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Research article

Unified relational-theoretic approach in metric-like spaces with an application

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Abstract: In this paper, we introduce a modified implicit relation and obtain some new fixed point results for σ-implicit type contractive conditions in relational metric-like spaces. We present some nontrivial examples to illustrative facts and compare our results with the related work. We also discuss sufficient conditions for the existence of a unique positive definite solution of the non-linear matrix equation $\mathcal{U} = \mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U})\mathcal{A}_i$, where $\mathcal{D}$ is an $n \times n$ Hermitian positive definite matrix, $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots$, $\mathcal{A}_m$ are $n \times n$ matrices, and $\mathcal{G}$ is a non-linear self-mapping of the set of all Hermitian matrices which is continuous in the trace norm. Finally, we discuss a couple of examples, convergence and error analysis, average CPU time analysis and visualization of solution in surface plot.

Keywords: positive definite matrix; nonlinear matrix equation; fixed point, relational metric space; metric-like space
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1. Introduction

In recent years, a number of mathematicians have obtained fixed point results for contraction type mappings in metric spaces equipped with a partial order. Some early results in this direction were established by Turinici in [26, 27]. This type of results have been reinvestigated by Ran and Reurings [24], and Nieto and Ródríguez-López [21, 22]. The results of Turinici [26, 27] were further extended and refined in [21, 22].

Recently, Samet and Turinici [25] established fixed point theorem for nonlinear contraction under symmetric closure of an arbitrary relation. Most recently, Ahmadullah et al. [1, 3] and Alam and Imdad [5] attempted a relation-theoretic analogue of the Banach contraction principle which in turn unifies some well-known relevant order-theoretic fixed point theorems.

A number of generalizations of metric spaces have been obtained by many authors in various ways. In [19], Matthews introduced the notion of a partial metric space (PMS, in short) and presented an analogue of the classical Banach contraction theorem in these spaces with applications denotational semantics of dataflow networks. In [12], Hitzler introduced dislocated metric spaces (also known as "metric-like space" by Amini-Harandi [9]) as a new generalization of the PMS. Amini-Harandi defined σ completeness of metric-like spaces. For more detail in these two spaces, one can refer [8, 10, 11, 13, 15, 20] references cited therein.

On other hand, Ahmadullah and Imdad [2] discussed the fixed point results for F-contraction mappings on relational metric spaces and applied to get solution of a nonlinear matrix equation. Further, this work is extended by Altun *et al.* [7] to relational PMSs and established some fixed point and property (P) results with some examples. Ahmadullah *et al.* [4] proved fixed point results on relational metric-like spaces.

In the present paper, we introduce a modified implicit relation and obtain some new fixed point results in relational metric-like spaces. Our results extend and generalize a number of fixed point theorems from the literature including certain results of Ahmadullah et al. [1, 4, 5]. This paper is organized in 7 sections. Particularly, in Section 3, we introduce and discuss a modified implicit relation. In Section 4, we introduce the notion of σ-implicit type self-mapping on relational set and prove a unified fixed point result, a periodic point result and discuss several special cases. In Section 5, we discuss some useful examples to illustrate our results. In Section 6, a sufficient condition for the existence of a unique positive definite solution of a non-linear matrix equation is discussed. In Sections 7, we visualize this procedure through convergence analysis using three different initializations and a solution graph.

2. Preliminaries

Throughout this article, the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ have their usual meanings, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

Definition 2.1. [19] For a set $\mathcal{E} \neq \emptyset$, define a partial metric $p : \mathcal{E}\times\mathcal{E} \to \mathbb{R}^+$ such that for all $\nu, \vartheta, \mu \in \mathcal{E}$:

(p_1) $\nu = \vartheta \Longleftrightarrow p(\nu, \nu) = p(\nu, \vartheta) = p(\vartheta, \vartheta)$,
(p_2) $p(\nu, \nu) \leq p(\nu, \vartheta)$,
(p_3) $p(\nu, \vartheta) = p(\vartheta, \nu)$,
(p_4) $p(\nu, \vartheta) \leq p(\nu, \mu) + p(\mu, \vartheta) - p(\mu, \nu)$.

In this case, the pair $(\mathcal{E}, p)$ is a PMS.

Definition 2.2. [9] For a set $\mathcal{E} \neq \emptyset$, define a metric-like $\sigma : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ such that for all $\nu, \vartheta, \mu \in \mathcal{E}$

(σ_1) $\sigma(\nu, \vartheta) = 0 \Rightarrow \nu = \vartheta$,
(σ_2) $\sigma(\nu, \vartheta) = \sigma(\vartheta, \nu)$,
(σ_3) $\sigma(\nu, \vartheta) \leq \sigma(\nu, \mu) + \sigma(\mu, \vartheta)$.

In this case, the pair $(\mathcal{E},\ \sigma)$ is a metric-like space (MLS, in short).

Remark 2.3. [4] In is noted that every metric spaces is PMS and every PMS is MLS but the converse is not necessarily true.

Definition 2.4. [9] Let $(\mathcal{E},\sigma)$ be a MLS and $\{\nu_n\}$ a sequence in $\mathcal{E}$. Then we say that

- the sequence $\{\nu_n\}$ converges to a point x in $\mathcal{E}$ if and only if $\lim_{n\to\infty}\sigma(\nu_n,\nu)=\sigma(\nu,\nu)$,
- the sequence $\{\nu_n\}$ is Cauchy in $\mathcal{E}$ if and only if $\lim_{n,m\to\infty}\sigma(\nu_n,\nu_m)$ is exist and finite,
- the MLS $(\mathcal{E},\sigma)$ is complete if every Cauchy sequence $\{\nu_n\}$ in $\mathcal{E}$ converges to a point ν in $\mathcal{E}$ with respect to τ_σ (topology generated by σ) such that

$$\lim_{n,m\to\infty}\sigma(\nu_n,\ \nu_m)=\sigma(\nu,\nu)=\lim_{n\to\infty}\sigma(\nu_n,\nu).$$

We call $(\mathcal{E},\mathfrak{R})$ a relational set if

(i) $\mathcal{E}\neq\emptyset$ is a set and
(ii) $\mathfrak{R}$ is a binary relation on $\mathcal{E}$.

In addition, if $(\mathcal{E},\sigma)$ is a MLS, we call $(\mathcal{E},\sigma,\mathfrak{R})$ a relational metric-like space (RMLS, for short). In the following, we present some elementary results on relational sets, see [4,5,16–18,25]).

Definition 2.5. Let $(\mathcal{E},\mathfrak{R})$ be a relational set, $(\mathcal{E},\sigma,\mathfrak{R})$ be an RMLS, and let $\mathcal{P}$ be a self-mapping on $\mathcal{E}$.

(1) $\nu\in\mathcal{E}$ is $\mathfrak{R}$-related to $\vartheta\in\mathcal{E}$ if and only if $(\nu,\vartheta)\in\mathfrak{R}$.
(2) The inverse of $\mathfrak{R}$ is denoted by $\mathfrak{R}^{-1}$ and is defined as $\mathfrak{R}^{-1}=\{(\nu,\vartheta)\in\mathcal{E}\times\mathcal{E}:(\vartheta,\nu)\in\mathfrak{R}\}$ and $\mathfrak{R}^s=\mathfrak{R}\cup\mathfrak{R}^{-1}$.
(3) The set $(\mathcal{E},\mathfrak{R})$ is said to be comparable if for all $\nu,\vartheta\in\mathcal{E},\ [\nu,\vartheta]\in\mathfrak{R}$, where $[\nu,\vartheta]\in\mathfrak{R}$ means that either $(\vartheta,\nu)\in\mathfrak{R}$ or $(\nu,\vartheta)\in\mathfrak{R}$.
(4) A sequence (ν_n) in $\mathcal{E}$ is said to be $\mathfrak{R}$-preserving if $(\nu_n,\ \nu_{n+1})\in\mathfrak{R},\ \forall n\in\mathbb{N}^*$.
(5) $\mathfrak{R}$ is said to be $\mathcal{P}$-closed if $(\nu,\vartheta)\in\mathfrak{R}\Rightarrow(\mathcal{P}\nu,\mathcal{P}\vartheta)\in\mathfrak{R}$. It is said to be weakly $\mathcal{P}$-closed if $(\nu,\vartheta)\in\mathfrak{R}\Rightarrow[\mathcal{P}\nu,\mathcal{P}\vartheta]\in\mathfrak{R}$.
(6) $(\mathcal{E},\sigma,\mathfrak{R})$ is said to be $\mathfrak{R}$-complete if every $\mathfrak{R}$-preserving Cauchy sequence $\{\nu_n\}$ in $\mathcal{E}$, there is some $x\in\mathcal{E}$ such that
$$\lim_{n,m\to\infty}\sigma(\nu_n,\ \nu_m)=\sigma(\nu,\ \nu)=\lim_{n\to\infty}\sigma(\nu_n,\ \nu).$$
Recall that the limit of a convergent sequence in metric-like spaces is not necessarily unique [9].
(7) $\mathcal{P}$ is said to be $\mathfrak{R}$-sequentially-continuous at ν if for any $\mathfrak{R}$-preserving sequence $(\nu_n)\to^{\tau_\sigma}\nu$, we get $\mathcal{P}(\nu_n)\to^{\tau_\sigma}\mathcal{P}(\nu)$ as $n\to\infty$. Moreover, $\mathcal{P}$ is said to be $\mathfrak{R}$-sequentially-continuous if it is $\mathfrak{R}$-sequentially-continuous at every point of $\mathcal{E}$.
(8) $\mathfrak{R}$ is said to be σ-self-closed if for every $\mathfrak{R}$-preserving sequence with $\nu_n\to^{\tau_\sigma}\nu$, there is a subsequence (ν_{n_k}) of (ν_n), such that $[\nu_{n_k},\nu]\in\mathfrak{R}$, for all $k\in\mathbb{N}^*$.
(9) A subset $\mathcal{Z}$ of $\mathcal{E}$ is called $\mathfrak{R}$-directed if for each $\nu,\vartheta\in\mathcal{Z}$, there exists $\mu\in\mathcal{E}$ such that $(\nu,\mu)\in\mathfrak{R}$ and $(\vartheta,\mu)\in\mathfrak{R}$. It is called $(\mathcal{P},\mathfrak{R})$-directed if for each $\nu,\vartheta\in\mathfrak{Z}$, there exists $\mu\in\mathcal{E}$ such that $(\nu,\mathcal{P}\mu)\in\mathfrak{R}$ and $(\vartheta,\mathcal{P}\mu)\in\mathfrak{R}$.
(10) For $\nu,\vartheta\in\mathcal{E}$, a path of length k (where k is a natural number) in $\mathfrak{R}$ from ν to ϑ is a finite sequence $\{\mu_0,\ \mu_1,\ \mu_2,\ \ldots,\ \mu_k\}\subset\mathcal{E}$ satisfying the following conditions:

(i) $z_0 = \nu$ and $\mu_k = \vartheta$,
(ii) $(\mu_i,\ \mu_{i+1}) \in \Re$ for each i $(0 \le i \le k-1)$,

then this finite sequence is called a path of length k joining ν to ϑ in $\Re$.

(11) If for a pair of $\nu, \vartheta \in \mathcal{E}$, there is a finite sequence $\{\mu_0,\ \mu_1,\ \mu_2,\ \ldots,\ \mu_k\} \subset \mathcal{E}$ satisfying the following conditions:

(i) $\mathcal{P}\mu_0 = \nu$ and $\mathcal{P}\mu_k = \vartheta$,
(ii) $(\mathcal{P}\mu_i,\ \mathcal{P}\mu_{i+1}) \in \Re$ for each i $(0 \le i \le k-1)$,

then this finite sequence is called a $\mathcal{P}$-path of length k joining ν to ϑ in $\Re$.

Notice that a path of length k involves $k+1$ elements of $\mathcal{E}$, although they are not necessarily distinct.

Remark 2.6. [4] Every complete MLS is an $\Re$-complete but the converse is not true. For the the universal relation $\Re$, the notion of $\Re$-completeness coincides with completeness, and $\Re$-continuity coincides with continuity. On an RMLS, every continuous mapping is $\Re$-continuous but not conversely.

We fix the following notation for a relational space $(\mathcal{E}, \Re)$, a self-mapping $\mathcal{T}$ on $\mathcal{E}$ and an $\Re$-directed subset $\mathfrak{D}$ of $\mathcal{E}$:

(i) $fix(\mathcal{T})$: the set of all fixed points of $\mathcal{T}$,
(ii) $\mathcal{X}(\mathcal{T}, \Re) : \{\nu \in \mathcal{E} : (\nu, \mathcal{T}\nu) \in \Re\}$,
(iii) $\mathfrak{P}(\nu, \vartheta, \Re)$: the class of all paths in $\Re$ from ν to ϑ in $\Re$, where $\nu, \vartheta \in \mathcal{E}$.

3. Implicit relations

In this section we introduce a modified version of implicit relation and examples discussed in [6,23].

Denote $\Phi := \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+\ \}$ satisfying the following conditions:

(i) φ is increasing and $\varphi(0) = 0$;
(ii) $\sum_{n=1}^{\infty} \varphi^n(\zeta) < \infty$, for $\zeta > 0$; where φ^n is the n-th iterate.

It should be noted that $\varphi(\zeta) < \zeta$ and the family $\Phi \neq \emptyset$.

Example 3.1. Consider $(\mathcal{E}, \sigma)$ with $\sigma(\nu, \vartheta) = \max\{\nu, \vartheta\}$, where $\mathcal{E} = [0, 1]$. Define the mapping $\varphi(\zeta) = \frac{\lambda\zeta}{5}$, where $0 < \lambda < 1$. Then we have $\varphi^n(\zeta) \le \frac{\lambda^n\zeta}{5^n}$. Therefore, $\sum_{n=1}^{\infty} \varphi^n(\zeta) = \sum_{n=1}^{\infty} \frac{\lambda^n\zeta}{5^n} < \infty$ and hence $\Phi \neq \emptyset$.

Let $\Upsilon := \{\mathcal{G} : \mathbb{R}_+^5 \to \mathbb{R}$ is continuous function $\}$ satisfying the following conditions:

$(\mathcal{G}_1)$ $\mathcal{G}$ is nonincreasing in fifth variable;
$(\mathcal{G}_2)$ there exists $\varphi \in \Phi$ such that for all $\zeta, \xi, \mu \ge 0$,

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \le 0, \text{ implies that } \zeta \le \varphi(\xi);$$

$(\mathcal{G}_3)$ $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) > 0$ for $\zeta > 0$.

Let $\Upsilon' := \{\mathcal{G} \in \Upsilon\}$ and following additional conditions hold:

$(\mathcal{G}_4)$ $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) > 0$, for all $\zeta > 0$.

Obviously Υ is more general than Υ'.

Example 3.2. Let $\mathcal{G}(r_1, r_2, r_3, r_4, r_5, r_6) = r_1 - a\max\{r_2, r_3, r_4\} + b(r_5 + r_6)$, $0 \le a < 1$ and $b > 0$.

($\mathcal{G}_2$) Then for $\zeta, \xi, \mu \ge 0$, we have

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta - a\max\{\xi, \xi, \zeta\} + b(\zeta + \xi + \mu).$$

If $\xi \le \zeta$, then $(1 - a)\zeta > 0$ and so, $\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) > 0$. Therefore

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \le 0 \Longrightarrow \zeta \le \varphi(\xi) \text{ where } \varphi(\xi) = h\xi \text{ and } h < 1.$$

($\mathcal{G}_3$) $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta + 2b\zeta > 0$ for all $\zeta > 0$.
($\mathcal{G}_4$) $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) = (1 - a + b)\zeta > 0$ for all $\zeta > 0$.

Example 3.3. Let $\mathcal{G}(r_1, r_2, r_3, r_4, r_5, r_6) = r_1^2 - ar_2^2 - b\frac{r_3^2+r_4^2}{r_5+r_6+1}, 0 < a, b < 1$ and $a + 2b < 1$.

($\mathcal{G}_2$) Then for $\zeta, \xi, \mu \ge 0$, we have

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta^2 - a\xi^2 - b\frac{\zeta^2 + \xi^2}{1 + \zeta + \xi + \mu}.$$

It is evident that $\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \le 0 \Longrightarrow \zeta \le \varphi(\xi)$ where $\varphi(\xi) = h\xi$ and $h = \sqrt{\frac{a+b}{1-b}} < 1$.
($\mathcal{G}_3$) For all $\zeta > 0$, $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta^2 > 0$.
($\mathcal{G}_4$) $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) = \frac{(1-b)\zeta^2+\zeta^3}{1+\zeta} > 0$, for all $\zeta > 0$.

Example 3.4. Let $\mathcal{G}(r_1, r_2, r_3, r_4, r_5, r_6) = r_1^2 - ar_2^2 - b\frac{r_3^2+r_4^2}{r_5^2+r_6^2+1}, 0 < a, b < 1$ and $a + 2b < 1$.
Proof is similar to Example 3.3.

Example 3.5. Let $\mathcal{G}(r_1, r_2, r_3, r_4, r_5, r_6) = r_1 - ar_2 - br_3 - c\frac{r_4 r_5}{r_5+r_6+1}, 0 < a, b, c < 1$ and $a + b + c < 1$.

($\mathcal{G}_2$) Then for $\zeta, \xi, \mu \ge 0$, we have

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta - a\xi - b\xi - c\frac{\zeta(\zeta + \xi)}{1 + \zeta + \xi + \mu}.$$

It is evident that $\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \le 0 \Longrightarrow \zeta \le \varphi(\xi)$ where $\varphi(\xi) = h\xi$ and $h = \left(\frac{a+b}{1-b}\right) < 1$.
($\mathcal{G}_3$) $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta > 0$ for all $\zeta > 0$.
($\mathcal{G}_4$) $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) = \frac{\zeta+(1-c)\zeta^2}{1+\zeta} > 0$ for all $\zeta > 0$.

Example 3.6. Let $\mathcal{G}(r_1, r_2, r_3, r_4, r_5, r_6) = r_1 - ar_2 - b\frac{r_3 r_6}{r_5+r_6+1} - cr_4$, $0 < a, b, c < 1$ and $a + b + c < 1$.

($\mathcal{G}_2$) Then for $\zeta, \xi, \mu \ge 0$, we have

$$\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) = \zeta - a\xi - b\frac{\zeta\xi}{1 + \zeta + \xi + \mu} - c\zeta.$$

It is evident that $\mathcal{G}(\zeta, \xi, \xi, \zeta, \zeta + \xi, \mu) \le 0 \Longrightarrow \zeta \le \varphi(\xi)$ where $\varphi(\xi) = h\xi$ and $h = \frac{a}{1-b-c} < 1$.
($\mathcal{G}_3$) $\mathcal{G}(\zeta, \zeta, 0, 0, \zeta, \zeta) = (1 - a)\zeta > 0$ for all $\zeta > 0$.
($\mathcal{G}_4$) $\mathcal{G}(\zeta, 0, 0, \zeta, \zeta, 0) = (1 - c)\zeta > 0$ for all $\zeta > 0$.

4. Results on metric-like spaces

4.1. Fixed point results under σ-implicit contractive mappings

We start with defining σ-implicit contractive mappings on RMLS under implicit relation $\mathcal{G} \in \Upsilon$.

Definition 4.1. Let $(\mathcal{E}, \sigma, \Re)$ be an *RMLS* and $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ be a given mapping. A mapping $\mathcal{T}$ is said to be a σ-implicit type mapping, if there exist $\mathcal{G} \in \Upsilon$, such that for $(\nu, \vartheta) \in \mathcal{E} \times \mathcal{E}$ with $(\nu, \vartheta) \in \Re$,

$$\mathcal{G}\Big(\sigma(\mathcal{T}\nu, \mathcal{T}\vartheta), \sigma(\nu, \vartheta), \sigma(\nu, \mathcal{T}\nu), \sigma(\vartheta, \mathcal{T}\vartheta), \sigma(\nu, \mathcal{T}\vartheta), \sigma(\vartheta, \mathcal{T}\nu) \Big) \leq 0. \tag{4.1}$$

Now, we are equipped to state and prove our first main result as follows:

Theorem 4.2. *Let $(\mathcal{E}, \sigma, \Re)$ be an RMLS and $\mathcal{T} : \mathcal{E} \to \mathcal{E}$. Assume that*

(a) there exists a subset $\mathcal{Y} \subseteq \mathcal{E}$ with $\mathcal{T}\mathcal{E} \subseteq \mathcal{Y}$ such that $(\mathcal{Y}, \sigma)$ is $\Re$-complete;
(b) $\mathcal{X}(\mathcal{T}, \Re) \neq \emptyset$;
(c) $\Re$ is $\mathcal{T}$-closed;
(d) $\mathcal{T}$ is σ-implicit type mapping;
(e) $\mathcal{T}$ is $\Re$-sequentially-continuous.

Then there exists a point $\nu_* \in fix(\mathcal{T})$.

Proof. Let $\nu_0 \in \mathcal{X}(\mathcal{T}, \Re)$. Define $\nu_{n+1} = \mathcal{T}\nu_n = \mathcal{T}^{n+1}\nu_0$ for all $n \in \mathbb{N}^*$. If there exists $n_0 \in \mathbb{N}^*$ such that $\mathcal{T}\nu_{n_0} = \nu_{n_0}$ then $\nu_{n_0} \in fix(\mathcal{T})$ and there is nothing to prove. Assume that $\nu_{n+1} \neq \nu_n$ for all $n \in \mathbb{N}^*$ so that $\sigma(\mathcal{T}\sigma_{n+1}, \mathcal{T}\sigma_n) > 0$. Since $(\nu_0,\ \mathcal{T}\nu_0) \in \Re$, by condition (b),

$$(\mathcal{T}\nu_0, \mathcal{T}^2\nu_0),\ (\mathcal{T}^2\nu_0, \mathcal{T}^3\nu_0),\ \cdots,\ (\mathcal{T}^n\nu_0, \mathcal{T}^{n+1}\nu_0) \in \Re,$$

and so

$$(\nu_n, \nu_{n+1}) \in \Re \text{ for all } n \in \mathbb{N}^*.$$

Hence $\{\nu_n\}$ is an $\Re$-preserving sequence. Using the condition (d) for $\nu = \nu_n$, $\vartheta = \nu_{n+1}$, we have (for all $n \in \mathbb{N}^*$)

$$\mathcal{G}\begin{pmatrix} \sigma(\mathcal{T}\nu_n, \mathcal{T}\nu_{n+1}), \sigma(\nu_n, \nu_{n+1}), \sigma(\nu_n, \mathcal{T}\nu_n), \\ \sigma(\nu_{n+1}, \mathcal{T}\nu_{n+1}), \sigma(\nu_n, \mathcal{T}\nu_{n+1}), \sigma(\nu_{n+1}, \mathcal{T}\nu_n) \end{pmatrix} \leq 0$$

or

$$\mathcal{G}\begin{pmatrix} \sigma(\nu_{n+1}, \nu_{n+2}), \sigma(\nu_n, \nu_{n+1}), \sigma(\nu_n, \nu_{n+1}), \\ \sigma(\nu_{n+1}, \nu_{n+2}), \sigma(\nu_n, \nu_{n+2}), \sigma(\nu_{n+1}, \nu_{n+1}) \end{pmatrix} \leq 0$$

Using property (σ_3) and $(\mathcal{G}_1)$, we can write

$$\mathcal{G}\begin{pmatrix} \sigma(\nu_{n+1}, \nu_{n+2}), \sigma(\nu_n, \nu_{n+1}), \sigma(\nu_n, \nu_{n+1}), \\ \sigma(\nu_{n+1}, \nu_{n+2}), \sigma(\nu_n, \nu_{n+1}) + \sigma(\nu_{n+1}, \nu_{n+2}), \sigma(\nu_{n+1}, \nu_{n+1}) \end{pmatrix} \leq 0.$$

It follows from $(\mathcal{G}_2)$ that there is $\varphi \in \Phi$ such that

$$\sigma(\nu_{n+1}, \nu_{n+2}) \leq \varphi(\sigma(\nu_n, \nu_{n+1})). \tag{4.2}$$

By property (σ_3), (4.2) and for all $n,\ m \in \mathbb{N}^*$ with $m > n$, we have

$$\sigma(\nu_n,\ \nu_m) \quad \leq \quad \sigma(\nu_n,\ \nu_{n+1}) + \sigma(\nu_{n+1},\ \nu_{n+2}) + \cdots + \sigma(\nu_{m-1},\ \nu_m)$$

$$
\begin{aligned}
&\leq (\varphi^n + \varphi^{n+1} + \cdots + \varphi^{m-1})\sigma(\nu_0, \mathcal{T}\nu_0) \\
&= \sum_{j=n}^{m-1} \varphi^j(\sigma(\nu_0,\ \mathcal{T}\nu_0)) \\
&= \sum_{j=1}^{m-1} \varphi^j(\sigma(\nu_0,\ \mathcal{T}\nu_0)) - \sum_{j=1}^{n-1} \varphi^j(\sigma(\nu_0,\ \mathcal{T}\nu_0)) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus $\lim_{n,m\to\infty} \sigma(\nu_n,\ \nu_m) = 0$. Therefore $\{\nu_n\}$ is Cauchy sequence in $\mathcal{Y}$. Hence, $\{\nu_n\}$ is an $\mathfrak{R}$-preserving Cauchy sequence in $\mathcal{Y}$. By $\mathfrak{R}$-completeness of $(\mathcal{Y}, \sigma)$, so the sequence $\{\nu_n\}$ converges to some ν_* in $(\mathcal{Y}, \sigma)$ i.e.,

$$\lim_{n\to\infty} \sigma(\nu_n, \nu_*) = \sigma(\nu_*, \nu_*) = \lim_{n,m\to\infty} \sigma(\nu_n,\ \nu_m) = 0. \tag{4.3}$$

By condition (e), $\nu_{n+1} = \mathcal{T}\nu_n \to^{\tau_\sigma} \mathcal{T}\nu_*$, so that

$$\lim_{n\to\infty} \sigma(\nu_{n+1}, \mathcal{T}\nu_*) = \lim_{n\to\infty} \sigma(\mathcal{T}\nu_n, \mathcal{T}\nu_*) = \sigma(\mathcal{T}\nu_*, \mathcal{T}\nu_*) = \lim_{n,m\to\infty} \sigma(\nu_n,\ \nu_m) = 0. \tag{4.4}$$

Using property (σ_3), (4.3) and (4.4), we have $\sigma(\nu_*, \mathcal{T}\nu_*) = 0$. Thus $fix(\mathcal{T}\} = \{\nu_*\}$. □

Theorem 4.3. *The conclusion of Theorem 4.2 remains true for $\mathcal{G} \in \Upsilon'$, if the condition (e) is replaced by*

(e') $\mathfrak{R}|_{\mathcal{Y}}$ is σ-self-closed.

Proof. By Theorem 4.2, we observe that the sequence $\{\nu_n\}$ is an $\mathfrak{R}$-preserving sequence in $\mathcal{Y}$ and $\nu_n \to^{\tau_\sigma} \nu_*$, and there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k},\ \nu_*] \in \mathfrak{R}$, for all $k \in \mathbb{N}^*$. Using (d) for $\nu = \nu_{n_k}$, $\vartheta = \nu_*$, we have

$$\mathcal{G}\begin{pmatrix} \sigma(\mathcal{T}\nu_{n_k}, \mathcal{T}\nu_*), \sigma(\nu_{n_k}, \nu_*), \sigma(\nu_{n_k}, \mathcal{T}\nu_{n_k}), \\ \sigma(\nu_*, \mathcal{T}\nu_*), \sigma(\nu_{n_k}, \mathcal{T}\nu_*), \sigma(\nu_*, \mathcal{T}\nu_{n_k}) \end{pmatrix} \leq 0.$$

Taking the limit as $k \to \infty$ and using (4.3), we have

$$\mathcal{G}\left(\sigma(\nu_*, \mathcal{T}\nu_*), 0, 0, \sigma(\nu_*, \mathcal{T}\nu_*), \sigma(\nu_*, \mathcal{T}\nu_*), 0) \right) \leq 0.$$

It follows from $(\mathcal{G}_4)$ that $\sigma(\nu_*, \mathcal{T}\nu_*) = 0$, that is, $fix(\mathcal{T}) = \nu_*$. □

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 4.2 and 4.3.

Theorem 4.4. *Let all the conditions of Theorem 4.2 or Theorem 4.3, and*

(f') $\mathfrak{P}(\mathcal{T}\varrho, \mathcal{T}\rho; \mathfrak{R}^s) \neq \emptyset$ for all $\varrho, \rho \in fix(\mathcal{T}) \subset \mathcal{E}$.

Then $\mathcal{T}$ has a unique fixed point.

Proof. By Theorem 4.2, $fix(\mathcal{T}) \neq \emptyset$. Consider two arbitrary elements $\varrho \neq \rho \in fix(\mathcal{T})$. Since $\mathfrak{P}(\varrho, \rho; \mathfrak{R}^s) \neq \emptyset$, there exists a path (say $\{\mu_0,\ \mu_1,\ \ldots,\ \mu_k\}$) of some finite length k in $\mathfrak{R}^s$ from ϱ to ρ (with $\mu_i \neq \mu_{i+1}$ for all $0 \leq i \leq k-1$). Then

$$\mu_0 = \varrho,\ \mu_k = \rho,\ (\mu_i, \mu_{i+1}) \in \Re^s \text{ for each } 0 \le i \le k-1.$$

As $\mu_i \in fix(\mathcal{T})$, $\mathcal{T}\mu_i = \mu_i$ for all $0 \le i \le k-1$. Also by (b), $[\mathcal{T}^n\mu_i, \mathcal{T}^n\mu_{i+1}] \in \Re$ for all $0 \le i \le k-1$. Using (d) for $\nu = \mathcal{T}^n\mu_i$, $\vartheta = \mathcal{T}^n\mu_{i+1}$, we have

$$\mathcal{G}\left(\begin{array}{c} \sigma(\mathcal{T}\mathcal{T}^n\mu_i, \mathcal{T}\mathcal{T}^n\mu_{i+1}), \sigma(\mathcal{T}^n\mu_i, \mathcal{T}^n\mu_{i+1}), \sigma(\mathcal{T}^n\mu_i, \mathcal{T}\mathcal{T}^n\mu_i), \\ \sigma(\mathcal{T}^n\mu_{i+1}, \mathcal{T}^n\mathcal{T}\mu_{i+1}), \sigma(\mathcal{T}^n\mu_i, \mathcal{T}\mathcal{T}^n\mu_{i+1}), \sigma(\mathcal{T}^n\mu_{i+1}, \mathcal{T}\mathcal{T}^n\mu_i) \end{array} \right) \le 0,$$

i.e.,

$$\mathcal{G}\left(\begin{array}{c} \sigma(\mu_i, \mu_{i+1}), \sigma(\mu_i, \mu_{i+1}), \sigma(\mu_i, \mu_i), \\ \sigma(\mu_{i+1}, \mu_{i+1}), \sigma(\mu_i, \mu_{i+1}), \sigma(\mu_{i+1}, \mu_i) \end{array} \right) \le 0$$

or

$$\mathcal{G}\left(\sigma(\mu_i, \mu_{i+1}), \sigma(\mu_i, \mu_{i+1}), 0, 0, \sigma(\mu_i, \mu_{i+1}), \sigma(\mu_{i+1}, \mu_i) \right) \le 0,$$

which is a contradiction to $(\mathcal{G}_3)$. Thus, $\mathcal{T}$ has a unique fixed point. □

If we set $\mathcal{Y} = \mathcal{E}$ in Theorem 4.2, we have consequences.

Corollary 4.5. *Let $(\mathcal{E}, \sigma, \Re)$ be an RMLS and $\mathcal{T} : \mathcal{E} \to \mathcal{E}$. Suppose that the following conditions hold:*

- *(a) $(\mathcal{E}, \sigma)$ is $\Re$-complete;*
- *(b) $\mathcal{X}(\mathcal{T}, \Re) \neq \emptyset$;*
- *(c) $\Re$ is $\mathcal{T}$-closed;*
- *(d) $\mathcal{T}$ is σ-implicit type mapping;*
- *(e) $\mathcal{T}$ is $\Re$-sequentially-continuous, or*
- *(e') $\Re|_{\mathcal{E}}$ is σ-self-closed with $\mathcal{G} \in \Upsilon'$.*

Then there exists a point $\nu_* \in fix(\mathcal{T})$. In addition, if $\mathfrak{P}(\varrho, \rho; \Re^s|_{\mathcal{E}}) \neq \emptyset$ for all $\varrho, \rho \in fix(\mathcal{T})$. Then $\mathcal{T}$ has a unique fixed point.

Theorem 4.6. *The conclusion of Theorem 4.4 remains true, if the condition (f) is replaced by*

- *(f') $\mathcal{T}(\mathcal{E})$ is $\Re^s$-directed;*
- *(f'') $\Re|_{\mathcal{T}(\mathcal{E})}$ is complete.*

Proof. Suppose that (f') holds. Then for each pair of points $(\varrho, \rho) \in \mathcal{T}(\mathcal{E})$, there exists $\omega \in \mathcal{E}$ such that $[\varrho, \omega] \in \Re$ and $[\rho, \omega] \in \Re$ so that the finite sequence $\{\varrho, \omega, \rho\}$ is a path of length 2 from ϱ to ρ in $\Re^s$. Thus, for each $\varrho, \rho \in \mathcal{T}(\mathcal{E})$, $\mathfrak{P}(\varrho, \rho; \Re^s) \neq \emptyset$ and hence result follows from Theorem 4.4.

Alternatively, we suppose that (f'') holds, then for each pair of points $(\varrho, \rho) \in \mathcal{T}(\mathcal{E})$, $[\varrho, \rho] \in \Re$, which implies that $\{\varrho, \rho\}$ is a path of length 1 from ϱ to ρ in $\Re^s$, so that $\mathfrak{P}(\varrho, \rho; \Re^s) \neq \emptyset$, for each $\varrho, \rho \in \mathcal{T}(\mathcal{E})$, and hence result follows from Theorem 4.4. □

4.2. Periodic point results

Definition 4.7. [14] A mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ is said to have the property (P) if it has no periodic points, i.e., if $fix(\mathcal{T}^n) = fix(\mathcal{T})$ for every $n \in \mathbb{N}$, where $fix(\mathcal{T}) := \{u \in \mathcal{E} : \mathcal{T}u = u\}$.

Theorem 4.8. *In addition to the assumptions of Theorem 4.2, if $\nu_* \in fix(\mathcal{T}^n)$ and $\nu_* \notin fix(\mathcal{T})$ implies $(\nu_*, \mathcal{T}\nu_*) \in \Re$, then $\mathcal{T}$ has the property (P).*

Proof. Let $\omega_0 \in \mathcal{X}(\mathcal{T}, \Re)$, i.e., $(\omega_0, \mathcal{T}\omega_0) \in \Re$, so by condition (b), we have $(\omega_n, \omega_{n+1}) \in \Re$, for all $n \in \mathbb{N}^*$. Define $\omega_{n+1} = \mathcal{T}\omega_n = \mathcal{T}^{n+1}\omega_0$, for all $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}^*$ such that $\mathcal{T}\omega_{n_0} = \omega_{n_0}$, then $\omega_{n_0} \in fix(\mathcal{T})$ and there is nothing to prove.

Assume that $\omega_{n+1} \neq \omega_n$, for all $n \in \mathbb{N}^*$. Then $(\omega_n, \omega_{n+1}) \in \Re$ (for all $n \in \mathbb{N}^*$). Using condition (d) for $\omega = \omega_n$ and $\vartheta = \mathcal{T}\omega_n$, we have

$$\mathcal{G}\begin{pmatrix} \sigma(\mathcal{T}\omega_n,\ \mathcal{T}^2\omega_n), \sigma(\omega_n, \mathcal{T}\omega_n),\ \sigma(\omega_n,\ \mathcal{T}\omega_n), \\ \sigma(\mathcal{T}\omega_n,\ \mathcal{T}^2\omega_n), \sigma(\omega_n,\ \mathcal{T}^2\omega_n), \sigma(\mathcal{T}\omega_n,\ \mathcal{T}\omega_n) \end{pmatrix} \leq 0,$$

i.e.,

$$\mathcal{G}\begin{pmatrix} \sigma(\omega_{n+1}, \omega_{n+2}), \sigma(\omega_n, \omega_{n+1}), \sigma(\omega_n, \omega_{n+1}), \\ \sigma(\omega_{n+1}, \omega_{n+2}), \sigma(\omega_n, \omega_{n+2}), \sigma(\omega_{n+1}, \omega_{n+1})) \end{pmatrix} \leq 0.$$

Using property (σ_3) and $(\mathcal{G}_1)$, we get

$$\mathcal{G}\begin{pmatrix} \sigma(\omega_{n+1}, \omega_{n+2}), \sigma(\omega_n, \omega_{n+1}), \sigma(\omega_n, \omega_{n+1}), \\ \sigma(\omega_{n+1}, \omega_{n+2}), \sigma(\omega_n, \omega_{n+1}) + \sigma(\omega_{n+1}, \omega_{n+2})), \sigma(\omega_{n+1}, \omega_{n+1})) \end{pmatrix} \leq 0.$$

It follows from $(\mathcal{G}_2)$ that there is $\varphi \in \Phi$ such that for all $n \in \mathbb{N}^*$

$$\sigma(\omega_{n+1}, \omega_{n+2}) \leq \varphi(\sigma(\omega_n, \omega_{n+1})). \tag{4.5}$$

Following the remaining proof of Theorem 4.2, $\{\omega_n\}$ is a Cauchy sequence in $(\mathcal{E}, \sigma)$. Using $\Re$-complete and $\Re$-continuity of $\mathcal{T}$, $\mathcal{T}\omega = \omega$. Finally, we show that $fix(\mathcal{T}^n) = fix(\mathcal{T})$ for any $n \in \mathbb{N}$. Let on contrary that $\omega \in fix(\mathcal{T}^n)$ and $\omega \notin fix(\mathcal{T})$ for some $n \in \mathbb{N}$. Then $\sigma(\omega, \mathcal{T}\omega) > 0$ and $(\omega, \mathcal{T}\omega) \in \Re$. By condition (b), we have $(\mathcal{T}^i\omega, \mathcal{T}^{i+1}\omega) \in \mathbb{R}$ for all $i \in \mathbb{N}^*$. From (4.5), we have

$$\begin{aligned} \sigma(\omega, \mathcal{T}\omega) &= \sigma(\mathcal{T}^n\omega, \mathcal{T}\mathcal{T}^n\omega) \\ &= \sigma(\omega_n, \omega_{n+1}) \\ &\leq \varphi(\sigma(\omega_{n-1}, \omega_n)) \\ &\ \vdots \\ &\leq \varphi^n(\sigma(\omega, \mathcal{T}\omega)), \end{aligned}$$

a contradiction, as $\varphi(\zeta) < \zeta$. Therefore, $fix(\mathcal{T}^n) = fix(\mathcal{T})$ for all $n \in \mathbb{N}$. □

Choosing $\mathcal{G} \in \Upsilon$ from Examples 3.2–3.6, we have the following consequences.

Corollary 4.9. *Let all the conditions of Theorems 4.2–4.8 be satisfied, except that the assumption of σ-implicit contractive mapping for $\mathcal{G} \in \Upsilon$ is replaced by either of the form*

(I)

$$\sigma(\mathcal{T}\vartheta, \mathcal{T}\nu) \leq a \max\{\sigma(\vartheta, \nu), \sigma(\vartheta, \mathcal{T}\vartheta), \sigma(\nu, \mathcal{T}\nu)\} - b[\sigma(\vartheta, \mathcal{T}\nu) + \sigma(\mathcal{T}\vartheta, \nu)],$$

where $0 \leq a < 1$, $b > 0$, *or*

(II)

$$[\sigma(\mathcal{T}\vartheta, \mathcal{T}\nu)]^2 \leq a[\sigma(\vartheta, \nu)]^2 + b\frac{[\sigma(\vartheta, \mathcal{T}\vartheta)]^2 + [\sigma(\nu, \mathcal{T}\nu)]^2}{1 + \sigma(\vartheta, \mathcal{T}\nu) + \sigma(\mathcal{T}\vartheta, \nu)},$$

where $0 < a, b < 1$, $a + 2b < 1$, $\sqrt{\frac{a+b}{1-b}} < 1$, *or*

(III)

$$[\sigma(\mathcal{T}\vartheta,\mathcal{T}\nu)]^2 \leq a[\sigma(\vartheta,\nu)]^2 + b\frac{[\sigma(\vartheta,\mathcal{T}\vartheta)]^2 + [\sigma(\nu,\mathcal{T}\nu)]^2}{1 + [\sigma(\vartheta,\mathcal{T}\nu)]^2 + [\sigma(\mathcal{T}\vartheta,\nu)]^2},$$

where $0 < a, b < 1$, $a + 2b < 1$, $\sqrt{\frac{a+b}{1-b}} < 1$*, or*

(IV)

$$\sigma(\mathcal{T}\vartheta,\mathcal{T}\nu) \leq a\sigma(\vartheta,\nu) + b\sigma(\vartheta,\mathcal{T}\vartheta) + c\frac{\sigma(\nu,\mathcal{T}\nu)\sigma(\vartheta,\mathcal{T}\nu)}{1 + \sigma(\vartheta,\mathcal{T}\nu) + \sigma(\mathcal{T}\vartheta,\nu)},$$

where $0 < a, b, c < 1$, $a + b + c < 1$*, or*

(V)

$$\sigma(\mathcal{T}\vartheta,\mathcal{T}\nu) \leq a\sigma(\vartheta,\nu) + b\frac{\sigma(\vartheta,\mathcal{T}\vartheta)\sigma(\mathcal{T}\vartheta,\nu)}{1 + \sigma(\vartheta,\mathcal{T}\nu) + \sigma(\mathcal{T}\vartheta,\nu)} + c\,\sigma(\nu,\mathcal{T}\nu),$$

where $0 < a, b, c < 1$, $a + b + c < 1$.

Then $fix(\mathcal{T})$ *is a singleton.*

5. Illustrative example

Example 5.1. Consider the set $\mathcal{E} = [0,\infty)$, and define a metric-like σ and mapping $\mathcal{T}$ by

$$\sigma(\nu,\vartheta) = \begin{cases} 3\nu, & \nu = \vartheta, \\ \max\{\nu,\vartheta\}, & \nu \neq \vartheta, \end{cases} \quad \text{and} \quad \mathcal{T}(\nu) = \begin{cases} \frac{\nu}{3}, & \nu \in [0,1], \\ 3\nu, & \nu > 1, \end{cases}$$

respectively. Define a binary relation $\Re$ by

$$(\nu,\vartheta) \in \Re \Leftrightarrow \nu,\vartheta \in \{0\} \cup \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \text{ for all } \nu,\vartheta \in \mathcal{E}.$$

Then $\mathcal{E}$ is $\mathcal{R}$-complete. Clearly, $\mathcal{X}(\mathcal{T},\Re) \neq \emptyset$ as $(\frac{1}{3},\frac{1}{3^2}) \in \Re$. Also $\mathcal{R}$ is $\mathcal{T}$-closed and $\mathcal{T}$ is $\mathcal{R}$-sequentially-continuous. Indeed, let $\{\nu_n\}$ be an arbitrary $\Re$-preserving sequence such that $\nu_n \to^{\tau_\sigma} \nu$ (for some $\nu \in \mathcal{E}$), that is, $\{\nu_n\}$ is a sequence in $[0,1]$ such that (ν_n,ν_{n+1}) for all n with $\lim_{n\to\infty}\sigma(\nu_n,\nu) = \sigma(\nu,\nu)$. Then $\nu \in [0,1]$ and

$$\begin{aligned}
\sigma(\mathcal{T}\nu,\mathcal{T}\nu) = \sigma(\frac{\nu}{3},\frac{\nu}{3}) = \nu &= \frac{1}{3}\sigma(\nu,\nu) \\
&= \frac{1}{3}\lim_{n\to\infty}\sigma(\nu_n,\nu) \\
&= \frac{1}{3}\left(\lim_{n\to\infty}\begin{cases} 3\nu_n, & \nu_n = \vartheta \\ \max\{\nu_n,\vartheta\}, & \nu_n \neq \vartheta \end{cases}\right) \\
&= \lim_{n\to\infty}\begin{cases} 3\frac{\nu_n}{3}, & \nu_n = \nu \\ \max\{\frac{\nu_n}{3},\frac{\nu}{3}\}, & \nu_n \neq \nu \end{cases} \\
&= \lim_{n\to\infty}\sigma(\nu_n,\nu)
\end{aligned}$$

This shows that $\mathcal{T}\nu_n \to^{\tau_\sigma} \mathcal{T}\nu$ and hence $\mathcal{T}$ is $\mathcal{R}$-sequentially-continuous.

By Example 3.2, (4.1) becomes

$$\sigma(\mathcal{T}\nu,\mathcal{T}\vartheta) \le a\ \max\{\sigma(\nu,\vartheta),\sigma(\nu,\mathcal{T}\nu),\sigma(\vartheta,\mathcal{T}\vartheta)\} - b\,[\sigma(\nu,\mathcal{T}\vartheta)+\sigma(\vartheta,\mathcal{T}\nu)]. \tag{5.1}$$

We show that $\mathcal{T}$ satisfies (5.1) by dividing the proof into four parts.

Take $(\nu,\vartheta) \in \Re$ and so $0 \le \vartheta, \nu \le \frac{1}{3}$.

• If $\nu = 0$ and $\vartheta = 1/3^n$ (similarly for $\vartheta = 0$ and $\nu = 1/3^n$), $n \in \mathbb{N}$, then (5.1) reduces to

$$\frac{1}{3^{n+1}} \le a \cdot \max\{\frac{1}{3^n}, 0, \frac{1}{3^n}\} - b\left[\frac{1}{3^{n+1}} + \frac{1}{3^n}\right],$$

that is,

$$\frac{1}{3^{n+1}} \le a.\frac{1}{3^n} - b\left[\frac{1}{3^{n+1}} + \frac{1}{3^n}\right].$$

• Let $\vartheta, \nu \in \{1/3^n \mid n \in \mathbb{N}\}$ with $0 < \vartheta < \nu$, i.e., $\vartheta \le \nu/3$. Then (5.1) reduces to

$$\begin{aligned}\frac{\nu}{3} &\le a\max\{\nu,\nu,\vartheta\} - b\left[\nu + \frac{\nu}{3}\right]\\ &= a.\nu - b\left[\nu + \frac{\nu}{3}\right].\end{aligned}$$

• Let $\nu = \vartheta = 0$. The condition (5.1) is trivially true.

• Let $\nu = \vartheta \neq 0$.

$$\nu \le a\ \max\{3\nu,\nu,\vartheta\} - b[\nu + \frac{\nu}{3}]$$

that is, $1 \le 3a - \frac{4b}{3}$.

It can be easily checked that the above inequalities hold true for $a = 4/5$ and $b = 1/100$. Thus $\mathcal{T}$ is a σ-implicit contractive mapping. Therefore, all conditions of Theorem 4.2 are satisfied and $\nu_* = 0$ is a unique fixed point of $\mathcal{T}$ in $\mathcal{E}$.

It can be noted that in this example condition (4.1) is not true if $(\nu,\vartheta) \notin \Re$. For instances, for $(\nu,\vartheta) = (2,3) \notin \Re$, (5.1) becomes

$$9 \nleq 9a - 15b$$

which is not true for any $a \in [0,1), b \ge 0$. Even the results are not true (when $(\nu,\vartheta) \notin \Re$) for the contraction conditions of the form

$$\sigma(\mathcal{T}\nu,\mathcal{T}\vartheta) \le k\max\left\{\sigma(\nu,\vartheta),\sigma(\nu,\mathcal{T}\nu),\sigma(\vartheta,\mathcal{T}\vartheta),\frac{1}{2}[\sigma(\nu,\mathcal{T}\vartheta)+\sigma(\vartheta,\mathcal{T}\nu)]\right\},$$

where $k \in (0,1)$.

6. Application

Let $\mathcal{M}(n)$ denote the set of all $n \times n$ matrices over $\mathbb{C}$ and $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all Hermitian (resp. positive semi-definite, positive definite) matrices from $\mathcal{M}(n)$. Also, we denote the element of $\mathcal{P}(n)$ as $\mathcal{X} > O$. For a matrix $\mathcal{B} \in \mathcal{H}(n)$, we will denote by $s(\mathcal{B})$ any of its singular values and by $s^+(\mathcal{B})$ the sum of all of its singular values, that is, the trace norm $\|\mathcal{B}\|_{tr} = s^+(\mathcal{B})$. For $C, \mathcal{D} \in \mathcal{H}(n)$, $C \geq \mathcal{D}$ (resp. $C \succ \mathcal{D}$) will mean that the matrix $C - \mathcal{D}$ is positive semi-definite (resp. positive definite).

The following lemmas are needed in the subsequent discussion.

Lemma 6.1. *[24] If $\mathcal{A} \geq O$ and $\mathcal{B} \geq O$ are $n \times n$ matrices, then*

$$0 \leq tr(\mathcal{A}\mathcal{B}) \leq \|\mathcal{A}\| tr(\mathcal{B}).$$

Lemma 6.2. *[24] If $\mathcal{A} \in \mathcal{H}(n)$ such that $\mathcal{A} \prec I_n$, then $\|\mathcal{A}\| < 1$.*

We establish the existence and uniqueness of the solution of the nonlinear matrix equation (NME)

$$\mathcal{U} = \mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i, \tag{6.1}$$

where $\mathcal{D}$ is a Hermitian positive definite matrix, $\mathcal{A}_i^*$ stands for the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{G}$ an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that $\mathcal{G}(O) = O$.

Theorem 6.3. *Consider NME (6.1). Assume that there exists a positive real number η such that*

(H_1) *there exists $\mathcal{D} \in \mathcal{P}(n)$ such that $\sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{D}) \mathcal{A}_i \succ 0$;*

(H_2) $\sum_{i=1}^{m} \mathcal{A}_i \mathcal{A}_i^* \prec \eta I_n$.

(H_3) *for every $\mathcal{U}$, $\mathcal{V} \in \mathcal{P}(n)$ such that $\mathcal{U} \preceq \mathcal{V}$ with*

$$\sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \neq \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i,$$

then for $a \in [0, 1), b \geq 0$, we have

$$\begin{aligned} &\left(|s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)| + |s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)| \right) \\ &\leq a \times \max \left\{ \begin{array}{c} (|s^+(\mathcal{U})| + |s^+(\mathcal{V})|), \\ (|s^+(\mathcal{U})| + |s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)|), \\ (|s^+(\mathcal{V})| + |s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)|) \end{array} \right\} \\ &-b \left[\begin{array}{c} (|s^+(\mathcal{U})| + |s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)|) \\ +(|s^+(\mathcal{V})| + |s^+(\mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)|) \end{array} \right]. \end{aligned}$$

Then the NME (6.1) has a unique solution. Moreover, the iteration

$$\mathcal{U}_n = \mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}_{n-1}) \mathcal{A}_i$$

where $\mathcal{U}_0 \in \mathcal{P}(n)$ satisfies

$$\mathcal{U}_0 \preceq \mathcal{D} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}_0) \mathcal{A}_i,$$

converges in the sense of trace norm $\|.\|_{tr}$ to the solution of the matrix equation (6.1).

Proof. Define a mapping $\mathcal{T}:\mathcal{P}(n)\to\mathcal{P}(n)$ by

$$\mathcal{T}(\mathcal{U})=\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{U})\mathcal{A}_i,\ \text{for all } \mathcal{U}\in\mathcal{P}(n),$$

and a binary relation

$$\Re=\{(\mathcal{U},\ \mathcal{V})\in\mathcal{P}(n)\times\mathcal{P}(n):\mathcal{U}\preceq\mathcal{V}\}.$$

Then the fixed point of the mapping $\mathcal{T}$ is a solution of the matrix equation (6.1). Notice that $\mathcal{T}$ is well defined, $\Re$-continuous and $\Re$ is $\mathcal{T}$-closed. Since

$$\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{D})\mathcal{A}_i\succ 0,$$

for some $\mathcal{D}\in\mathcal{P}(n)$, we have $(\mathcal{D},\mathcal{T}(\mathcal{D}))\in\Re$ and hence $\mathcal{P}(n)(\mathcal{T};\Re)\neq\emptyset$.

Let $\sigma:\mathcal{P}(n)\times\mathcal{P}(n)\to\mathbb{R}_+$ be defined by

$$\sigma(\mathcal{U},\mathcal{V})=(\|\mathcal{U}\|+\|\mathcal{V}\|)\text{ for all }\mathcal{U},\mathcal{V}\in\mathcal{P}(n).$$

Then $(\mathcal{P}(n),\sigma,\Re)$ is a complete $\Re$-relational metric-like space.

Now

$$\begin{aligned}\sigma(\mathcal{T}(\mathcal{U}),\mathcal{T}(\mathcal{V}))&=(\|\mathcal{T}(\mathcal{U})\|+\|\mathcal{T}(\mathcal{V})\|)\\&=\left(|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{U})\mathcal{A}_i)|+|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{V})\mathcal{A}_i)|\right)\\&\leq a\ \max\left\{\begin{array}{c}(|s^+(\mathcal{U})|+|s^+(\mathcal{V})|),\\(|s^+(\mathcal{U})|+|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{U})\mathcal{A}_i)|),\\(|s^+(\mathcal{V})|+|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{V})\mathcal{A}_i)|)\end{array}\right\}\\&\quad-b\left[\begin{array}{c}(|s^+(\mathcal{U})|+|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{V})\mathcal{A}_i)|)\\+(|s^+(\mathcal{V})|+|s^+(\mathcal{D}+\sum_{i=1}^{m}\mathcal{A}_i^*\mathcal{G}(\mathcal{U})\mathcal{A}_i)|)\end{array}\right]\\&=a\ \max\left\{(\|\mathcal{U}\|+\|\mathcal{V}\|),(\|\mathcal{U}\|+\|\mathcal{T}(\mathcal{U})\|),(\|\mathcal{V}\|+\|\mathcal{T}(\mathcal{V})\|)\right\}\\&\quad-b\left[(\|\mathcal{U}\|+\|\mathcal{T}(\mathcal{V})\|)+(\|\mathcal{V}\|+\|\mathcal{T}(\mathcal{U})\|)\right]\\&=a\ \max\left\{\sigma(\mathcal{U},\mathcal{V}),\sigma(\mathcal{U},\mathcal{T}(\mathcal{U})),\sigma(\mathcal{V},\mathcal{T}(\mathcal{V}))\right\}\\&\quad-b[\sigma(\mathcal{U},\mathcal{T}(\mathcal{V}))+\sigma(\mathcal{V},\mathcal{T}(\mathcal{U}))].\end{aligned}\tag{6.2}$$

Consider $\mathcal{G}\in\Upsilon$ as

$$\mathcal{G}(\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5,\hbar_6)=\hbar_1-a\max\{\hbar_2,\hbar_3,\hbar_4\}+b[\hbar_5+\hbar_6]$$

where $a\in[0,1), b\geq 0, \hbar_i\in\mathbb{R}_+$ $(i=1,2,3,4,5)$. Then (6.2) reduces to

$$\mathcal{G}\left(\begin{array}{c}\varrho(\mathcal{T}(\mathcal{U}),\mathcal{T}(\mathcal{V})),\varrho(\mathcal{U},\mathcal{V}),\varrho(\mathcal{U},\mathcal{T}(\mathcal{U})),\\\varrho(\mathcal{V},\mathcal{T}(\mathcal{V})),\varrho(\mathcal{U},\mathcal{T}(\mathcal{V})),\varrho(\mathcal{V},\mathcal{T}(\mathcal{U}))\end{array}\right)\leq 0.$$

Thus all the hypotheses of Theorem 4.2 are satisfied and thus there exists $\hat{\mathcal{U}}\in\mathcal{P}(n)$ such that $\mathcal{T}(\hat{\mathcal{U}})=\hat{\mathcal{U}}$, and hence the matrix equation (6.1) has a solution in $\mathcal{P}(n)$. Furthermore, due to the existence of the least upper bound and greatest lower bound for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(n)$, we have $\mathfrak{P}(\mathcal{U},\mathcal{V};\Re|_{\mathcal{P}(n)})\neq\emptyset$ for all $\mathcal{U},\mathcal{V}\in\mathcal{P}(n)$. Hence, by Theorem 4.4, $\mathcal{T}$ has a unique fixed point and so we conclude that the matrix equation (6.1) has a unique solution in $\mathcal{P}(n)$. □

7. Numerical experiments

In this section, we consider some numerical results. All experiments were run on a macOS Mojave version 10.14.6 CPU @1.6 GHz intel core i5 8GB with MATLAB R2020b as the programming language (Online). In particular, *rand*(n) syntax is used to generate random square matrix of order n. The number of necessary iterations is denoted by Iter. No., initial matrix is denoted by Int. Mat., Dimension is denoted by Dim., Minimum eigenvalue of a matrix is denoted by Min(Eng) and the trace norm of the residual is denoted by Res (Res($\mathcal{X}$) = $\|\mathcal{X}_{n+1} - \mathcal{X}_n\|_{tr}$). We have assigned $tol = 10^{-10}$ in all studies.

Example 7.1. Consider matrices with randomly generated coefficients by

$$\mathcal{A}_1 = \mathcal{K}_1^{-(1/2)}\tilde{\mathcal{A}}\mathcal{K}_1^{-(1/2)}, \qquad \mathcal{A}_2 = \mathcal{K}_2^{-(1/2)}\tilde{\mathcal{B}}\mathcal{K}_2^{-(1/2)},$$
$$\mathcal{A}_3 = \mathcal{K}_3^{-(1/2)}\tilde{\mathcal{C}}\mathcal{K}_3^{-(1/2)},$$

where

$$\mathcal{K}_1 = \mathcal{I} + \tilde{\mathcal{A}}^*\tilde{\mathcal{A}}, \ \ \mathcal{K}_2 = \mathcal{I} + \tilde{\mathcal{B}}^*\tilde{\mathcal{B}}, \ \ \mathcal{K}_3 = \mathcal{I} + \tilde{\mathcal{C}}^*\tilde{\mathcal{C}},$$

and

$$(\tilde{\mathcal{A}})_{ij} = \frac{170}{i+j-1}, \ \ \tilde{\mathcal{B}} = \frac{1}{4}\tilde{\mathcal{A}}, \ \ (\tilde{\mathcal{C}})_{ij} = \frac{230}{i+j-1}.$$

Take $n = 4$ and

$$\mathcal{D} = \begin{bmatrix} 0.002001134214072 & 0.000001512247318 & 0.000001890089141 & 0.000002267448494 \\ 0.000001512247318 & 0.002002041472151 & 0.000002570454118 & 0.000003098868007 \\ 0.000001890089141 & 0.000002570454118 & 0.002003250525038 & 0.000003929941830 \\ 0.000002267448494 & 0.000003098868007 & 0.000003929941830 & 0.002004760277918 \end{bmatrix}.$$

We take $\eta = 4$, $a = 0.7, b = 0.001$ and $\mathcal{G}(\mathcal{U}) = \mathcal{U}^{5/2}$ to test hypothesis of Theorem 6.3. The numerical results are given in Table 1.

Table 1. Three initalization based analysis.

Int. Matrix	$\mathcal{G}(U)$	η	Dim	Iter no.	CPU(Sec)	Error(×1.e-15)	Min(Eig)
U_0	$U^{5/2}$	4	4	5	0.26442327	0.025142	0.002
V_0	$V^{5/2}$	4	4	5	0.02830777	0.022915	0.002
W_0	$W^{5/2}$	4	4	5	0.00484885	0.027327	0.002

We use the initial values

$$\mathcal{U}_0 = \begin{bmatrix} 0.017646443918273 & 0.023512262592047 & 0.029297021502829 & 0.034881823092682 \\ 0.023512262592047 & 0.031715226102244 & 0.039815171016495 & 0.047680128367051 \\ 0.029297021502829 & 0.039815171016495 & 0.050211413538305 & 0.060332232799390 \\ 0.034881823092682 & 0.047680128367051 & 0.060332232799390 & 0.072697850175567 \end{bmatrix},$$

$$\mathcal{V}_0 = \begin{bmatrix} 0.008823221959136 & 0.011756131296024 & 0.014648510751414 & 0.017440911546341 \\ 0.011756131296024 & 0.015857613051122 & 0.019907585508248 & 0.023840064183525 \\ 0.014648510751414 & 0.019907585508248 & 0.025105706769152 & 0.030166116399695 \\ 0.017440911546341 & 0.023840064183525 & 0.030166116399695 & 0.036348925087784 \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 0.004411610979568 & 0.005878065648012 & 0.007324255375707 & 0.008720455773170 \\ 0.005878065648012 & 0.007928806525561 & 0.009953792754124 & 0.011920032091763 \\ 0.007324255375707 & 0.009953792754124 & 0.012552853384576 & 0.015083058199847 \\ 0.008720455773170 & 0.011920032091763 & 0.015083058199847 & 0.018174462543892 \end{bmatrix},$$

where $\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0 \in \mathcal{P}(4)$.

After 5 successive iterations, we obtain the following pds

$$\hat{\mathcal{X}} =$$

$$\begin{bmatrix} 0.002001137516642 & 0.000001516648878 & 0.000001895574989 & 0.000002273982415 \\ 0.000001516648878 & 0.002002047409975 & 0.000002577909508 & 0.000003107798330 \\ 0.000001895574989 & 0.000002577909508 & 0.002003259927822 & 0.000003941241995 \\ 0.000002273982415 & 0.000003107798330 & 0.000003941241995 & 0.002004773895785 \end{bmatrix}.$$

The graphical view of convergence and $\hat{\mathcal{X}}$ are shown in Figures 1 and 2, respectively:

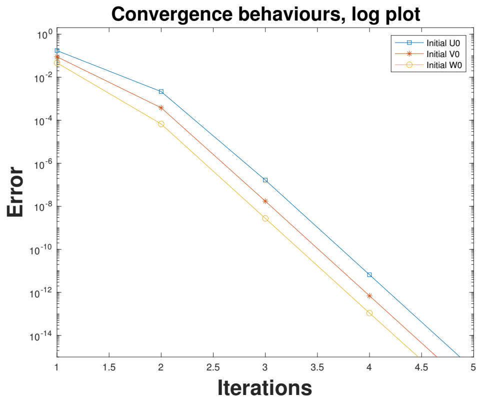

Figure 1. Convergence analysis graph.

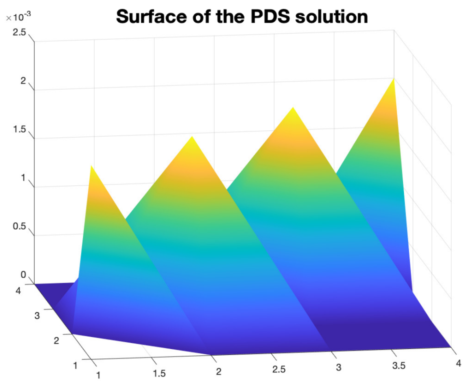

Figure 2. Solution gragh.

Example 7.2. Consider the following matrices $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{D} \in \mathcal{M}(4)$:

$$\mathcal{A}_1 = \begin{bmatrix} 0.0120 + 0.0049i & 0.0110 - 0.0017i & 0.0231 - 0.0009i & 0.0289 + 0.0039i \\ 0.0420 - 0.0015i & 0.0113 + 0.0006i & 0.0319 + 0.0008i & 0.0216 - 0.0022i \\ 0.0428 - 0.0008i & 0.0268 + 0.0041i & 0.0459 + 0.0047i & 0.0442 + 0.0027i \\ 0.0482 - 0.0023i & 0.0381 + 0.0016i & 0.0081 - 0.0016i & 0.0197 + 0.0048i \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.0486 + 0.0007i & -0.0167 + 0.0006i & -0.0094 + 0.0004i & 0.0393 + 0.0029i \\ -0.0155 + 0.0012i & 0.0056 + 0.0015i & 0.0081 + 0.0027i & -0.0217 + 0.0023i \\ -0.0076 + 0.0007i & 0.0413 + 0.0023i & 0.0467 + 0.0003i & 0.0269 + 0.0017i \\ -0.0232 + 0.0000i & 0.0161 + 0.0016i & -0.0157 + 0.0004i & 0.0484 + 0.0021i \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0.0066 + 0.0012i & 0.0064 + 0.0011i & 0.0043 + 0.0023i & 0.0290 + 0.0029i \\ 0.0118 + 0.0042i & 0.0150 + 0.0011i & 0.0268 + 0.0032i & 0.0228 + 0.0022i \\ 0.0073 + 0.0043i & 0.0227 + 0.0027i & 0.0025 + 0.0046i & 0.0171 + 0.0044i \\ 0.0001 + 0.0048i & 0.0156 + 0.0038i & 0.0041 + 0.0008i & 0.0205 + 0.0020i \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} 0.0020 + 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0020 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0020 + 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0020 + 0.0000i \end{bmatrix}.$$

To see the convergence of the sequence $\{\mathcal{U}_n\}$ defined in (6.3), we start with the following initializations:

$$\mathcal{U}_0 = \begin{bmatrix} 0.0060 + 0.0000i & 0.0002 + 0.0001i & 0.0027 + 0.0001i & 0.0024 + 0.0002i \\ 0.0002 - 0.0001i & 0.0042 + 0.0000i & 0.0047 - 0.0001i & 0.0024 + 0.0000i \\ 0.0027 - 0.0001i & 0.0047 + 0.0001i & 0.0113 + 0.0000i & 0.0058 + 0.0001i \\ 0.0024 - 0.0002i & 0.0024 - 0.0000i & 0.0058 - 0.0001i & 0.0077 + 0.0000i \end{bmatrix},$$

$$\mathcal{V}_0 = \begin{bmatrix} 0.0030 + 0.0000i & 0.0001 + 0.0001i & 0.0014 + 0.0001i & 0.0012 + 0.0001i \\ 0.0001 - 0.0001i & 0.0021 + 0.0000i & 0.0023 - 0.0000i & 0.0012 + 0.0000i \\ 0.0014 - 0.0001i & 0.0023 + 0.0000i & 0.0057 + 0.0000i & 0.0029 + 0.0000i \\ 0.0012 - 0.0001i & 0.0012 - 0.0000i & 0.0029 - 0.0000i & 0.0038 + 0.0000i \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 0.0015 + 0.0000i & 0.0001 + 0.0000i & 0.0007 + 0.0000i & 0.0006 + 0.0001i \\ 0.0001 - 0.0000i & 0.0011 + 0.0000i & 0.0012 - 0.0000i & 0.0006 + 0.0000i \\ 0.0007 - 0.0000i & 0.0012 + 0.0000i & 0.0028 + 0.0000i & 0.0014 + 0.0000i \\ 0.0006 - 0.0001i & 0.0006 - 0.0000i & 0.0014 - 0.0000i & 0.0019 + 0.0000i \end{bmatrix},$$

where $\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0 \in \mathcal{P}(4)$.

We take $\eta = 4$, $a = 0.8, b = 0.01$ and $\mathcal{G}(\mathcal{U}) = \mathcal{U}^{7/2}$ to test hypothesis of Theorem 6.3. The numerical results are given in Table 2.

Table 2. Three initialization based analysis.

Int. Matrix	$\mathcal{G}(U)$	η	Dim	Iter no.	CPU	Error	Min(Eig)
U_0	$U^{7/2}$	4	4	3	0.019032	0.2112	0.002
V_0	$V^{7/2}$	4	4	3	0.015490	0.0195	0.002
W_0	$W^{7/2}$	4	4	3	0.014904	0.0017	0.002

The positive definite solution (after 3 iterations) is given by

$$\hat{X} = \begin{bmatrix} 0.0020 - 0.0000i & 0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 - 0.0000i & 0.0020 - 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0020 - 0.0000i & 0.0000 - 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0020 + 0.0000i \end{bmatrix}.$$

The graphical view of convergence is shown in Figure 3 below:

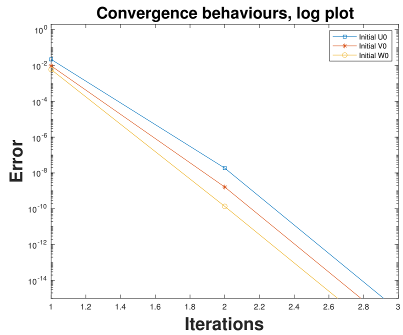

Figure 3. Convergence analysis graph.

Remark 7.3. It is clear from the last columns of Tables 1 and 2 that the solutions of NMEs in Examples 7.1 and 7.2 are positive definite because the minimum eigenvalues of solutions for any initial matrices are positive. Also it is evident from the convergence plots that for any initial guess, we get convergence parallel. It can also be visualized from the surface plot of solution in the first example, as it is pointed upwards.

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Conflict of interest

The authors declare that they have no competing interests.

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