau$$

and hence, by induction,

$$F(s^n\lambda_n) \leq F(s^{n-1}\lambda_{n-1}) - \frac{3}{2}\tau \leq \ldots \leq F(\lambda_0) - \frac{3}{2}n. \tag{3.7}$$

Thus we get

$$\lim_{n\to\infty} F(s^n\lambda_n) = -\infty.$$

and so

$$\lim_{n\to\infty} s^n\lambda_n = 0.$$

From condition (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n\lambda_n)^k F(s^n\lambda_n) = 0.$$

Multiplication of (3.7) with $(s^n\lambda_n)^k$ yields

$$0 \leq (s^n\lambda_n)^k F(s^n\lambda_n) + \frac{3}{2}n(s^n\lambda_n)^k\tau \leq (s^n\lambda_n)^k F(\lambda_0).$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} n(s^n\lambda_n)^k = 0.$$

Now, following the proof of [22, Theorem 3.2], we can show that $\{u_n\}$ is a Cauchy O-sequence. Since $(W, \perp, d, s)$ is O-complete, there exists $t \in W$ such that

$$\lim_{n\to\infty} u_n = t.$$

Since H is $\perp$-continuous, we have

$$Ht = H(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} u_{n+1} = t$$

and so t is a fixed point of H. Let $h, g \in W$ be two fixed points of H and suppose that $H^n h = h \neq g = H^n g$ for all $n \in \mathbb{N}$. By choice of u_0, we obtain

$$\Big(u_0 \perp h \quad and \quad u_0 \perp g\Big) \quad or \quad \Big(h \perp u_0 \quad and \quad g \perp u_0\Big).$$

Since H is $\perp$-preserving, we have

$$\Big(H^n u_0 \perp H^n h \quad and \quad H^n u_0 \perp H^n g\Big) or \Big(H^n h \perp H^n u_0 \quad and \quad H^n g \perp H^n u_0\Big)$$

for all $n \in \mathbb{N}$. Now

$$\begin{aligned} d(h, g) &= d(H^n h, H^n g) \\ &\leq s\Big(d(H^n h, H^n u_0) + d(H^n u_0, H^n g)\Big). \end{aligned}$$

As $n \to \infty$, we obtain $d(h, g) \leq 0$. Thus $h = g$. Hence H has a unique fixed point in W. □

Theorem 3.5. *Let $(W, \perp, d, s)$ be an O-complete orthogonal b-metric space, u_0 be an orthogonal element and $H : W \to W$ be a mapping. Suppose that there exist $F \in \Im$ and $\tau > 0$ such that the following conditions hold:*

(i) H is $\perp$-preserving;
(ii) H is a Kannan $H_\perp$-contractive type mapping;
(iii) H is $\perp$-continuous.

Then H is a Picard operator.

Proof. The result follows from the proof of Theorem 3.4. □

Theorem 3.6. *Let $(W, \perp, d, s)$ be a boundedly compact orthogonal b-metric space, u_0 be an orthogonal element and $H : W \to W$ be a mapping. Suppose that there exist $F \in \Im$ and $\tau > 0$ such that the following conditions hold:*

(i) H is $\perp$-preserving;
(ii) H is a Kannan $H_\perp$-contractive type mapping;
(iii) H is $\perp$-continuous.

Then H is a Picard operator.

Proof. Since $(W, \perp)$ is an O-set,

$$\exists u_0 \in W : (\forall u \in W, u \perp u_0) \quad \text{or} \quad (\forall u \in W, u_0 \perp u).$$

It follows that $u_0 \perp Hu_0$ or $Hu_0 \perp u_0$. Let $u_n = Hu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_n = u_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that u_n is a fixed point of H. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $d(u_n, u_{n+1})$ by λ_n for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since H is $\perp$-preserving, we have

$$u_n \perp u_{n+1} \quad \text{or} \quad u_{n+1} \perp u_n$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{u_n\}$ is an O-sequence. Since H is a Kannan $H_\perp$-contractive type mapping, we have

$$\begin{aligned}
\tau + F(s\lambda_n) = F(sd(H^n u_0, H^{n+1} u_0)) &= F(sd(H(H^{n-1} u_0), H(H^n u_0))) \\
&\leq \frac{1}{2}\{F(d(H^{n-1} u_0, H^n u_0)) + F(d(H^n u_0, H^{n+1} u_0))\} \\
&= \frac{1}{2}\{F(q\lambda_{n-1}) + F(q\lambda_n)\} \leq \frac{1}{2} F(q\lambda_{n-1}) + \frac{1}{2} F(\lambda_n).
\end{aligned}$$

This implies

$$\tau + F(s\lambda_n) \leq F(\lambda_{n-1}), \quad \forall n \in \mathbb{N}$$

The rest of the proof is similar to that of Theorem 3.5, by using the fact that a Cauchy O-sequence is bounded. The uniqueness of the fixed point is derived from (3.4). □

Theorem 3.7. *Let $(W, \perp, d, s)$ be an O-complete orthogonal b-metric space and u_0 be an orthogonal element. A mapping $H : W \to W$ is said to be an $H_\perp$-A.R mapping on $(W, \perp, s)$ if there are $F \in \mathfrak{I}$ and $\tau > 0$ such that for all $u, v \in W$ with $u \perp v$ and $[d(Hu, Hv) > 0$, $d(u, Hu)d(v, Hv) \neq 0$ implies*

$$\tau + F(sd(Hu, Hv)) \leq F(d(u, Hu)) + F(d(v, Hv)) \tag{3.8}$$

and $d(u, Hu)d(v, Hv) = 0$ implies

$$\tau + F(sd(Hu, Hv)) \leq F(d(u, Hv)) + F(d(v, Hu)). \tag{3.9}$$

Then the following conditions hold:

(i) H is $\perp$-preserving;
(ii) H is an $H_\perp$-A.R mapping;
(iii) H is $\perp$-continuous.

Then H has a unique fixed point $t \in W$ and for every $u \in W$, the sequence $\{H^n u\}$ converges to t.

Proof. Since $(W, \perp)$ is an O-set,

$$\exists u_0 \in W : (\forall u \in W, u \perp u_0) \quad \text{or} \quad (\forall u \in W, u_0 \perp u).$$

It follows that $u_0 \perp Hu_0$ or $Hu_0 \perp u_0$. Let $u_n = Hu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_n = u_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that u_n is a fixed point of H. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $d(u_n, u_{n+1})$ by λ_n for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since H is $\perp$-preserving, we have

$$u_n \perp u_{n+1} \quad \text{or} \quad u_{n+1} \perp u_n$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{u_n\}$ is an O-sequence. Since H is an $H_\perp$-A. R mapping, we have for all m, n with $n < m \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} \tau + F(sd(u_{n+1}, u_{m+1})) &\leq F(d(H^n u_0, H^{n+1} u_0)) + F(d(H^m u_0, H^{m+1} u_0)) \\ &= F(\lambda_n) + F(\lambda_m). \end{aligned}$$

Thus we get

$$\lim_{n\to\infty} F(sd(u_{n+1}, u_{m+1})) = -\infty$$

or

$$\lim_{n\to\infty} sd(u_{n+1}, u_{m+1}) = 0,$$

showing that $\{u_n\}$ is a Cauchy O-sequence. The completeness of W ensures the existence of $t \in W$ such that

$$\lim_{n\to\infty} u_n = t.$$

Since H is $\perp$-continuous, we have

$$Ht = H(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} u_{n+1} = t$$

and so t is a fixed point of H. Let $h, g \in W$ be two fixed points of H and suppose that $H^n h = h \neq g = H^n g$ for all $n \in \mathbb{N}$. By choice of u_0, we obtain

$$\Big(u_0 \perp h \quad and \quad u_0 \perp g\Big) \quad or \quad \Big(h \perp u_0 \quad and \quad g \perp u_0\Big).$$

Since H is $\perp$-preserving, we have

$$\Big(H^n u_0 \perp H^n h \quad and \quad H^n u_0 \perp H^n g\Big) or \Big(H^n h \perp H^n u_0 \quad and \quad H^n g \perp H^n u_0\Big)$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} d(h, g) &= d(H^n h, H^n g) \\ &\leq s\Big(d(H^n h, H^n u_0) + d(H^n u_0, H^n g)\Big). \end{aligned}$$

As $n \to \infty$, we obtain $d(h, g) \leq 0$. Thus $h = g$. Hence H has a unique fixed point in W. □

Theorem 3.8. *Let $(W, \perp, d, s)$ be an O-complete orthogonal b-metric space, u_0 be an orthogonal element and $H : W \to W$ be surjective mapping. Suppose that there exist $F \in \mathfrak{I}$ and $\tau > 0$ such that the following conditions hold:*

(i) H is $\perp$-preserving;
(ii) H is an $H_\perp$-expanding type mapping;
(iii) H is $\perp$-continuous.

Then H has a unique fixed point $t \in W$ and for every $u \in W$, the sequence $\{H^n u\}$ converges to t.

Proof. By Lemma 2.15, there exists a mapping $H^* : W \to W$ such that $H \circ H^*$ is the identity mapping in W. Since $(W, \perp)$ is an O-set,

$$\exists u_0 \in W : (\forall u \in W, u \perp u_0) \quad \text{or} \quad (\forall u \in W, u_0 \perp u).$$

It follows that $u_0 \perp Hu_0$ or $Hu_0 \perp u_0$. Let $u_n = Hu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_n = u_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that u_n is a fixed point of H. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $d(u_n, u_{n+1})$ by λ_n for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\lambda_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since H is $\perp$-preserving and also from (3.5), we have

$$\tau + F(sd(H^* u, H^* v)) \leq \frac{1}{3}\{F(d(u, v)) + F(d(u, H^* u)) + F(d(v, H^* v))\}$$

for $d(u, Hu)d(v, Hv) \neq 0$ and

$$\tau + F(sd(H^* u, H^* v)) \leq \frac{1}{3}\{F(d(u, v)) + F(d(u, H^* v)) + F(d(v, H^* u))\}$$

for $d(u, Hu)d(v, Hv) = 0$, showing that H^* is an $H_\perp$-contractive type mapping.

Let $h, g \in W$ be two fixed points of H and suppose that $H^n h = h \neq g = H^n g$ for all $n \in \mathbb{N}$. By choice of u_0, we obtain

$$\Big(u_0 \perp h \quad \textit{and} \quad u_0 \perp g\Big) \quad \textit{or} \quad \Big(h \perp u_0 \quad \textit{and} \quad g \perp u_0\Big).$$

Since H is $\perp$-preserving, we have

$$\Big(H^n u_0 \perp H^n h \quad \textit{and} \quad H^n u_0 \perp H^n g\Big) \textit{or} \Big(H^n h \perp H^n u_0 \quad \textit{and} \quad H^n g \perp H^n u_0\Big)$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} d(h, g) &= d(H^n h, H^n g) \\ &\leq s\Big(d(H^n h, H^n u_0) + d(H^n u_0, H^n g)\Big). \end{aligned}$$

As $n \to \infty$, we obtain $d(h, g) \leq 0$. Thus $h = g$. Hence H has a unique fixed point in W. □

Example 3.9. Let $W = [0, 1] \cup [2, 3]$ and $d : W \times W \to [0, \infty)$ be a mapping defined by

$$d(h, g) = \max\{h, g\}^2$$

for all $h, g \in W$. Define the binary relation $\perp$ on W by $h \perp g$ if $hg \leq (h \vee g)$, where $h \vee g = h \quad or \quad g$. Then (W, d) is an O-complete b-metric space. Define the mapping $G : W \to W$ by

$$Gg = \begin{cases} 1 & \text{if } g \in [0, 1] \\ \frac{1}{g} & \text{if } g \in [2, 3]. \end{cases}$$

Clearly, G are $\perp$-preserving and $\perp$-continuous. Now, let us consider the mapping F defined by $F(t) = \ln t$. Then G is a Kannan $H_\perp$-contraction with $\tau = \ln 2$. Let $h \perp g$. Without loss of generality, we may assume that $hg \le h$. Note that $d(u,Gu)d(v,Gv) \neq 0$ implies

$$\tau + F(sd(Gu,Gv)) \le \frac{1}{2}\{F(d(u,Gu)) + F(d(v,Gv))\},\ \ \forall u,v \in W$$

is equivalent to

$$s^2 d(Gu,Gv)^2 \le e^{-2\tau}(d(u,Gu)d(v,Gv)),\ \ \forall u,v \in W. \tag{3.10}$$

Now we consider the following cases:
Case 1: Let $u = 0$ and $v \in [0,1]$. Then

$$d(Gu,Gv)^2 = 1,\ \ d(u,Gu) = 1 \text{ and } d(v,Gv) = v^2.$$

It is clear that (3.10) is satified.
Case 2: Let $u = 0$ and $v \in [2,3]$. Then

$$d(Gu,Gv)^2 = \frac{1}{v^4},\ d(u,Gu) = \frac{1}{v^2} \text{ and } d(v,Gv) = v^2.$$

It is clear that (3.10) is satisfied.
Case 3: Let $u \in [0,1]$ and $v \in (0,1]$. Then

$$d(Gu,Gv)^2 = 1,\ \ d(u,Gu) = 1 \text{ and } d(v,Gv) = 1.$$

It is clear that (3.10) is satisfied.
Case 4: Let $u \in [2,3]$ and $v \in [0,1]$. Then

$$d(Gu,Gv)^2 = 1, d(u,Gu) = u^2 \text{ and } d(v,Gv) = 1.$$

It is clear that (3.10) is satisfied.

Note that $d(u,Gu)d(v,Gv) = 0$ implies

$$\tau + F(sd(Gu,Gv)) \le \frac{1}{2}\{F(d(u,Gv)) + F(d(v,Gu))\},\ \ \forall u,v \in W,$$

which is equivalent to

$$s^2 d(Gu,Gv)^2 \le e^{-2\tau}(d(u,Gv)d(v,Gu)),\ \ \forall u,v \in W. \tag{3.11}$$

Now we consider the following cases:
Case 1: Let $u = 0$ and $v \in [0,1]$. Then

$$d(Gu,Gv)^2 = 1,\ \ d(u,Gv) = 1 \text{ and } d(v,Gu) = 1.$$

It is clear that (3.11) is satisfied.
Case 2: Let $u = 0$ and $v \in [2,3]$. Then

$$d(Gu, Gv)^2 = 1,\ d(u, Gv) = \frac{1}{v^2} \text{ and } d(v, Gu) = v^2.$$

It is clear that (3.11) is satisfied.
Case 3: Let $u \in [0,1]$ and $v \in (0,1]$. Then

$$d(Gu, Gv)^2 = 1,\ \ d(u, Gv) = 1 \text{ and } d(v, Gu) = 1.$$

It is clear that (3.11) is satisfied.
Case 4: Let $u \in [2,3]$ and $v \in [0,1]$. Then

$$d(Gu, Gv)^2 = 1, d(u, Gv) = u^2 \text{ and } d(v, Gu) = v^2.$$

It is clear that (3.11) is satisfied.
Therefore, all the conditions of Theorem 3.5 are satisfied. Hence we can conclude that G has a unique fixed point in W, that is, a point $u = 1$.

4. Conclusions

In this paper, we proved fixed point theorems for an orthogonal F-contractive type mapping, an orthogonal Kannan F-contractive type mapping and an orthogonal F-expanding type mapping in O-complete b-metric spaces.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Conflicts of interest

The authors declare that they have no competing interests.

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