Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2947–2955.
DOI: 10.3934/math.2021178
Received: 30 October 2020
Accepted: 04 January 2021
Published: 08 January 2021

Research article

On δb-open continuous functions

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Abstract: In this paper, we define an almost δb-continuity, which is a weaker form of R-map and we investigate and obtain its some properties and characterizations. Finally, we show that a function $f : (X, \tau) \to (Y, \varphi)$ is almost δb-continuous if and only if $f : (X, \tau_s) \to (Y, \varphi_s)$ is b-continuous, where τ_s and φ_s are semiregularizations of τ and φ, respectively.

Keywords: R-map; almost δb-continuity; b-continuity; semi-regularization
Mathematics Subject Classification: 54B05, 54B10, 54C08, 54C10, 54D10

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1. Introduction

By using various forms of open sets, several authors defined and investigated some properties of them. Levine [11] (resp., Andrijević [1] and El-Atik [5]) introduced semi open (resp., b-open or γ-open) sets. On the other hand, Velićko [19] studied δ-open sets which are stronger than the open sets. In 1993, Raychaudhuri et al. [17] defined δ-preopen sets and in 1997, Park et al. [16] defined δ-semi-open sets. Since then modifications of δ-open sets have been widely studied. Noiri [15] investigated δ-preopen and δ-semi-open sets. Recently, Magharabi and Mubarki [6] introduced the z-open sets and investigated its some properties. The notion of -open sets was renamed as δb-open sets by Kaymakci [8]. Noiri [15] proved that the connectedness for semi-open sets and δ-semi-open sets coincide. Besides, since the continuity is an important concept in general topology, many authors studied various types of continuity. Of course its weak forms and strong forms are important, too. It is well-known that Carnahan [2] studied R-maps. Noiri [14] introduced δ-continuous functions. Munshi and Bassan [12] defined almost semi-continuous functions. Munshi and Bassan [13] studied super continuous functions. Ekici [3] introduced and investigated almost δ-semicontinuity. Recently, Keskin and Noiri [10] studied almost b-continuous functions.

The aim of this work is to introduce one class of functions, namely, almost δb-continuous functions by using δb-open sets. We investigate several properties of this class. The class of almost δ-b-continuity is a generalization of almost δ-semicontinuity. At the same time, the class of almost b-continuity is a generalization of the almost δb-continuity.

This paper consists of five sections. In Section 2, we give some notations and preliminaries. In Section 3, we introduce almost δb-continuous functions. Also, we obtain some its characterizations and basic properties. In Section 4, we investigate some relationships among several functions; almost b-continuous, almost semi-continuous, almost δ-semicontinuous, almost δb-continuous, δ-continuous and R-maps. In Section 5, we obtain some relationships between almost δb-continuity and separation axioms.

2. Preliminaries

Let (X, τ) be a topological space and A be a subset of (X, τ). The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. The regular open sets are important several in branches of mathematics such as real analysis, functional analysis and topology. Recall that a subset A of a space (X, τ) is said to be regular open (resp., regular closed) [18] if $A = Int(Cl(A))$ (resp., $A = Cl(Int(A))$). A point $x \in X$ is called a δ-cluster point of A [19] if $A \cap Int(Cl(U)) \neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$Cl(A)$). If $A = \delta$-$Cl(A)$), then A is said to be δ-closed. The complement of a δ-closed set is said to be δ-open. The set $\{ x \in X \mid x \in V \subset A$ for some regular open set V of $X \}$ is called the δ-interior of A and is denoted by δ-$Int(A)$.

Throughout this paper, we will use $RO(X)$ (resp., $\delta O(X)$ and $RC(X)$) as the family of all regular open (resp., δ-open and regular closed) sets of a space (X, τ).

As a sequel, we need the following definition.

Definition 2.1. A subset A of a space (X, τ) is said to be

(1) semi open [11] if $A \subset Cl(Int(A))$,

(2) δ-semi open [16] if $A \subset Cl(\delta\text{-}Int(A))$,

(3) b-open [1] (γ-open [5]) if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,

(4) δb-open [8] (z-open [6]) if $A \subset Int(Cl(A)) \cup Cl(\delta\text{-}Int(A))$.

The complement of a δb open set is said to be *δb-closed* ([8]). If A is a subset of a space (X, τ), then the *δb-closure* of A, denoted by δ_b-$Cl(A)$, is the smallest *δb-closed* set containing A ([8]). The family of all *δb-open*, *δb-closed*, *δ-semiopen*, *semi open* and *b-open* sets of a space (X, τ) will be denoted by $\delta BO(X)$, $\delta BC(X)$, $\delta SO(X)$, $SO(X)$ and $BO(X)$, respectively.

W note that each one of the converses of these implications is not true in general. One can find them in related references.

$$\begin{tikzcd}
\textit{regular}\text{ open} \arrow[r] & \delta\text{-open} \arrow[r] & \delta\text{-semiopen} \arrow[r] \arrow[d] & \delta b\text{-open} \arrow[d] \\
 & & \text{semi-open} \arrow[r] & \text{b-open}
\end{tikzcd}$$

Diagram I

3. Almost δb-continuous functions

In this section, we introduce almost δ-b-continuous functions. Then we will obtain some characterizations and properties of these functions.

Definition 3.1. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be almost δb-continuous if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset V$.

The next statement is obvious, so its proof is omitted.

Theorem 3.2. *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is almost δb-continuous;

(2) For each $x \in X$ and $V \in \varphi$ containing $f(x)$, there exists a subset $U \in \delta BO(X)$ containing x such that $f(U) \subset Int(Cl(V))$;

(3) $f^{-1}(V) \in \delta BO(X)$ for every $V \in RO(Y)$;

(4) $f^{-1}(F) \in \delta BC(X)$ for every $F \in RC(Y)$.

Theorem 3.3. *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is almost δ-b-continuous;

(2) $f(bCl_\delta(A)) \subset \delta\text{-}Cl((f(A)))$ for every subset A of X;

(3) $\delta_b\text{-}Cl(f^{-1}(B)) \subset f^{-1}(\delta\text{-}Cl(B))$ for every subset B of Y;

(4) $f^{-1}(F) \in \delta BC(X)$ for every δ-closed set F of Y;

(5) $f^{-1}(V) \in \delta BO(X)$ for every δ-open set V of Y.

Proof. (1) $\Longrightarrow$ (2) Let A be a subset of X. Since $\delta\text{-}Cl((f(A)))$ is a δ-closed set in Y, it is denoted by $\cap\{F_\alpha : F_\alpha \in RC(Y, \varphi), \alpha \in \Delta\}$, where Δ is an index set. Then we have $A \subset f^{-1}(\delta\text{-}Cl((f(A)))) = \cap\{f^{-1}(F_\alpha) : \alpha \in \Delta\} \in \delta BC(X)$ by Theorem 3.2. So we obtain $\delta_b\text{-}Cl(A) \subset f^{-1}(\delta\text{-}Cl((f(A))))$ and hence $f(bCl_\delta(A)) \subset \delta\text{-}Cl((f(A)))$.

(2) $\Longrightarrow$ (3) Let B be a subset of Y. We have $f\left(\delta_b\text{-}Cl(f^{-1}(B))\right) \subset \delta\text{-}Cl(f\left(f^{-1}(B)\right)) \subset \delta\text{-}Cl(B)$ and hence $\delta_b\text{-}Cl(f^{-1}(B)) \subset f^{-1}(\delta\text{-}Cl(B))$.

(3) $\Longrightarrow$ (4) Let F be any δ-closed set of Y. We have $\delta_b\text{-}Cl(f^{-1}(F)) \subset f^{-1}(\delta\text{-}Cl(F)) = f^{-1}(F)$ and $f^{-1}(F)$ is δb-closed in X.

(4) $\Longrightarrow$ (5) Let V be any δ-open set of Y. Using (4), we have that $f^{-1}(Y\text{-}V) = X\text{-}f^{-1}(V) \in \delta BC(X)$ and so $f^{-1}(V) \in \delta BO(X)$.

(5) $\Longrightarrow$ (1) Let V be any *regular* open set of Y. Since V is δ-open set in Y, $f^{-1}(V) \in \delta BO(X)$ and hence by Theorem 3.2, f is almost δb-continuous. □

Lemma 3.4. ([6]) *Let (X, τ) be a topological space. If $A \in \delta O(X)$ and $B \in \delta BO(X)$, then $A \cap B \in \delta BO(X)$.*

Lemma 3.5. ([8]) *Let A and B be subsets of a space (X, τ). If $A \in \delta O(X)$ and $B \in \delta BO(X)$, then $A \cap B \in \delta BO(A)$.*

Theorem 3.6. *If $f : X \to Y$ is almost δb-continuous and A is a δ-open subspace of X, then the restriction $f|_A$ is almost δ-b-continuous.*

Proof. Let V be any regular open set of Y. Then we have $f^{-1}(V) \in \delta BO(X)$ by Theorem 3.2. Therefore, we have $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in \delta BO(A)$ by Lemma 3.5. Hence $f|_A$ is almost δb-continuous. □

Theorem 3.7. *If $f : X \to Y$ is a constannt function that maps all of X into a single point r of Y, then f is almost δb-continuous.*

Proof. Let U be a regular open set of Y. Then the subset $f^{-1}(U)$ is either X or $\emptyset$ depending on whether U contains r or not. In either case, it is δb-open in X. Hence f is almost δb-continuous. □

Theorem 3.8. *If A is a δb-open subspace in X, then the inclusion function $J : A \to X$ is almost δb-continuous.*

Proof. Let U be a regular open set in X. Then U is δb-open in X and $J^{-1}(U) = U \cap A$. Using Lemma 3.5, we deduce that $J^{-1}(U)$ is δb-open in A. Hence J is almost δb-continuous. □

Theorem 3.9. *Let $f : X \to Y$ be an almost δb-continuous function. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the image of f is almost δb-continuous.*

Proof. Let V be a regular open subset of Z. Then $V = Z \cap U$ for a regular open set U of Y. By assumption that $f(X) \subset Z$, we deduce that $f^{-1}(U) = g^{-1}(V)$. Since f is almost δb-continuous, $f^{-1}(U) \in \delta BO(X)$. Therefore, $g^{-1}(V) \in \delta BO(X)$. Hence g is almost δb-continuous. □

Theorem 3.10. *The pasting lemma holds for almost δb-continuous functions.*

Let Δ be an index set and $\{X_\alpha : \alpha \in \Omega\}$ and $\{Y_\alpha : \alpha \in \Omega\}$ be any two families indexed by Ω. For each $\alpha \in \Omega$, let $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ be a function. The product space $\Pi\{X_\alpha : \alpha \in \Omega\}$ will be denoted by ΠX_α and the product function $\Pi f_\alpha : \Pi X_\alpha \longrightarrow \Pi Y_\alpha$ is simply denoted by $f : \Pi X_\alpha \longrightarrow \Pi Y_\alpha$.

Theorem 3.11. *If $f : X \longrightarrow \Pi Y_\alpha$ is almost δ-b-continuous, then $\rho_\alpha \circ f : X \longrightarrow Y_\alpha$ is almost δb-continuous for each $\alpha \in \Delta$, where ρ_α is the projection of ΠY_α onto Y_α.*

Proof. Let V_α be any regular open set of Y_α. Since ρ_α is continuous and open, it is obvious that it is an R-map. Therefore, we have that $\rho_\alpha^{-1}(V_\alpha) \in RO(\Pi Y_\alpha)$. On the other hand, since f is almost δ-b-continuous, we obtain $f^{-1}\left(\rho_\alpha^{-1}(V_\alpha)\right) = (\rho_\alpha \circ f)^{-1}(V_\alpha) \in \delta BO(X)$ by Theorem 3.2. Hence $\rho_\alpha \circ f$ is almost δb-continuous. □

Remark 3.12. What is about the converse? If for all $\alpha \in \Omega, \rho_\alpha \circ f : X \to Y_\alpha$ is almost δb-continuous, then is the function $f : X \to \Pi Y_\alpha$ almost δb-continuous? If it does not hold for infinite products, what is about finite products?

The answer is yes. We can give an answer of this question. To prove that the converse side holds we need the following results.

Theorem 3.13. *The product function $f : \Pi X_\alpha \longrightarrow \Pi Y_\alpha$ is almost δb-continuous if and only if $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ is almost δb-continuous for each $\alpha \in \Omega$.*

Proof. (Necessity) Let V_β be any regular open set of Y_β, where β is an arbitrary fixed index. Then we have $\Pi Y_\gamma \times V_\beta$ is regular open in ΠY_γ where $\gamma \in \Delta$ and $\gamma \neq \beta$. Therefore, we obtain $f^{-1}\left(\Pi Y_\gamma \times V_\beta\right) = \Pi Y_\gamma \times f_\beta^{-1}\left(V_\beta\right)$ is a δb-open subset of ΠX_α and hence f_β is almost δb-continuous.

(Sufficiency) Let $\{x_\alpha\}$ be any point of ΠX_α and G be any regular open set of ΠY_α containing $f(\{x_\alpha\})$. There exists a finite subset Δ_0 of Δ such that $V_\beta \in RO\left(Y_\beta\right)$ for each $\beta \in \Delta_0$ and $\{f(\{x_\alpha\})\} \in \Pi\{V_\beta : \beta \in \Delta_0\} \times \Pi\{Y_\gamma : \gamma \in \Delta \backslash \Delta_0\} \subset G$. Therefore, there exists $U_\beta \in \delta BO(X_\beta)$ containing $\{x_\beta\}$ such that

$f_\beta\left(U_\beta\right) \subset V_\beta$ for each $\beta \in \Delta_0$. Consequently, we obtain $U = \Pi\{U_\beta : \beta \in \Delta_0\} \times \Pi\{X_\gamma : \gamma \in \Delta\backslash\Delta_0\}$ is δ-b-open in ΠX_α containing $\{x_\alpha\}$ and $f(U) \subset G$. This shows that f is almost δb-continuous. □

Let $\Delta X = \{(x, x, , x, \ldots,) : x \in X\}$ be the diagonal subspace of the product space of any number copies of a topological space X. Then we claim the following.

Theorem 3.14. *(i) The subspace* $\Delta X = \{(x, x, , x, \ldots,) : x \in X\}$ *is* δ-b*-open in* $\prod X$.

(ii) The inclusion map $J : \Delta X \to \prod X$ *is almost* δ-b*-continuous.*

Theorem 3.15. *Let* $f : (X, \tau) \longrightarrow (Y, \varphi)$ *be a function and* $g : (X, \tau) \longrightarrow (X \times Y, \tau \times \varphi)$ *be the graph function defined by* $g(x) = (x, f(x))$ *for every* $x \in X$. *Then*

g *is almost* δ-b*-continuous if and only if* f *is almost* δ-b*-continuous.*

Proof. (Necessity) Let $x \in X$ and $V \in RO(Y)$ containing $f(x)$. Then we have $g(x) = (x, f(x)) \in (X \times V) \in RO(X \times Y)$. Since g is almost δ-b-continuous, there exists a δ-b-open set U of X containing x such that $g(U) \subset X \times V$. Therefore, we deduce that $f(U) \subset V$ and hence f is almost δ-b-continuous.

(Sufficiency) Let $x \in X$ and W be a regular open set of $X{\times}Y$ containing $g(x)$. There exist $U_1 \in RO(X)$ and $V \in RO(Y)$ such that $(x, f(x)) \in (U_1 \times V) \subset W$. Since f is almost δ-b-continuous, there exists $U_2 \in \delta BO(X)$ such that $x \in U_2$ and $f(U_2) \subset V$. If we take $U = U_1 \cap U_2$, then we obtain $x \in U \in \delta BO(X)$ and $g(U) \subset (U_1 \times V) \subset W$. Hence g is almost δ-b-continuous. □

4. Relationships

In this section, we investigate the relationships between different types of continuous functions.

Definition 4.1. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be δcontinuous [14]. (resp., almost semicontinuous [12], almost δsemicontinuous [3], almost $b$continuous [10]) if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta O(X)$ (resp., $U \in SO(X)$, $U \in \delta SO(X)$, $U \in BO(X)$) containing x such that $f(U) \subset V$.

The following diagram is an extended form of the second diagram in [3].

$$\begin{tikzcd}
R\text{-map} \arrow[r] & \delta\text{-continuous} \arrow[r] & \text{almost } \delta\text{-semicontinuous} \arrow[r] \arrow[d] & \text{almost } \delta b\text{-continuous} \arrow[d] \\
& & \text{almost semicontinuous} \arrow[r] & \text{almost } b\text{-continuous}
\end{tikzcd}$$

Diagram II

Remark 4.2. None of the converses of these implications is true in general as shown in the following examples. The other examples are in related references.

Example 4.3. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a, b\}, \{a, b, c\}\}$, $\varphi = \{\varnothing, X, \{a\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$. A function $f : (X, \tau) \longrightarrow (X, \varphi)$ is defined as follows: $f(a) = f(b) = a$, $f(c) = c$ and $f(d) = d$. Then f is almost δ-b-continuous but is not almost δ-semicontinuous, because $A = \{a, b, c\}$ is a δ-b-open set of X which is not δ-semi-open.

Example 4.4. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$. A function $f : (X, \tau) \longrightarrow (X, \tau)$ is defined as follows: $f(a) = a$, $f(b) = c$, $f(c) = b$ and $f(d) = d$. Then f is almost b-continuous but is not almost δ-b-continuous, because $A = \{a, b\}$ is a b-open set of X which is not δ-b-open.

Now, we recall the following definition.

Definition 4.5. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be δ-b-irresolute if $f^{-1}(V)$ is a δb-open set in X for every δb-open set V of Y.

The next statement is obvious, so its proof is omitted.

Theorem 4.6. *Let* $f : (X, \tau) \longrightarrow (Y, \varphi)$ *and* $g : (Y, \varphi) \longrightarrow (Z, \psi)$ *be functions. For the composition* $g \circ f : (X, \tau) \longrightarrow (Z, \psi)$*, the following properties hold:*
(1) If f *is almost* δb*-continuous and* g *is an* R*-map, then* $g \circ f$ *is almost* δb*-continuous.*
(2) If f *is* δb*-irresolute and* g *is almost* δb*-continuous, then* $g \circ f$ *is almost* δb*-continuous.*

Recall that a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be δb-continuous [9] if for each $x \in X$ and each $V \in \varphi$ containing $f(x)$, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset V$.

It is known that the family of all regular open sets in a space (X, τ) forms a base for a topology τ_s which is called the semiregularization [18]. So we have the following properties.

Theorem 4.7. *For a function* $f : (X, \tau) \longrightarrow (Y, \varphi)$*, the following properties hold:*
(1) f *is super continuous if and only if* $f : (X, \tau_s) \longrightarrow (Y, \varphi)$ *is continuous.*
(2) f *is* δ*-*b*-continuous if and only if* $f : (X, \tau_s) \longrightarrow (Y, \varphi)$ *is* b*-continuous.*

In [8], as related to semiregularization, the following result was proved.

Lemma 4.8. *[8] A subset* A *is* δ*-*b*-open in* (X, τ) *if and only if* A *is* b*-open in* (X, τ_s)*.*

Now, we present the main result.

Theorem 4.9. *Let* $f : (X, \tau) \longrightarrow (Y, \varphi)$ *be a function. Then the following properties are equivalent:*
(1) f *is almost* δb*-continuous.*
(2) $f : (X, \tau_s) \longrightarrow (Y, \varphi)$ *is almost* b*-continuous.*
(3) $f : (X, \tau) \longrightarrow (Y, \varphi_s)$ *is* δb*-continuous.*
(4) $f : (X, \tau_s) \longrightarrow (Y, \varphi_s)$ *is* b*-continuous.*

Proof. (1) $\Longrightarrow$ (2) Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be almost δb-continuous and $A \in RO(Y)$. According to (1), $f^{-1}(A)$ is δb-open in X. By Lemma 4.8, we have that $f^{-1}(A)$ is b-open in (X, τ_s). Hence f is almost b-continuous.

(2) $\Longrightarrow$ (3) Let $V \in \varphi_s$. Then there exists a regular open set $(U_\alpha)_{\alpha \in \Delta}$ such that $V = \cup U_\alpha$. Since $f : (X, \tau_s) \longrightarrow (Y, \varphi)$ is almost b-continuous, by the hypothesis, $f^{-1}(U_\alpha)$ is b-open in (X, τ_s). Then $f^{-1}(U_\alpha)$ is δb-open in (X, τ) for each $\alpha \in \Delta$ by Lemma 4.8. Thus $f^{-1}(V)$ is δb-open in (X, τ). Hence $f : (X, \tau) \longrightarrow (Y, \varphi_s)$ is δ-b-continuous.

(3) $\Longrightarrow$ (4) The proof is similar to (1) $\Longrightarrow$ (2).

(4) $\Longrightarrow$ (1) Let $f : (X, \tau_s) \longrightarrow (Y, \varphi_s)$ be b-continuous and $A \in RO(Y)$. By definition of semiregularization, $A \in \varphi_s$. According to (4), $f^{-1}(A)$ is b-open in (X, τ_s). By Lemma 4.8, we have that $f^{-1}(A)$ is δb-open in (X, τ). Therefore, $f : (X, \tau) \longrightarrow (Y, \varphi)$ is almost δb-continuous by Theorem 3.2. □

We can obtain another proof of Theorem 3.11 and another proof of Theorem 3.13 by using Theorem 4.9 and the following lemma.

Lemma 4.10. *([7]) Let* $\{X_\alpha : \alpha \in \Omega\}$ *be any family of spaces indexing by* Ω*. Then* ΠX_α*,* $(\Pi X_\alpha)_s = \Pi\,(X_\alpha)_s$.

Theorem 4.11. *If* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is almost* δb*-continuous and* (Y,φ) *is semi-regular, then* f *is* δb*-continuous.*

Proof. Let $x \in X$ and let V be an open set in Y containing $f(x)$. Then there exists a regular open set W of Y such that $f(x) \in W \subset V$ by using the semiregularity of Y. Since f is almost δ-b-continuous, there exists $U \in \delta BO(X)$ containing x such that $f(U) \subset Int(Cl(W)) = W \subset V$. Hence f is δ-b-continuous. □

5. Separation axioms and δb-ocontinuity

In this section, we investigate a relationship between almost δb-continuous functions and separation axioms.

Definition 5.1. A topological space (X,τ) is said to be $r - T_1$ [4] (resp., δbT_1 [8]) if for each pair of distinct points x and y in X, there exist regular open (resp., δb-open) sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$.

Theorem 5.2. *If* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is an almost* δb*-continuous injective and* (Y,φ) *is r-*T_1*, then* (X,τ) *is* δb*-*T_1.

Proof. Let (Y,φ) be $r - T_1$ and x, y be distinct points of (X,τ). Since f is injective, there exist regular open subsets A, B in (Y,φ) such that $f(x) \in A$, $f(y) \notin A$, $f(y) \in B$ and $f(x) \notin B$. Since f is almost δ-b-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are δb-open subsets of X such that $x \in f^{-1}(A)$, $y \notin f^{-1}(A)$, $y \in f^{-1}(B)$ and $x \notin f^{-1}(B)$. Hence (X,τ) is $\delta b - T_1$. □

Corollary 5.3. *[8] If* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is* δb*-continuous injective and* (Y,φ) *is* T_1*, then* (X,τ) *is* $\delta b - T_1$.

Definition 5.4. [8] A topological space (X,τ) is said to be $\delta b - T_2$ if for each pair of distinct points x and y in X, there exist disjoint δb-open sets A and B such that $x \in A$ and $y \in B$.

Theorem 5.5. *If* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is almost b-continuous injective and* (Y,φ) *is* T_2*, then* (X,τ) *is* $\delta b - T_2$.

Proof. Let x and y be any pair of distinct points of X. Since f is injective, there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. The sets $Int(Cl(A))$ and $Int(Cl(B))$ are disjoint regular open sets in (Y,φ). Since f is almost b-continuous, $f^{-1}(Int(Cl(A)))$ and $f^{-1}(Int(Cl(B)))$ are b-open in (X,τ) containing x and y, respectively. But $f^{-1}(Int(Cl(A))) \cap f^{-1}(Int(Cl(B))) = \emptyset$. Hence by [8, Theorem 4.5], (X,τ) is δ-b-T_2. □

Corollary 5.6. *[8] If* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is* δb*-continuous injective and* (Y,φ) *is* T_2*, then* (X,τ) *is* δb*-*T_2.

Theorem 5.7. *If* $f, g : X \to Y$ *are almost* δb*-continuous functions and* Y *is Hausdorff, then*

$$\{x \in X : f(x) = g(x)\}$$

is δb*-closed.*

Theorem 5.8. *Let* $(X,\tau_1),(X,\tau_2)$ *be two topological spaces such that* $\tau_2 \subset \tau_1$*. Then the identity map* $i:(X,\tau_1)\to(X,\tau_2)$ *is almost* δb*-continuous.*

Proof. Let U be a regular open subset of X with respect to τ_2. By assumption U is also a regular open subset in X with respect to τ_1 and, thus it is a δb-open subset in X. Hence the identity map i is almost δb-continuous. □

6. Conclusions

In this paper, we have defined an almost δb-continuity, which is a weaker form of R-map and we have investigated and obtained its some properties and characterizations. Finally, we showed that a function $f:(X,\tau)\to(Y,\varphi)$ is almost δb-continuous if and only if $f:(X,\tau_s)\to(Y,\varphi_s)$ is b-continuous, where τ_s and φ_s are semiregularizations of τ and φ, respectively.

Conflict of interest

The authors declare that they have no competing interests.

Acknowledgements

The first author was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-145-130-1437).

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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