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Research article

Approximation of involution in multi-Banach algebras: Fixed point technique

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Abstract: In this research work, we demonstrate the Hyers-Ulam stability for Cauchy-Jensen functional equation in multi-Banach algebras by the fixed point technique. In fact, we prove that for a function which is approximately Cauchy-Jensen in multi Banach algebra, there is a unique involution near it. Next, we show that under some conditions the involution is continuous, the multi-Banach algebra becomes multi-C^*-algebra and the Banach algebra is self-adjoint.

Keywords: multi-Banach algebra; Hyers-Ulam stability; functional equation; fixed point technique; C^*-algebra
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1. Introduction

Ulam [25] suggested a problem of stability on group homomorphisms in metric groups in 1940. Hyers is the first mathematician who answered the question of Ulam in 1941. He demonstrated the following theorem in [12].

Theorem 1.1. *Let M and N be two Banach spaces and $f : M \to N$ be a mapping such that*

$$\|f(m+n) - f(m) - f(n)\| \leq \delta$$

for some $\delta > 0$ and all $m, n \in M$. Then there exists a unique additive mapping $A : M \to N$ such that

$$\|f(m) - A(m)\| \leq \delta$$

for all $m \in M$.

The results of Hyers' stability theorem were developed by other mathematicians. Recently, numerous consequences concerning the stability of various functional equations in different normed spaces and various control functions have been obtained.

The problem of stability of some functional equations have been widely explored by direct methods and there are numerous exciting outcomes regarding this problem [1, 2, 9, 13–16, 19, 21, 23, 26]. The fixed point approach has been used to study the Hyers-Ulam stability investigations. The relationship between Hyers-Ulam stability and fixed point theory has been defined in [3, 4, 20, 22, 24].

Definition 1.2. [22] A function $d : M \times M \to [0, \infty]$ is named a generalized metric on the set M if d satisfies the following conditions

(a) for each $m, n \in M$, $d(m, n) = 0$ if and only if $m = n$;

(b) for all $m, n \in M$, $d(m, n) = d(n, m)$;

(c) for all $m, n, l \in M$, $d(m, l) \leq d(m, n) + d(n, l)$.

Notice that the just generalized metric significant difference from the metric is that the generalized metric range contains the infinity.

Theorem 1.3. *[22] Let (M, d) be a complete generalized metric space and $J : M \to M$ be a contractive mapping with Lipschitz constant $L < 1$. Then for every $m \in M$, either*

$$d(J^{s+1}m, J^s m) = \infty$$

for all nonnegative integers s or there exists an integer $s_0 \geq 0$ so that

(a) *For all $s \geq s_0$, $d(J^s m, J^{s+1}m) < \infty$;*

(b) *$J^s(m) \to n^*$, where n^* is a fixed point of J;*

(c) *n^* is the unique fixed point of J in the set $N = \{n \in M \ : \ d(J^{s_0}(m), n) < \infty\}$;*

(d) *For each $n \in N$, $d(n^*, n) \leq \frac{1}{1-L} d(Jn, n)$.*

Dales and Polyakov [8] introduced the concept of multi-normed spaces. We have collected some properties of multi-normed spaces which will be used in this article. We refer readers to [8, 17, 18] for more details.

2. Multi-Banach algebras

Suppose that $(A, \|.\|)$ is a complex normed space and $k \in \mathbb{N}$. The linear space $A \oplus \cdots \oplus A$ consists of k-tuples $(x_1, \cdots, x_k)$ denote by A^k, where $x_1, \cdots, x_k \in A$. The linear operations on A^k are defined coordinate-wise. The zero element of either A or A^k is denoted by 0. The set $\{1, 2, \cdots, k\}$ is indicated by $\mathbb{N}_k$ and we denote by Σ_k the group of permutations on k symbols.

Definition 2.1. [8] A multi-norm on $\{A^k : k \in \mathbb{N}\}$ is *a sequence*

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N}),$$

such that $\|\cdot\|_k$ is a norm on A^k for every $k \in \mathbb{N}$ with $\|\cdot\|_1 = \|\cdot\|$:

$(M1)$ $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$ for every $\sigma \in \Sigma_k$ and $x_1, \cdots, x_k \in A$.

$(M2)$ $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|(x_1, \cdots, x_k)\|_k$ for all $\alpha_1, \cdots, \alpha_k \in \mathbb{C}$ and $x_1, \cdots, x_k \in A$.

$(M3)$ $\|(x_1, \cdots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ for all $x_1, \cdots, x_{k-1} \in A$.

$(M4)$ $\|(x_1, \cdots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1}$ for all $x_1, \ldots, x_{k-1} \in A$.

With these properties, we say that $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Lemma 2.2. *[18] Assume that* $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ *is a multi-normed space and let* $k \in \mathbb{N}$. *Then*

(i) $\|(x, \ldots, x)\|_k = \|x\|$ *for all* $x \in A$.

(ii) $\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, \cdots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|$
for all $x_1, \cdots, x_k \in A$.

By the second part of the above lemma, we conclude that, if $(A, \|\cdot\|)$ is a Banach space, then $(A^k, \|\cdot\|_k)$ is a Banach space for every $k \in \mathbb{N}$. In this case, $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 2.3. [8] We define the sequence $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ on A^k as follows:

$$\left\|(x_1, \ldots, x_k)\right\|_k := \max_{i\in\mathbb{N}_k} \|x_i\| \quad \text{for all} \quad (x_1, \ldots, x_k \in A).$$

This is a minimum multi-norm on A^k.

Definition 2.4. [8, 17] Suppose that $(A, \|\cdot\|)$ is a normed algebra such that $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. Then $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a multi-normed algebra if

$$\left\|(a_1b_1, \ldots, a_kb_k)\right\|_k \leq \left\|(a_1, \ldots, a_k)\right\|_k \cdot \left\|(b_1, \ldots, b_k)\right\|_k,$$

for all $k \in \mathbb{N}$ and $a_1, \cdots, a_k, b_1, \cdots, b_k \in A$. Moreover, the multi-normed algebra $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra if $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

It is obvious that every Banach algebra is a multi-Banach algebra with minimum multi-norm.

A mapping $^* : A \to A$, denoted by

$$^*(x) = x^*,$$

is an involution on A if

(1) $x^{**} = x$,

(2) $(\lambda x + \mu y)^* = \overline{\lambda}x^* + \overline{\mu}y^*$, for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$,

(3)$(xy)^* = y^*x^*$, for all $x, y \in A$.

Definition 2.5. Assume that $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra. A multi-C^*-algebra is a complex multi-Banach algebra $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ with an involution * satisfying

$$\left\|(a_1^*a_1, \cdots, a_k^*a_k)\right\|_k = \left\|(a_1, \cdots, a_k)\right\|_k^2,$$

for all $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in A$.

3. Main results

In all sections of this article, wherever needed, we assume that

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{S}^1_{\frac{1}{n_0}} := \{e^{i\theta};\ 0 \leq \theta \leq \tfrac{2\pi}{n_0}\}$$

for $n_0 \in \mathbb{N}$. It is obvious that $\mathbb{S}^1 = \mathbb{S}^1_{\frac{1}{1}}$. Furthermore, suppose $\Big((A^k, \|.\|_k) : k \in \mathbb{N}\Big)$ is a multi-Banach algebra. Let $I := \frac{2+\alpha}{\alpha}$ for real α greater than or equal to 2. When A is a Banach algebra, for a mapping $f : A \to A$, we define

$$E_{\mu\eta\nu}f(x, y, z) = \alpha\overline{\mu}f\left(\frac{x+y}{\alpha} + z\right) - f(\mu x) - f(\eta y) - \alpha f(\nu z),$$

for all $x, y, z \in A$ and $\mu, \nu, \eta \in \mathbb{S}^1_{\frac{1}{n_0}}$.

An algebra A is called C^*-algebra if it is a Banach algebra with an involution * such that $\|xx^*\| = \|x\|^2$. We suggest [6] as a reference for stability in Banach algebras and C^*-algebras.

Lemma 3.1. *Let I be a positive real number and $\varphi : A^{3k} \to [0, \infty)$ be a function such that there exists $L < 1$ satisfying*

$$I^{-1}\varphi(Ix_1, Iy_1, Iz_1, \cdots, Ix_k, Iy_k, Iz_k) \leq L\varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k) \tag{3.1}$$

for all $x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k \in A$. Then

$$\lim_{j\to\infty} I^{-j}\varphi(I^j x_1, I^j y_1, I^j z_1, \cdots, I^j x_k, I^j y_k, I^j z_k) = 0.$$

Proof. Putting $x_i = Ix_i$, $y_i = Iy_i$ and $z_i = Iz_i$ for $1 \leq i \leq k$ in (3.1), we have

$$I^{-1}\varphi(Ix_1, Iy_1, Iz_1, \cdots, Ix_k, Iy_k, Iz_k) \leq L\varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k).$$

Replacing x_i, y_i, z_i by Ix_i, Iy_i, Iz_i, respectively, in the above inequality, we get

$$I^{-2}\varphi(I^2x_1, I^2y_1, I^2z_1, \cdots, I^2x_k, I^2y_k, I^2z_k) \leq L^2\varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k).$$

By induction, we obtain

$$I^{-n}\varphi(I^nx_1, I^ny_1, I^nz_1, \cdots, I^nx_k, I^ny_k, I^nz_k) \leq L^n\varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k).$$

Since $L < 1$, we get

$$\lim_{n\to\infty} I^{-n}\varphi(I^nx_1, I^ny_1, I^nz_1, \cdots, I^nx_k, I^ny_k, I^nz_k) = 0,$$

as desired. □

Theorem 3.2. *Let $((A^n, \|.\|_n) : n \in \mathbb{N})$ be a multi-Banach algebra and $f : A \to A$ be a mapping and $\varphi : A^{3k} \to [0, \infty)$ be a function satisfying (3.1) and*

$$\|(E_{\mu\eta\nu}f(x_1, y_1, z_1), \cdots, E_{\mu\eta\nu}f(x_k, y_k, z_k))\|_k \leq \varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k), \tag{3.2}$$

$$\|(f(x_1y_1) - f(y_1)f(x_1), \cdots, f(x_ky_k) - f(y_k)f(x_k))\|_k \leq \varphi(x_1, y_1, 0, \cdots, x_k, y_k, 0), \tag{3.3}$$

$$\lim_{m\to\infty} I^{-m}f(I^m \lim_{n\to\infty} I^{-n}f(I^nx)) = x \tag{3.4}$$

for all $x, x_1, y_1, z_1, \cdots, x_k, y_k, z_k \in A$ and $\mu, \eta, \nu \in \mathbb{S}^1_{\frac{1}{n_0}}$. Then there exists a unique involution mapping $F : A \to A$ which satisfies

$$\|(F(x_1) - f(x_1), \cdots, F(x_k) - f(x_k))\|_k \leq \frac{1}{(1 - L)(2 + \alpha)}\varphi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k). \tag{3.5}$$

Furthermore, (a) *if*

$$\Big| \|(f(x_1), f(x_2), \ldots, f(x_k))\|_k - \|(x_1, x_2, \ldots, x_k)\|_k \Big| \leq \varphi(x_1, x_1, x_1, \ldots, x_k, x_k, x_k) \tag{3.6}$$

for all $x_1, x_2, \ldots, x_n \in A$, then the involution mapping $F : A \to A$ is continuous;
(b) if

$$\left| \|(x_1 f(x_1), \ldots, x_k f(x_k))\|_k - \|(x_1, \ldots, x_k)\|_k^2 \right| \leq \varphi(x_1, x_1, x_1, \ldots, x_k, x_k, x_k) \tag{3.7}$$

for all $x_1, x_2, \ldots, x_k \in A$, then A is a C^*-algebra with involution $x^* = F(x)$ for all $x \in A$.

Proof. Let (Δ, d) be a generalized metric space given by the following definition:

$$\begin{aligned} \Delta &= \{g : A \to A;\ \ g(0) = 0\}, \\ d(g, h) &= \inf\{c \in [0, \infty)\ :\ \|(g(x_1) - h(x_1), \cdots, g(x_k) - h(x_k))\|_k \\ &\leq c\varphi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k)\} \end{aligned} \tag{3.8}$$

for all $x_1, x_2, \ldots x_k \in A$. We can show that the metric space (Δ, d) is complete (see [5, Theorem 2.5]). Next, we define an operator

$$\begin{aligned} \Upsilon &: \Delta \to \Delta \\ (\Upsilon g)(x) &:= I^{-1} g(Ix), \end{aligned} \tag{3.9}$$

for all $x \in A$. We claim that the operator Υ is strictly contractive on Δ. For this, assume that $d(g, h) = \kappa$ for any $g, h \in \Delta$, for some $\kappa \in \mathbb{R}$, we have

$$\|(g(x_1) - h(x_1), \cdots, g(x_k) - h(x_k))\|_k \leq \kappa\varphi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k),$$

for all $x_1, x_2, \cdots, x_k \in A$. Replacing $x_1, x_2, \cdots, x_k$ by $Ix_1, Ix_2, \cdots, Ix_k$, respectively, in the above inequality, we get

$$\|(g(Ix_1) - h(Ix_1), \ldots, g(Ix_k) - h(Ix_k))\|_k \leq \kappa\varphi(Ix_1, Ix_1, Ix_1, \ldots, Ix_k, Ix_k, Ix_k)$$

and by (3.1), we obtain

$$\begin{aligned} \|(I^{-1}g(Ix_1) - I^{-1}h(Ix_1), \cdots, I^{-1}g(Ix_k) - I^{-1}h(Ix_k))\|_k &\leq \kappa I^{-1}\varphi(Ix_1, Ix_1, Ix_1, \cdots, Ix_k, Ix_k, Ix_k) \\ &= \kappa L\varphi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k). \end{aligned}$$

So $d(\Upsilon g, \Upsilon h) \leq \kappa L$. Hence we get

$$d(\Upsilon g, \Upsilon h) \leq Ld(g, h).$$

Then Υ is strictly contactive on Δ by Lipschitz constant $L < 1$.
Letting $\mu = \eta = \nu = 1$ and $y_i = z_i := x_i$ for $1 \leq i \leq k$ in (3.2), we obtain

$$\|(E_{\mu\eta\nu} f(x_1, x_1, x_1), \ldots, E_{\mu\eta\nu} f(x_k, x_k, x_k))\|_k \leq \varphi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k), \tag{3.10}$$

for all $x_1, x_2, \cdots, x_k \in A$. Note that

$$\begin{aligned} E_{\mu,\eta,\nu} f(x_1, x_1, x_1) &= \alpha f\left(\frac{x_1 + x_1}{\alpha} + x_1\right) - f(x_1) - f(x_1) - \alpha f(x_1) \\ &= f\left(\frac{2 + \alpha}{\alpha} x_1\right) - (2 + \alpha) f(x_1). \end{aligned}$$

By (3.10), we have

$$\left\|\left(\alpha f\left(\frac{2+\alpha}{\alpha}x_1\right)-(2+\alpha)f(x_1),\ldots,\alpha f\left(\frac{2+\alpha}{\alpha}x_k\right)-(2+\alpha)f(x_k)\right)\right\|_k \leq \varphi(x_1,x_1,x_1,\cdots,x_k,x_k,x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$. Thus we get

$$\left\|(I^{-1}f(Ix_1)-f(x_1),\cdots,I^{-1}f(Ix_k)-f(x_k))\right\|_k \leq \frac{1}{\alpha+2}\varphi(x_1,x_1,x_1,\cdots,x_k,x_k,x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$. It follows from (3.9) that

$$\left\|(\Upsilon f(x_1)-f(x_1),\cdots,\Upsilon f(x_k)-f(x_k))\right\|_k \leq \frac{1}{\alpha+2}\varphi(x_1,x_1,x_1,\cdots,x_k,x_k,x_k),$$

for all $x_1, x_2, \ldots, x_k \in A$, and by (3.8), we get

$$d(\Upsilon f, f) \leq \frac{1}{\alpha+2} < \infty. \tag{3.11}$$

The conditions of the fixed point theorem are estabilished, so there exists a mapping $F : A \to A$ such that $F(0) = 0$ and the following conditions hold.
(i) F is a fixed point of Υ. This means that

$$(\Upsilon F)(Ix) = F(x) \Rightarrow I^{-1}F(Ix) = F(x) \Rightarrow F(Ix) = IF(x)$$

and thus

$$F\left(\frac{\alpha+2}{\alpha}x\right) = \frac{\alpha+2}{\alpha}F(x)$$

for all $x \in A$. Moreover, the mapping F is a unique fixed point in the set

$$\chi = \{g \in \Delta : \ d(g, f) < \infty\}.$$

Note that $n_0 = 0$ in Theorem 1.3. From (3.8), there is $c \in [0, \infty)$ satisfying

$$\|(F(x_1)-f(x_1),\ldots,F(x_k)-f(x_k))\|_k \leq c\varphi(x_1,x_1,x_1,\ldots,x_k,x_k,x_k).$$

(ii) The sequence $\{\Upsilon^n f\}$ converges to F. This implies that

$$F(x) = \lim_{n\to\infty} I^{-n}f(I^n x) \quad \text{or} \quad F(x) = \lim_{n\to\infty}\left(\frac{\alpha}{2+\alpha}\right)^n f\left(\left(\frac{2+\alpha}{\alpha}\right)^n x\right). \tag{3.12}$$

(iii) We obtain

$$d(F, f) \leq \frac{1}{1-L}d(\Upsilon F, f) \leq \frac{1}{(1-L)(2+\alpha)}$$

and so (3.5) holds. By Lemma 3.1, (3.1), (3.2) and (3.11), we have

$$\left\|\left(\alpha F\left(\frac{x_1+y_1}{\alpha}+z_1\right)-F(x_1)-F(y_1)-\alpha F(z_1),\cdots,\alpha F\left(\frac{x_k+y_k}{\alpha}+z_k\right)-F(x_k)-F(y_k)-\alpha F(z_k)\right)\right\|_k$$

$$
\begin{aligned}
&= \Big\|(\alpha \lim_{n\to\infty} I^{-n} f\left(I^n\left(\frac{x_1+y_1}{\alpha}+z_1\right)\right) - \lim_{n\to\infty} I^{-n} f(I^n x_1) - \lim_{n\to\infty} I^{-n} f(I^n y_1) - \alpha \lim_{n\to\infty} I^{-n} f(I^n z_1), \cdots, \\
&\alpha \lim_{n\to\infty} I^{-n} f\left(I^n\left(\frac{x_k+y_k}{\alpha}+z_k\right)\right) - \lim_{n\to\infty} I^{-n} f(I^n x_k) - \lim_{n\to\infty} I^{-n} f(I^n y_k) - \alpha \lim_{n\to\infty} I^{-n} f(I^n z_k))\Big\|_k \\
&= \lim_{n\to\infty} I^{-n}\Big[\Big\|(\alpha f\left(\frac{I^n x_1 + I^n y_1}{\alpha} + I^n z_1\right) - f(I^n x_1) - f(I^n y_1) \\
&-\alpha f(I^n z_1), \ldots, \alpha f\left(\frac{I^n x_k + I^n y_k}{\alpha} + I^n z_k\right) - f(I^n x_k) - f(I^n y_k) - \alpha f(I^n z_k))\Big\|_k\Big] \\
&\leq \lim_{n\to\infty} I^{-n}\varphi(I^n x_1, I^n y_1, I^n z_1, \cdots, I^n x_k, I^n y_k, I^n z_k) = 0 \ \text{(by Lemma 3.1)}.
\end{aligned}
$$

So we obtain

$$
\Big\|(\alpha F\left(\frac{x_1+y_1}{\alpha}+z_1\right) - F(x_1) - F(y_1) - \alpha F(z_1), \cdots, \alpha F\left(\frac{x_k+y_k}{\alpha}+z_k\right) - F(x_k) - F(y_k) - \alpha F(z_k))\Big\|_k = 0.
$$

Replacing $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \cdots, y_k$ and $z_1, z_2, \cdots, z_k$ with x, y, z, respectively, we have

$$
\left\|\alpha F\left(\frac{x+y}{\alpha}+z\right) - F(x) - F(y) - \alpha F(z)\right\| = 0
$$

and thus

$$
\alpha F\left(\frac{x+y}{\alpha}+z\right) = F(x) + F(y) + \alpha F(z)
$$

and so F is an additive mapping, that is,

$$
F(x+y) = F(x) + F(y) \tag{3.13}
$$

for all $x, y \in A$ (see [11]).

We will show that $F(xy) = F(y)F(x)$ for all $x, y \in A$. Substituting $x_1, x_2, \cdots ., x_k$ with $I^n x_1, I^n x_2, \ldots, I^n x_k$ and $y_1, y_2, \cdots, y_k$ with $I^n y_1, I^n y_2, \cdots, I^n y_k$ in (3.3) and dividing on I^{-2n}, we obtain

$$
\begin{aligned}
I^{-2n}\big\|(f(I^{2n} x_1 y_1) - f(I^n y_1) f(I^n x_1), \cdots, f(I^{2n} x_k y_k) - f(I^n y_k) f(I^n x_k))\big\|_k \\
\leq I^{-2n}\varphi(I^n x_1, I^n y_1, 0, \cdots, I^n x_k, I^n y_k, 0).
\end{aligned}
$$

Letting $n \to \infty$, we have

$$
\begin{aligned}
&\big\|F(x_1 y_1) - F(y_1)F(x_1), \cdots, F(x_k y_k) - F(y_k)F(x_k)\| \\
&\leq \lim_{n\to\infty} I^{-2n}\varphi(I^n x_1, I^n y_1, 0, \cdots, I^n x_k, I^n y_k, 0) = 0 \ \text{(by Lemma 3.1)}.
\end{aligned}
$$

Replacing $x_1, x_2, \cdots, x_k$ and $y_1, y_2, \cdots, y_k$ with x, y, respectively in the above inequality, we get

$$
\|F(xy) - F(y)F(x)\| = 0 \Rightarrow F(xy) = F(y)F(x) \tag{3.14}
$$

for all $x, y \in A$. We will show that $F(\mu x) = \overline{\mu} F(x)$ for all $x \in A$ and all $\mu \in \mathbb{S}^1_{\frac{1}{n_0}}$.

The method applied in [10] is used to continue the proof. Substituting $x_i = y_i = z_i := I^n x_i$ for every $1 \leq i \leq k$ and $\eta = \nu := \mu$, we obtain

$$
\left\|(E_{\mu\eta\nu} f(I^n x_1, I^n x_1, I^n x_1), \ldots, E_{\mu\eta\nu} f(I^n x_k, I^n x_k, I^n x_k))\right\|_k
$$

$$
\begin{aligned}
&= \left\|\left(\alpha\overline{\mu} f\left(\frac{I^n x_1 + I^n x_1}{\alpha} + I^n x_1\right) - f(\mu I^n x_1) - f(\mu I^n x_1) - \alpha f(\mu I^n x_1), \cdots ,\right.\right.\\
&\qquad \left.\left.\alpha\overline{\mu} f\left(\frac{I^n x_k + I^n x_k}{\alpha} + I^n x_k\right) - f(\mu I^n x_k) - f(\mu I^n x_k) - \alpha f(\mu I^n x_k)\right)\right\|_k\\
&= \left\|\left(\alpha\overline{\mu} f(I^{n+1} x_1) - (\alpha + 2) f(\mu I^n x_1), \cdots , \overline{\mu} f(I^{n+1} x_k) - (\alpha + 2) f(\mu I^n x_k)\right)\right\|_k\\
&\leq \varphi(I^n x_1, I^n x_1, I^n x_1, \cdots , I^n x_k, I^n x_k, I^n x_k).
\end{aligned}
$$

In particular, if $\mu = 1$, then

$$
\left\|\left(\alpha f(I^{n+1} x_1) - (\alpha + 2) f(I^n x_1), \ldots , f(I^{n+1} x_k) - (\alpha + 2) f(I^n x_k)\right)\right\|_k \leq \varphi(I^n x_1, I^n x_1, I^n x_1, \cdots , I^n x_k, I^n x_k, I^n x_k).
$$

We have

$$
\begin{aligned}
&\left\|\left((2 + \alpha) f(\mu I^n x_1) - (2 + \alpha)\overline{\mu} f(I^n x_1), \cdots , (2 + \alpha) f(\mu I^n x_k) - (2 + \alpha)\overline{\mu} f(I^n x_k)\right)\right\|_k\\
&= \left\|\left((2 + \alpha) f(\mu I^n x_1) - \alpha\overline{\mu} f(I^{n+1} x_1) + \alpha\overline{\mu} f(I^{n+1} x_1) - (2 + \alpha)\overline{\mu} f(I^n x_1), \ldots ,\right.\right.\\
&\qquad \left.\left.(2 + \alpha) f(\mu I^n x_k) - \alpha\overline{\mu} f(I^{n+1} x_k) + \alpha\overline{\mu} f(I^{n+1} x_k) - (2 + \alpha)\overline{\mu} f(I^n x_k)\right)\right\|_k\\
&\leq \left\|\left((2 + \alpha) f(\mu I^n x_1) - \alpha\overline{\mu} f(I^{n+1} x_1), \cdots , (2 + \alpha) f(\mu I^n x_k) - \alpha\overline{\mu} f(I^{n+1} x_k)\right)\right\|_k\\
&\qquad + \left\|\left(\alpha\overline{\mu} f(I^{n+1} x_1) - (2 + \alpha)\overline{\mu} f(I^n x_1), \cdots , \alpha\overline{\mu} f(I^{n+1} x_k) - (2 + \alpha)\overline{\mu} f(I^n x_k)\right)\right\|_k\\
&\leq \left\|\left((2 + \alpha) f(\mu I^n x_1) - \alpha\overline{\mu} f(I^{n+1} x_1), \ldots , (2 + \alpha) f(\mu I^n x_k) - \alpha\overline{\mu} f(I^{n+1} x_k)\right)\right\|_k\\
&\qquad + |\mu| \left\|\left(\alpha f(I^{n+1} x_1) - (2 + \alpha) f(I^n x_1), \cdots , \alpha f(I^{n+1} x_k) - (2 + \alpha) f(I^n x_k)\right)\right\|_k\\
&\leq 2\varphi(I^n x_1, I^n x_1, I^n x_1, \cdots , I^n x_k, I^n x_k, I^n x_k)
\end{aligned}
$$

for all $x_1, x_2, \ldots , x_k \in A$. This implies that

$$
\begin{aligned}
&\left\|\left(I^{-n} f(\mu I^n x_1) - \overline{\mu} I^{-n} f(I^n x_1), \cdots , I^{-n} f(\mu I^n x_k) - \overline{\mu} I^{-n} f(I^n x_k)\right)\right\|_k\\
&\leq I^{-n} \left\|\left(f(\mu I^n x_1) - \overline{\mu} f(I^n x_1), \cdots , f(\mu I^n x_k) - \overline{\mu} f(I^n x_k)\right)\right\|_k\\
&\leq \frac{2}{2 + \alpha} I^{-n} \varphi(I^n x_1, I^n x_1, I^n x_1, \ldots , I^n x_k, I^n x_k, I^n x_k)\\
&\leq I^{-n} \varphi(I^n x_1, I^n x_1, I^n x_1, \cdots , I^n x_k, I^n x_k, I^n x_k)
\end{aligned}
$$

for all $x_1, x_2, \cdots , x_k \in A$. As $n \to \infty$, we have

$$
\lim_{n\to\infty} \left\|\left(I^{-n} f(I^n x_1) - I^{-n}\overline{\mu} f(I^n x_1), \ldots , I^{-n} f(I^n x_k) - I^{-n}\overline{\mu} f(I^n x_k)\right)\right\|_k = 0,
$$

by Lemma 3.1. Therefore,

$$
\|(F(\mu x_1) - \overline{\mu} F(x_1), \cdots , F(\mu x_k) - \overline{\mu} F(x_k))\|_k = 0.
$$

Letting $x_1 = x_2 = \cdots = x_k = x$, we get
$$F(\mu x) = \overline{\mu} F(x) \tag{3.15}$$
for all $x \in A$ and $\mu \in \mathbb{S}^1_{\frac{1}{n_0}}$. Now, we will show that the above equality holds for all $\mu \in \mathbb{C}$. To prove this, we can consider the following statements:

(1) If $\mu \in \mathbb{S}^1$ then there exists $\theta \in [0, 2\pi]$ such that $\mu = e^{i\theta}$. Putting $\mu_1 = e^{i\frac{\theta}{n_0}} \in \mathbb{S}^1_{\frac{1}{n_0}}$, by (3.15), we get
$$\mu_1^{n_0} = \mu \Rightarrow F(\mu x) = \overline{\mu_1}^{n_0} F(x) = \overline{\mu} F(x)$$
for all $x \in A$.

(2) If $\mu \in n\mathbb{S}^1 = \{nz; \ \ z \in \mathbb{S}^1\}$ for some $n \in \mathbb{N}$, then by additivity of F, we have
$$F(\mu x) = F(nzx) = \overline{z} F(nx) = n\overline{z} F(x) = \overline{\mu} F(x).$$

(3) Let $t \in (0, \infty)$. Then there exists a positive integer n such that the point $(t, 0)$ lies in the interior of circle with center at origin and radius n. Putting $t_1 := t + i\sqrt{n^2 - t^2}$ and $t_2 := t - i\sqrt{n^2 - t^2}$, we have $t = \frac{t_1+t_2}{2}$ and $t_1, t_2 \in n\mathbb{S}^1$. It follows that
$$\begin{aligned} F(tx) = F\left(\frac{t_1 + t_2}{2} x\right) &= F(\frac{t_1}{2} x + \frac{t_2}{2} x) = F(\frac{t_1}{2} x) + F(\frac{t_2}{2} x) \\ &= \frac{\overline{t_1}}{2} F(x) + \frac{\overline{t_2}}{2} F(x) \\ &= \frac{t_1 + t_2}{2} F(x) = tF(x) \end{aligned}$$
for all $x \in A$. On the other hand, for any $\lambda \in \mathbb{C}$, there is $\theta \in [0, 2\pi]$ such that $\lambda = |\lambda| e^{i\theta}$. It follows that
$$F(\lambda x) = F(|\lambda| e^{i\theta} x) = |\lambda| e^{\overline{i\theta}} F(x) = |\lambda| e^{-i\theta} F(x) = \overline{\lambda} F(x)$$
for all $x \in A$. Hence, for any case, we get
$$F(\lambda x) = \overline{\lambda} F(x) \tag{3.16}$$
for all $\lambda \in \mathbb{C}$. Using the assumption (3.4), we have
$$F(F(x)) = F(\lim_{n\to\infty} I^{-n} f(I^n x)) = \lim_{m\to\infty} I^{-m} f(I^m \lim_{n\to\infty} I^{-n} f(I^n x)) = x$$
and thus
$$F(F(x)) = x. \tag{3.17}$$
Hence $F(x)$ is an involution for A, since
$$\begin{aligned} F(x + y) &= F(x) + F(y), && \text{i.e., } (x + y)^* = x^* + y^* \text{ by (3.13)}, \\ F(xy) &= F(y)F(x), && \text{i.e., } (xy)^* = y^* x^* \text{ by (3.14)}, \\ F(\lambda x) &= \overline{\lambda} F(x), && \text{i.e., } (\lambda x)^* = \overline{\lambda} x^* \text{ by (3.16)}, \end{aligned}$$

$$F(F(x)) = x, \qquad \text{i.e., } (x^*)^* = x \text{ by (3.17).}$$

Therefore A is a Banach algebra with an involution $F(x)$. To prove (a) of Theorem 3.2, replacing $x_1, x_2, \cdots, x_k$ by $I^n x_1, I^n x_2, \cdots, I^n x_k$ in (3.6) and dividing both sides on I^n, we have

$$I^{-n}\left|\left\|(f(I^n x_1), f(I^n x_2), \cdots, f(I^n x_k))\right\|_k - \left\|(I^n x_1, I^n x_2, \ldots, I^n x_k)\right\|_k\right|$$
$$\leq I^{-n}\varphi(I^n x_1, I^n x_1, I^n x_1, \cdots, I^n x_k, I^n x_k, I^n x_k).$$

Letting $n \to \infty$, we have

$$\left|\|(F(x_1), F(x_2), \ldots, F(x_k))\|_k - \|(x_1, x_2, \cdots, x_k)\|_k\right| \leq \lim_{n\to\infty} I^{-n}\varphi(I^n x_1, I^n x_1, I^n x_1, \cdots, I^n x_k, I^n x_k, I^n x_k) = 0.$$

Replacing $x_1, x_2, \cdots, x_k$ by x, we get

$$|\|F(x)\| - \|x\|| = 0 \quad \Longrightarrow \quad \|F(x)\| = \|x\|.$$

This implies that the involution $F(x)$ is isometric. Let $\{x_n\}$ be an arbitrary sequence in A such that $x_n \to x$. Then

$$\|x_n - x\| = \|F(x_n - x)\| = \|F(x_n) - F(x)\|,$$

which indicates that the involution $F(x)$ is continuous. To show that A is a C^*-algebra, we must prove that

$$\|xF(x)\| = \|x\|^2.$$

For this purpose, we use the assumption (3.7) and the equality (3.12). If

$$\left|\left\|(x_1 f(x_1), \cdots, x_k f(x_k))\right\|_k - \left\|(x_1, x_2, \cdots, x_k)\right\|_k^2\right| \leq \varphi(x_1, x_1, x_1, \ldots, x_k, x_k, x_k),$$

for all $x_1, x_2, \cdots, x_k \in A$, then by replacing $x_1, x_2, \cdots, x_k$ by $I^n x_1, I^n x_1, \cdots, I^n x_k$ and dividing both sides of the above inequality by I^{-2n}, we obtain

$$I^{-2n}\left|\left\|(I^n x_1 f(I^n x_1), \cdots, I^n x_k f(I^n x_k))\right\|_k - \left\|(I^n x_1, I^n x_2, \ldots, I^n x_k)\right\|_k^2\right|$$
$$\leq I^{-2n}\varphi(I^n x_1, I^n x_1, I^n x_1, \cdots, I^n x_k, I^n x_k, I^n x_k).$$

Thus

$$\left|\left\|(I^{-n} x_1 f(I^n x_1), \ldots, I^{-n} x_k f(I^n x_k))\right\|_k - \left\|(x_1, x_2, \cdots, x_k)\right\|_k^2\right| \leq I^{-2n}\varphi(I^n x_1, I^n x_1, I^n x_1, \cdots, I^n x_k, I^n x_k, I^n x_k).$$

Letting $n \to \infty$, we get

$$\left|\left\|(x_1 F(x_1), \ldots, x_k F(x_k))\right\|_k - \left\|(x_1, x_2, \cdots, x_k)\right\|_k^2\right| \leq \lim_{n\to\infty} I^{-2n}\varphi(I^n x_1, I^n x_1, I^n x_1, \ldots, I^n x_k, I^n x_k, I^n x_k) = 0.$$

Therefore,

$$\left\|(x_1 F(x_1), \cdots, x_k F(x_k))\right\|_k = \left\|(x_1, x_2, \cdots, x_k)\right\|_k^2.$$

Replacing $x_1, x_2, \cdots, x_k$ by x in last inequality, we have

$$\|xF(x)\| = \|x\|^2$$

for all $x \in A$. Then A is a C^*-algebra with $F(x)$ as an involution. □

Corollary 3.3. *Let $f : A \to A$ be a mapping such that*

$$\left\|\left(E_{\mu\eta\nu}f(x_1, y_1, z_1), \cdots, E_{\mu\eta\nu}f(x_k, y_k, z_k)\right)\right\|_k \leq \epsilon\Big(\sum_{i=1}^{k} \|x_i\|^p + \|y_i\|^p + \|z_i\|^p\Big),$$

$$\|(f(x_1y_1) - f(y_1)f(x_1), \cdots, f(x_ky_k) - f(y_k)f(x_k))\|_k \leq \epsilon\Big(\sum_{i=1}^{k} \|x_i\|^p + \|y_i\|^p\Big),$$

$$\lim_{m\to\infty} I^{-m} f(I^m \lim_{n\to\infty} I^{-n} f(I^n x)) = x$$

for all $x, x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k \in A$ and $\mu, \eta, \nu \in \mathbb{S}^1_{\frac{1}{n_0}}$, $p \in [0, 1)$ and $\epsilon \in [0, \infty)$. Then there is a unique involution mapping $F : A \to A$ such that

$$\|(F(x_1) - f(x_1), \cdots, F(x_k) - f(x_k))\|_k \leq \frac{3\epsilon}{\alpha(I - I^p)} \sum_{i=1}^{k} \|x_i\|^p.$$

Furthermore, (a) *if*

$$\|(f(x_1) - x_1, \ldots, f(x_k) - x_k)\|_k \leq \epsilon \sum_{i=1}^{k} \|x_i\|^p, \tag{3.18}$$

for all $x_1, x_2, \cdots, x_n \in A$, then $A = A_{sa}$, which means that for every $x \in A$, we have $x = F(x)$; (b) *if*

$$\Big|\|(x_1f(x_1), \cdots, x_kf(x_k))\|_k - \|(x_1, \cdots, x_k)\|_k^2\Big| \leq 3\epsilon \sum_{i=1}^{k} \|x_i\|^p,$$

for all $x_1, x_2, \cdots, x_k \in A$, then A is a C^*-algebra with involution $x^* = F(x)$ for all $x \in A$.

Proof. By taking

$$\varphi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k) = \sum_{i=1}^{k} (\|x_i\|^p + \|y_i\|^p + \|z_i\|^p),$$

in Theorem 3.2, this corollary is proved. Note that

$$\lim_{j\to\infty} I^{-j}\varphi(I^jx_1, I^jy_1, I^jz_1, \cdots, I^jx_k, I^jy_k, I^jz_k) = \lim_{j\to\infty} I^{(p-1)j} \sum_{i=1}^{k} (\|x_i\|^p + \|y_i\|^p + \|z_i\|^p) = 0.$$

Furthermore, putting $I^n x_i$ by x_i for all $1 \leq i \leq k$ in (3.18), we get

$$\left\|\left(f(I^nx_1) - I^nx_1, f(I^nx_2) - I^nx_2, \ldots, f(I^nx_k) - I^nx_k\right)\right\|_k \leq \epsilon \sum_{i=1}^{k} \|I^nx_i\|^p.$$

Dividing both sides of the above inequality by I^n, we obtain

$$\left\|\left(I^{-n}f(I^nx_1) - x_1, I^{-n}f(I^nx_2) - x_2, \cdots, I^{-n}f(I^nx_k) - x_k\right)\right\|_k \leq \epsilon I^{n(1-p)} \sum_{i=1}^{k} \|x_i\|^p.$$

Letting $n \to \infty$, we have

$$\|(F(x_1) - x_1, F(x_2) - x_2, \ldots, F(x_k) - x_k)\|_k = 0.$$

Replacing $x_1, x_2, \cdots, x_k$ by x in the above equality, we obtain

$$F(x) = x \qquad \text{(i.e., } \; x^* = x\text{)}$$

for all $x \in A$. The rest of the proof is similar to the previous one. □

Theorem 3.4. *Suppose that $f : A \to A$ is a mapping which satisfies (3.2), (3.3) and (3.4). Let $\psi : A^{3k} \to [0, \infty)$ be a function such that there is $L < 1$ satisfying*

$$\psi(x_1, y_1, z_1, \cdots, x_k, y_k, z_k) \le I^{-2} L \psi(Ix_1, Iy_1, Iz_1, \cdots, Ix_k, Iy_k, Iz_k),$$

for all $x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k \in A$ and $\mu, \eta, \nu \in \mathbb{S}^1_{\frac{1}{n_0}}$. Then there exists a unique involution mapping $F : A \to A$ which satisfies

$$\|(F(x_1) - f(x_1), \cdots, F(x_k) - f(x_k))\|_k \le \frac{LI^{-2}}{\alpha(1 - L)} \psi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k).$$

Furthermore, (a) if

$$\|(f(x_1) - x_1, \cdots, f(x_k) - x_k)\|_k \le \psi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k)$$

for all $x_1, x_2, \cdots, x_n \in A$, then $A = A_{sa}$, that is, for every $x \in A$, we have $x = F(x)$; (b) if

$$\left| \|(x_1 f(x_1), \cdots, x_k f(x_k))\|_k - \|(x_1, \cdots, x_k)\|_k^2 \right| \le \psi(x_1, x_1, x_1, \cdots, x_k, x_k, x_k)$$

for all $x_1, x_2, \cdots, x_n \in A$, then A is C^*-algebra with $x^* = F(x)$ as an involution.

Proof. The proof of this theorem is the same as in the previous one. We prove some of the parts and refer the rest on to the readers.
The linear mapping $\Upsilon : \Delta \to \Delta$ is defined by

$$\Upsilon g(x) := Ig(I^{-1}x),$$

for all $x \in A$. It is easy to show that the operator Υ is strictly contractive. Putting $\mu = \eta = \nu = 1$ and replacing x_i, y_i, z_i by $I^{-i}x_i$ in (3.2) for all $1 \le i \le k$, we get

$$\begin{aligned}
&\left\| (E_{\mu\eta\nu} f(I^{-1}x_1, I^{-1}x_1, I^{-1}x_1), \ldots, E_{\mu\eta\nu} f(I^{-1}x_k, I^{-1}x_k, I^{-1}x_k)) \right\|_k \\
&= \left\| (\alpha f(x_1) - (2 + \alpha) f(I^{-1}x_1), \ldots, \alpha f(x_k) - (2 + \alpha) f(I^{-1}x_k)) \right\|_k \\
&\le \psi(I^{-1}x_1, I^{-1}x_1, I^{-1}x_1, \ldots, I^{-1}x_k, I^{-1}x_k, I^{-1}x_k).
\end{aligned}$$

Therefore,

$$\left\| (f(x_1) - If(I^{-1}x_1), \ldots, f(x_k) - If(I^{-1}x_k)) \right\|_k \;\; \le \;\; \frac{1}{\alpha} \psi(I^{-1}x_1, I^{-1}x_1, I^{-1}x_1, \ldots, I^{-1}x_k, I^{-1}x_k, I^{-1}x_k),$$

$$\leq \frac{1}{\alpha}I^2\psi(x_1, x_1, x_1, \ldots, x_k, x_k, x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$. Hence

$$\|(f(x_1) - \Upsilon f(x_1), \ldots, f(x_k) - \Upsilon f(x_k))\|_k \leq \frac{1}{\alpha}I^2\psi(x_1, x_1, x_1, \ldots, x_k, x_k, x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$. We obtain

$$d(f, \Upsilon f) \leq \frac{L}{I^2\alpha}.$$

There exists a mapping $F : A \to A$ with $F(0) = 0$, which is a fixed point of Υ and $\Upsilon^n f \to F$, that is,

$$F(x) = \lim_{n\to\infty} I^n f(I^{-n}x).$$

The remainder is similar to the proof of Theorem 3.2. □

Theorem 3.5. *Let A be a Banach algebra and $\left(\left(A^k\|\cdot\|_k\right) : k \in \mathbb{N}\right)$ be a multi-Banach algebra. Let $p \in \mathbb{N}$ with $p \geq 2$ and assume that there are $0 \leq L \leq 1$ and functions $\Gamma, \varphi : A^{2k} \longrightarrow [0, \infty)$ satisfying*

$$\varphi(x_1, y_1, \cdots, x_k, y_k) \leq \frac{L}{p}\varphi(px_1, py_1, \ldots, px_k, py_k),$$

$$\lim_{n\to\infty} \frac{1}{p^{2n}}\Gamma\left(\frac{x_1}{p^n}, \frac{y_1}{p^n}, \cdots, \frac{x_k}{p^n}, \frac{y_k}{p^n}\right) = 0$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. Assume that $f : A \to A$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}\|(f(\mu x_1 + \nu y_1) - \bar{\mu}f(x_1) - \bar{\nu}f(y_1), \cdots, f(\mu x_k + \nu y_k) - \bar{\mu}f(x_k) - \bar{\nu}f(y_k))\|_k \\ \leq \varphi(x_1, y_1, \cdots, x_k, y_k),\end{aligned} \tag{3.19}$$

$$\|(f(x_1y_1) - f(y_1)f(x_1), \cdots, f(x_ky_k) - f(y_k)f(x_k))\|_k \leq \Gamma(x_1, y_1, \cdots, x_k, y_k), \tag{3.20}$$

$$\lim_{n\to\infty} p^n f\left(\frac{1}{p^n}\lim_{m\to\infty} p^m f\left(\frac{1}{p^m}x\right)\right) = x \tag{3.21}$$

for all $\mu = 1, i$ and $\nu = 1, i$ and for all $x, x_1, x_2, \cdots, x_k, y_1, y_2, \ldots, y_k \in A$. Suppose that for any fixed $x \in A$ the function $t \longmapsto f(tx)$ is continuous on $\mathbb{R}$. Then there is a unique involution $T : A \to A$ such that

$$\|(f(x_1) - T(x_1), \cdots, f(x_k) - T(x_k))\|_k \leq \frac{1}{1-L}\Psi(x_1, x_2, \cdots, x_k),$$

where

$$\Psi(x_1, x_2, \cdots, x_k) := \sum_{j=1}^{p-1}\varphi\left(\frac{jx_1}{p}, \frac{x_1}{p}, \cdots, \frac{jx_k}{p}, \frac{x_k}{p}\right)$$

for all $x_1, x_2, \cdots, x_k \in A$.

Furthermore, if

$$\left|\|(x_1 f(x_1), \cdots, x_k f(x_k))\|_k - \|(x_1, \ldots, x_k)\|_k^2\right| \leq \Gamma(x_1, x_1, \cdots, x_k, x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$, then A is a C^*-algebra with involution $x^* = T(x)$ for all $x \in A$. Also, if

$$\left|\|(f(x_1), \cdots, f(x_k))\|_k - \|(x_1, \cdots, x_k)\|_k\right| \leq \varphi(x_1, x_1, \cdots, x_k, x_k)$$

for all $x_1, x_2, \cdots, x_k \in A$, then the involution mapping $T : A \to A$ is continuous.

Proof. Putting $\mu = \nu = 1$ and $y_1 = x_1, y_2 = x_2, \cdots, y_k = x_k$ in (3.19), we have

$$\|(f(2x_1) - 2f(x_1), \cdots, f(2x_k) - 2f(x_k))\|_k \le \varphi(x_1, x_1, \cdots, x_k, x_k),$$

for all $x_1, x_2, \cdots, x_k \in A$. By using induction, we get

$$\|(f(px_1) - pf(x_1), \ldots, f(px_k) - pf(x_k))\|_k \le \sum_{j=1}^{p-1} \varphi(jx_1, x_1, \cdots, jx_k, x_k).$$

Replacing x_j by $\frac{x_j}{p}$ in the above inequality for $1 \le j \le k$, we get

$$\left\|\left(f(x_1) - pf\left(\frac{x_1}{p}\right), \cdots, f(x_k) - pf\left(\frac{x_k}{p}\right)r\right)\right\|_k \le \Psi(x_1, \cdots, x_k). \tag{3.22}$$

Set $\Delta := \{g : A \to A, g(0) = 0\}$ and define

$$\begin{aligned}&d : \Delta \times \Delta \longrightarrow [0, \infty),\\ &d(g, h) = \inf\{c > 0,\ \|(g(x_1) - h(x_1), \cdots, g(x_k) - h(x_k))\|_k \le c\Psi(x_1, \cdots, x_k)\}\end{aligned}$$

for all $x_1, x_2, \cdots, x_k \in A$. We can easily show that (Δ, d) is a complete metric space. We assert that the mapping $J : \Delta \to \Delta$ defined by $Jg(x) = pg(\frac{x}{p})$ is a strictly contractive mapping. Let $g, h \in \Delta$ with $d(g, h) < \infty$ and $d(g, h) < c$. Thus we have

$$\|(g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k))\|_k \le c\Psi(x_1, \cdots, x_k)$$

for all $x_1, x_2, \cdots, x_k \in A$. Therefore,

$$\begin{aligned}\left\|\left(pg\left(\frac{x_1}{p}\right) - ph\left(\frac{x_1}{p}\right), \ldots, pg\left(\frac{x_k}{p}\right) - ph\left(\frac{x_k}{p}\right)\right)\right\|_k &\le pc\Psi\left(\frac{1}{p}x_1, \cdots, \frac{1}{p}x_k\right)\\ &\le cL\psi(x_1, \cdots, x_k).\end{aligned}$$

Thus $d(Jg, Jh) \le cL$ and so $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in \Delta$. Hence J is a strictly contractive mapping with Lipschitz constant L. By (3.22), we have

$$d(f, Jf) \le 1 < \infty. \tag{3.23}$$

The conditions of the fixed point theorem are satisfied. There exists $n_0 \in \mathbb{N}$ such that the sequence $\{J^n f\}$ converges to a fixed point T of J and therefore $T\left(\frac{x}{p}\right) = \frac{1}{p}T(x)$. Also T is a unique fixed point of J in the set $\chi = \{g \in \Delta : d(J^{n_0} f, g) < \infty\}$ and

$$d(g, T) \le \frac{d(g, Jf)}{1 - L} \quad (g \in \chi).$$

Since $\lim_{n\to\infty} d(J^n f, T) = 0$, we have

$$\lim_{n\to\infty} p^n f\left(\frac{x}{p^n}\right) = T(x) \quad (x \in A).$$

From (3.23), we have $d(f,T) \le \frac{1}{1-L}$. Hence

$$\|(f(x_1) - T(x_1), \cdots, f(x_k) - T(x_k))\|_k \le \frac{1}{1-L}\varphi(x_1, x_1, \cdots, x_k, x_k).$$

Replacing $x_1, x_2, \cdots, x_k$ by $\frac{x}{p^n}$ and $y_1, y_2, \cdots, y_k$ by $\frac{y}{p^n}$ and multiplying both sides by p^n, we get

$$\begin{aligned}&\left\|\left(p^n f\left(\mu\frac{x_1}{p^n} + \nu\frac{y_1}{p^n}\right) - p^n\bar{\mu} f\left(\frac{x_1}{p^n}\right) - p^n\bar{\nu} f\left(\frac{y_1}{p^n}\right), \cdots, p^n f\left(\mu\frac{x_k}{p^n} + \nu\frac{y_k}{p^n}\right) - p^n\bar{\mu} f\left(\frac{x_k}{p^n}\right) - p^n\bar{\nu} f\left(\frac{y_k}{p^n}\right)\right)\right\|_k \\ &\le p^n\varphi\left(\frac{x_1}{p^n}, \frac{y_1}{p^n}, \cdots, \frac{x_k}{p^n}, \frac{y_k}{p^n}\right).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}&\|(T(\mu x_1 + \nu y_1) - \bar{\mu}T(x_1) - \bar{\nu}T(y_1), \cdots, T(\mu x_k + \nu y_k) - \bar{\mu}T(x_k) - \bar{\nu}T(y_k))\|_k \\ &\qquad \le \lim_{n\to\infty} p^n\varphi\left(\frac{x_1}{p^n}, \frac{y_1}{p_n}, \cdots, \frac{x_k}{p_n}, \frac{y_k}{p_n}\right) = 0.\end{aligned}$$

Therefore,

$$\|(T(\mu x_1 + \nu y_1) - \bar{\mu}T(x_1) - \bar{\nu}T(y_1), \cdots, T(\mu x_k + \nu y_k) - \bar{\mu}T(x_k) - \bar{\nu}T(y_k))\|_k = 0.$$

Replacing $x_1, x_2, \cdots, x_k$ by x and $y_1, y_2, \cdots, y_k$ by y, we have

$$T(\mu x + \nu y) = \bar{\mu}T(x) + \bar{\nu}T(y) \tag{3.24}$$

for all $x, y \in A$ and $\mu, \nu = 1, i$.

Next, putting $x_1 := \frac{x_1}{p^n}, \cdots, x_k := \frac{x_k}{p^n}$ and $y_1 := \frac{y_1}{p^n}, \cdots, y_k := \frac{y_k}{p^n}$ in (3.20), we obtain

$$\left\|\left(f\left(\frac{x_1}{p^n}\cdot\frac{y_1}{p^n}\right) - f\left(\frac{y_1}{p^n}\right)f\left(\frac{x_1}{p^n}\right), \ldots, f\left(\frac{x_k}{p^n}\cdot\frac{y_k}{p^n}\right) - f\left(\frac{y_k}{p^n}\right)f\left(\frac{x_k}{p^n}\right)\right)\right\|_k \le \Gamma\left(\frac{x_1}{p^n}, \frac{y_1}{p^n}, \ldots, \frac{x_k}{p^n}, \frac{y_k}{p^n}\right).$$

Multiplying both sides in p^{2n}, we get

$$\begin{aligned}&\left\|\left(p^{2n} f\left(\frac{x_1y_1}{p^{2n}}\right) - p^n f\left(\frac{y_1}{p^n}\right)p^n f\left(\frac{x_1}{p^n}\right), \ldots p^{2n} f\left(\frac{x_ky_k}{p^{2n}}\right) - p^n f\left(\frac{y_k}{p^n}\right)p^n f\left(\frac{x_k}{p}\right)\right)\right\|_k \\ &\le p^{2n}\Gamma\left(\frac{x_1}{p^n}, \frac{y_1}{p^n}, \ldots, \frac{x_k}{p^n}, \frac{y_k}{p^n}\right).\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$\left\|T(x_1y_1) - T(x_1)T(y_1), \cdots, T(x_ky_k) - T(x_k)T(y_k)\right\|_k \le \lim_{n\to\infty} p^{2n}\Gamma\left(\frac{x_1}{p^n}, \frac{y_1}{p^n}, \cdots, \frac{x_k}{p^n}, \frac{y_k}{p^n}\right) = 0.$$

Therefore, we get

$$T(xy) = T(x)T(y) \tag{3.25}$$

for all $x, y \in A$. We must show that T is linear. For this, we can use the method of [7]. Fix $x_0 \in A$ and $\rho \in A'$(dual space of A). We define the mapping $\Phi; \mathbb{R} \to \mathbb{R}$ as follows:

$$\Phi(t) := \rho(T(tx_0)) = \lim_{n\to\infty} p^n \rho(f(p^{-n}tx_0)).$$

So
(i) Φ is an additive mapping, i.e., for any $a, b \in \mathbb{R}$, we have

$$\Phi(a + b) = \Phi(a) + \Phi(b).$$

It is easy to show that Φ is additive by (3.24).

(ii) Φ is a Borel function. Put $\Phi_n(t) = p^n \rho(f(p^{-n}tx_0))$. Then $\Phi_n(t)$ are continuous, and $\Phi(t)$ is the pointwise limit of $\Phi_n(t)$ and thus $\Phi(t)$ is a Borel function.

Also, we know that if $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a function such that Φ is additive and measurable, then Φ is continuous. Note that, if we replace $\mathbb{R}^n$ by any separable, locally compact abelian group and this statement is true. Therefore $\Phi(t)$ is a continuous function. Let $a \in \mathbb{R}$. Then $a = \lim_{n\to\infty} r_n$, where $\{r_n\}$ is a sequence of rational numbers. Hence

$$\Phi(at) = \Phi(t \lim_{n\to\infty} r_n) = \lim_{n\to\infty}(\Phi(tr_n)) = \lim_{n\to\infty} r_n \Phi(t) = a\Phi(t).$$

Thus Φ is $\mathbb{R}$-linear and therefore $T(ax) = aT(x)$ for all $a \in \mathbb{R}$. For $\varrho = a_1 + ia_2 \in \mathbb{C}$, where $a_1, a_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(\varrho x) &= T(a_1 x + ia_2 x) = T(a_1 x) + T(ia_2 x) = T(a_1 x) + \bar{i}T(a_2 x) \\ &= T(a_1 x) - iT(a_2 x) = a_1 T(x) - ia_2 T(x) = \overline{\varrho} T(x). \end{aligned}$$

Then T is a $\mathbb{C}$-linear mapping. Using the assumption (3.21), we have

$$T(T(x)) = T(\lim_{m\to\infty} p^m f(p^{-m}x)) = \lim_{n\to\infty} p^n f(p^{-n} \lim_{m\to\infty} p^m f(p^{-m}x)) = x. \tag{3.26}$$

Now, we can say that $T(x)$ is an involution for A (see (3.24), (3.25), (3.26) and (3.26)). □

4. Conclusion

In this research work, we demonstrated the stability for Cauchy-Jensen functional equation in multi-Banach algebra by using the fixed point technique. In fact, we proved that for a function which is approximately Cauchy-Jensen in multi-Banach algebra, there is a unique involution near it. Next, we showed that under some conditions the involution is continuous, the multi-Banach algebra becomes multi-C^*-algebra and the Banach algebra is self-adjoint.

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Conflict of interest

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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