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On n -polynomial p -convex functions and some related inequalities

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Abstract

In this paper, we introduce a new class of convex functions, so-called n -polynomial p -convex functions. We discuss some algebraic properties and present Hermite–Hadamard type inequalities for this generalization. Moreover, we establish some refinements of Hermite–Hadamard type inequalities for this new class.

Keywords: Convex function; p -convex function; n -polynomial convexity; n -polynomial p -convex functions; Hermite–Hadamard type inequality

1 Introduction

Some geometric properties of convex sets and, to a lesser extent, of convex functions were studied before 1960 by outstanding mathematicians, first of all by Hermann Minkowski and Werner Fenchel. At the beginning of 1960 convex analysis was greatly developed in the works of R. Tyrrell Rockafellar and Jean-Jacques Morreau who initiated a systematic study of this new field. There are several books devoted to different aspects of convex analysis and optimization. See [1–6].

Let $I = [c, d] \subset \mathbb{R}$ be an interval. Then a real-valued function $\psi : I \rightarrow \mathbb{R}$ is said to be convex on I if

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y) \quad (1.1)$$

holds for all x, y in I and $t \in (0, 1)$. The function $\psi : I \rightarrow \mathbb{R}$ is said to be concave if inequality (1.1) is reversed. For more on convexity, see [7–12].

The idea of convexity is not new one even it occurs in some other form in Archimede's treatment of orbit length. Nowadays, the application of several works on convexity can be directly or indirectly seen in various subjects like real analysis, functional analysis, linear algebra, and geometry. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes. In the last few decades, the subject of convex analysis got rapid development because of its geometry and its role in the optimization. The deep relation between convex analysis and fractional calculus can never be ignored. For recent work on fractional calculus, we refer to [13–17].

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Let $\psi : I \rightarrow \mathbb{R}$ be a convex function, then for all $x, y \in I$ and $t \in (0, 1)$, the following holds:

$$\psi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d \psi(x) dx \leq \frac{\psi(c) + \psi(d)}{2}. \tag{1.2}$$

For the extended version of the above inequality, see [18, 19].

In [20], Lipot Fejér presented an extended version of (1.2) inequality known as Fejér inequality or a weighted version of the Hermite–Hadamard inequality. If $\psi : I \rightarrow R$ is a convex function, then

$$\psi\left(\frac{a+b}{2}\right) \int_c^d w(x) dx \leq \frac{1}{d-c} \int_c^d w(x)\psi(x) dx \leq \frac{\psi(c) + \psi(d)}{2} \int_c^d w(x) dx, \tag{1.3}$$

where $c \leq d$, and $w : I \rightarrow R$ is nonnegative, integrable, and symmetric about $\frac{c+d}{2}$.

The present paper is organized as follows:

First we give some preliminary material and basic definition for n -polynomial p -convex function. In the second section we give some basic results for our newly defined generalization. Next we develop Hermite–Hadamard type inequality. In the last section, we give some theorems related to our work.

2 Preliminaries

We start with some basic definitions.

Definition 2.1 (p -convex set [21]) The interval I is said to be a p -convex set if $[(tx^p + (1-t)y^p)^{\frac{1}{p}}] \in I$ for all $x, y \in I, p > 0$ and $t \in [0, 1]$.

Definition 2.2 ($(p$ -convex function) [22]) A function $\psi : I \rightarrow R$ is said to be p -convex if the following inequality

$$\psi\left[(tx^p + (1-t)y^p)^{\frac{1}{p}}\right] \leq t\psi(x) + (1-t)\psi(y) \tag{2.1}$$

holds for all $x, y \in I = [c, d]$ and $t \in [0, 1]$ where $p > 0$.

It can be easily seen that, for $p = 1$, p -convexity is reduced to the classical convexity of functions defined on $I \subset (0, \infty)$.

Now we recall the definition of harmonically convex function as follows.

Definition 2.3 (Harmonic convex function [23]) Let $I \subset \mathbb{R}$ be an interval. Then a real-valued function $\psi : I \rightarrow \mathbb{R}$ is said to be harmonically convex if

$$\psi\left(\frac{xy}{tx + (1-t)y}\right) \leq t\psi(y) + (1-t)\psi(x) \tag{2.2}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [24] n -polynomial convexity has been defined.

Definition 2.4 (*n*-polynomial convex function) Let $n \in \mathbb{N}$. A nonnegative function $\psi : I \rightarrow R$ is called *n*-polynomial convex function if, for every $x, y \in I$ and $t \in [0, 1]$,

$$\psi(tx + (1-t)y) \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y). \tag{2.3}$$

We will denote by POLC(I) the class of all *n*-polynomial convex functions on interval *I*.

We note that every *n*-polynomial convex function is an *h*-convex function with the function $h(t) = \sum_{s=1}^n [1 - (1-t)^s]$.

In [25] *n*-polynomial harmonically convexity has been defined.

Definition 2.5 (*n*-polynomial harmonic convex function) Let $n \in \mathbb{N}$. A nonnegative function $\psi : I \rightarrow R$ is called *n*-polynomial harmonically convex function if, for every $x, y \in I$ and $t \in [0, 1]$,

$$\psi\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(y) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(x). \tag{2.4}$$

From Definition 2.5, for $n = 2$, we can see that the class of *n*-polynomial harmonically convex functions satisfies the inequality

$$\psi\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{3t - t^2}{2} \psi(y) + \frac{2 - t - t^2}{2} \psi(x) \tag{2.5}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Now we are going to introduce a new generalization of *n*-polynomial convex function.

Definition 2.6 (*n*-polynomial *p*-convex function) Let $n \in \mathbb{N}$. A nonnegative function $\psi : I \rightarrow R$ is called *n*-polynomial *p*-convex function if, for every $x, y \in I, p > 0$ and $t \in [0, 1]$,

$$\psi\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y) \tag{2.6}$$

holds.

Remark 2.7 1: if we put $p = -1$, then (2.6) is reduced to (2.4) *n*-polynomial harmonic convex function [25].

2: if we put $p = 1$, then (2.6) is reduced to (2.3) *n*-polynomial convex function [24].

3 Basic results

In this section we derive some basic results and propositions related to our new generalization.

The following proposition shows the linearity of *n*-polynomial *p*-convex function.

Proposition 3.1 Let $\phi : I \rightarrow \mathbb{R}$ be a nonnegative *n*-polynomial *p*-convex function, and where for $n \in \mathbb{N}, x, y \in I, p > 0$ and $t \in [0, 1]$, then $\psi + \phi$ is an *n*-polynomial *p*-convex function.

Proof Let ψ and ϕ be two n -polynomial p -convex functions, then for all $x, y \in I, p > 0$ and $t \in [0, 1]$ we have

$$\begin{aligned}
 & (\psi + \phi)\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] \\
 &= \psi\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] + \phi\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] \\
 &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y) \\
 &\quad + \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \phi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \phi(y) \\
 &= \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] (\psi + \phi)(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] (\psi + \phi)(y), \tag{3.1}
 \end{aligned}$$

this assures the n -polynomial p -convexity of $\psi + \phi$. □

Now we will discuss the scalar multiplication of n -polynomial p -convex function.

Proposition 3.2 *Let $\psi : I \rightarrow \mathbb{R}$ be a nonnegative n -polynomial p -convex function and $\lambda > 0$, where for $n \in \mathbb{N}, x, y \in I, p > 0$ and $t \in [0, 1]$, then $\lambda\psi : I \rightarrow \mathbb{R}$ is also an n -polynomial p -convex function.*

Proof Let ψ be an n -polynomial p -convex function, then for all $x, y \in I, p > 0$ and $t \in [0, 1]$, where $\lambda > 0$, we have

$$\begin{aligned}
 (\lambda\psi)\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] &= \lambda\left[\psi\left(\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right)\right] \\
 &\leq \lambda\left[\frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y)\right] \\
 &= \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] (\lambda\psi)(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] (\lambda\psi)(y), \tag{3.2}
 \end{aligned}$$

which shows that $\lambda\psi$ is also an n -polynomial p -convex function. □

Proposition 3.3 *Let $\psi : I \rightarrow \mathbb{R}$ be a nonnegative n -polynomial p -convex function, and where for $n \in \mathbb{N}, x, y \in I, p > 0$ and $t \in [0, 1]$, then*

$$\psi = \max\{\psi_i, i = 1, 2, 3, \dots, n\}$$

is also an n -polynomial p -convex function.

Proof Take any $x, y \in R^n$ and $t \in [0, 1]$. Denote $\psi = \max \psi_i$, where $i = 1, 2, 3, \dots, n$,

$$\begin{aligned}
 \psi\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right] &= \max\{\psi_i\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right], i = 1, 2, 3, \dots, n\} \\
 &= \psi_w\left[\left(tx^p + (1-t)y^p\right)^{\frac{1}{p}}\right]
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi_w(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi_w(y) \\
 &= \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \max \psi_i(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \max \psi_i(y) \\
 &= \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y), \tag{3.3}
 \end{aligned}$$

$\implies \psi = \max\{\psi_i, i = 1, 2, 3, \dots, n\}$ is also an n -polynomial p -convex function.

This completes the proof. □

Proposition 3.4 *Let $\psi_i : R^n \rightarrow \bar{R}$ for $i \in I$ be a collection of n polynomial p -convex functions. Then the supremum function*

$$\psi(x) = \sup \psi_i(x), \quad i \in I,$$

is also n polynomial p -convex function.

Hint. If $\psi(x) = \sup \psi_i(x), i \in I$, then $\psi(x) \geq \psi_i(x), \forall i \in I$.

Proof Fix $x, y \in R^n, p > 0$ and $t \in [0, 1]$, then for every $i \in I$ we have

$$\begin{aligned}
 \psi_i[(kx^p + (1-k)y^p)^{\frac{1}{p}}] &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi_i(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi_i(y) \\
 &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y), \tag{3.4}
 \end{aligned}$$

which implies in turn that

$$\begin{aligned}
 \psi[(tx^p + (1-t)y^p)^{\frac{1}{p}}] &= \sup_{i \in I} [\psi_i(tx^p + (1-t)y^p)^{\frac{1}{p}}] \\
 &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(x) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(y). \tag{3.5}
 \end{aligned}$$

This justifies the convexity of supremum function. □

Remark 3.5 1: If we insert $p = -1$ in Proposition 3.4, then we will get the result for an n -polynomial harmonically convex function [25, Theorem 2.2].

4 Hermite–Hadamard type inequality for n -polynomial p -convex function

The goal of this paper is to establish some inequalities of Hermite–Hadamard type for n -polynomial p -convex function. Throughout the section, $L[c, d]$ will represent the space of (Lebesgue) integrable functions on $[c, d] \subseteq \mathbb{R}$. For more on Hermite–Hadamard type inequality, see [23, 26–29].

Theorem 4.1 (Hermite–Hadamard type inequality) *Let $\psi : [c, d] \rightarrow \mathbb{R}$ be an n -polynomial p -convex function. If $c < d$ and $\psi \in L[c, d]$, where $p > 0$, then the following Hermite–*

Hadamard type inequalities hold:

$$\begin{aligned} \frac{1}{2} \left(\frac{n}{n+2^{-n}-1} \right) \psi \left[\left(\frac{c^p+d^p}{2} \right)^{\frac{1}{p}} \right] &\leq \frac{p}{d^p-c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \\ &\leq \left(\frac{\psi(c)+\psi(d)}{n} \right) \sum_{s=1}^n \frac{s}{s+1}. \end{aligned} \tag{4.1}$$

Proof Fix $x, y \in R^n$, $p > 0$, and $t \in [0, 1]$, then for every $i \in I$, by the definition of n -polynomial p -convex function of ψ , we have

$$\begin{aligned} \psi \left[\left(\frac{c^p+d^p}{2} \right)^{\frac{1}{p}} \right] &= \psi \left[\left(\frac{[tc^p+(1-t)d^p]+[(1-t)c^p+td^p]}{2} \right)^{\frac{1}{p}} \right] \\ &= \psi \left[\left(\frac{[tc^p+(1-t)d^p]}{2} + \frac{[(1-t)c^p+td^p]}{2} \right)^{\frac{1}{p}} \right] \\ &\leq \frac{1}{n} \sum_{s=1}^n \left[1 - \left(1 - \frac{1}{2} \right)^s \right] \psi(tc^p+(1-t)d^p) \\ &\quad + \frac{1}{n} \sum_{s=1}^n \left[1 - \left(\frac{1}{2} \right)^s \right] \psi(td^p+(1-t)c^p). \end{aligned} \tag{4.2}$$

Integration in the last inequality with respect to $t \in [0, 1]$ yields that

$$\begin{aligned} \psi \left[\left(\frac{c^p+d^p}{2} \right)^{\frac{1}{p}} \right] &\leq \frac{1}{n} \sum_{s=1}^n \left[1 - \left(1 - \frac{1}{2} \right)^s \right] \int_0^1 \psi(tc^p+(1-t)d^p) dt \\ &\quad + \frac{1}{n} \sum_{s=1}^n \left[1 - \left(\frac{1}{2} \right)^s \right] \int_0^1 \psi(td^p+(1-t)c^p) dt \\ &= \frac{1}{n} \sum_{s=1}^n \left[1 - \left(\frac{1}{2} \right)^s \right] \left[\int_0^1 \psi(tc^p+(1-t)d^p) dt + \int_0^1 \psi(td^p+(1-t)c^p) dt \right]. \end{aligned} \tag{4.3}$$

After solving the above inequality (4.3), we get

$$\psi \left[\left(\frac{c^p+d^p}{2} \right)^{\frac{1}{p}} \right] \leq \frac{2p}{d^p-c^p} \left(\frac{n+2^{-n}-1}{n} \right) \int_c^d \frac{\psi(x)}{x^{1-p}} dx \tag{4.4}$$

$$\left(\frac{n}{2(n+2^{-n}-1)} \right) \psi \left[\left(\frac{c^p+d^p}{2} \right)^{\frac{1}{p}} \right] \leq \frac{p}{d^p-c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx, \tag{4.5}$$

which is the left-hand side of the theorem.

To prove the right-hand side of the theorem, take

$$\frac{p}{d^p-c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx = \int_0^1 \psi(tc^p+(1-t)d^p)^{\frac{1}{p}} dt, \tag{4.6}$$

since ψ is an n -polynomial p -convex function:

$$\begin{aligned}
 & \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \\
 & \leq \int_0^1 \left[\frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \psi(c) + \frac{1}{n} \sum_{s=1}^n [1 - t^s] \psi(d) \right] dt \\
 & = \frac{\psi(c)}{n} \int_0^1 \sum_{s=1}^n [1 - (1-t)^s] dt + \frac{\psi(d)}{n} \int_0^1 [1 - t^s] dt \\
 & = \frac{\psi(c)}{n} \sum_{s=1}^n \int_0^1 [1 - (1-t)^s] dt + \frac{\psi(d)}{n} \sum_{s=1}^n \int_0^1 [1 - t^s] dt \\
 & = \left[\frac{\psi(c) + \psi(d)}{n} \right] \sum_{s=1}^n \frac{s}{s+1}, \tag{4.7}
 \end{aligned}$$

which is the right-hand side of the theorem. □

Remark 4.2 Imposing some condition on Theorem (4.1), we get a different version of Hermite–Hadamard type inequality.

1. For $n = 1$ and $p = 1$, we obtain Hermite–Hadamard type inequality (1.2) for classical convex functions.
2. For $p = -1$, we obtain Hermite–Hadamard type inequality for n -polynomial harmonically convex function [25].
3. For $p = 1$, we obtain Hermite–Hadamard type inequality for n -polynomial classical convex function [24].

5 New inequalities for n -polynomial p -convex function

In this section, we establish new estimates that refine Hermite–Hadamard inequality for a function whose first derivative is absolute value, raised to a certain power which is greater than one.

In [26] the following lemma is given, which will be helpful for generating refinements of Hermite–Hadamard type inequality.

Lemma 5.1 ([26]) *Let $\psi : I = [c, d] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° with $c < d$. If $\psi' \in L[c, d]$, then*

$$\begin{aligned}
 & \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \\
 & = \frac{d^p - c^p}{2p} \int_0^1 M_p^{-1}(c, d; t) (1 - 2t) \psi'(M_p(c, d; t)) dt, \tag{5.1}
 \end{aligned}$$

where $M_p^{-1}(c, d; t) = [tc^p + (1 - t)d^p]^{\frac{1}{p}-1}$.

Theorem 5.2 *Let $\psi : I \rightarrow \mathbb{R}$ be a differentiable function on I° , $c, d \in I^\circ$ with $c < d$ and assume that $\psi' \in L[c, d]$. If ψ' is an n -polynomial p -convex function on the interval $[c, d]$,*

then the following inequality holds for $t \in [0, 1]$:

$$\left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \leq \frac{d^p - c^p}{2np} \sum_{s=1}^n [|\psi'(c)|C_1(t, s) + |\psi'(d)C_2(t, s)|],$$

where

$$C_1(t, s) = \int_0^1 |1 - 2t[1 - (1 - t)^s]|tc^p + (1 - t)d^p|^{\frac{1}{p}-1} dt \tag{5.2}$$

and

$$C_2(t, s) = \int_0^1 |1 - 2t[1 - t^s]|tc^p + (1 - t)d^p|^{\frac{1}{p}-1} dt. \tag{5.3}$$

Proof The definition of n -polynomial convexity and Lemma 5.1 yields the following:

$$\begin{aligned} & \left| \psi'(tc^p + (1 - t)d^p)^{\frac{1}{p}} \right| \\ & \leq \frac{1}{n} \sum_{s=1}^n [1 - (1 - t)^s] |\psi'(c)| + \frac{1}{n} \sum_{s=1}^n [1 - (t)^s] |\psi'(d)|. \end{aligned} \tag{5.4}$$

We get

$$\begin{aligned} & \left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \\ & \leq \left| \frac{d^p - c^p}{2p} \int_0^1 (1 - 2t)(tc^p + (1 - t)d^p)^{\frac{1}{p}-1} \psi'(tc^p + (1 - t)d^p)^{\frac{1}{p}} dt \right| \\ & \leq \frac{d^p - c^p}{2np} \left(|\psi'(c)| \int_0^1 |1 - 2t|tc^p + (1 - t)d^p|^{\frac{1}{p}-1} \sum_{s=1}^n [1 - (1 - t)^s] dt \right) \\ & \quad + \frac{d^p - c^p}{2np} \left(|\psi'(d)| \int_0^1 |1 - 2t|tc^p + (1 - t)d^p|^{\frac{1}{p}-1} \sum_{s=1}^n [1 - (t)^s] dt \right) \\ & = \frac{d^p - c^p}{2np} \sum_{s=1}^n [|\psi'(c)|C_1(t, s) + |\psi'(d)C_2(t, s)|]. \end{aligned} \tag{5.5}$$

This completes the proof. □

Remark 5.3 1. For $p = 1$, we have [24, Theorem 5].

Corollary 5.4 *If we take $n = 1$ and $p = 1$ in inequality (4.1), we get the following inequality:*

$$\left| \frac{\psi(c) + \psi(d)}{2} - \frac{1}{d - c} \int_c^d \psi(x) dx \right| \leq \frac{d - c}{2} \left(\frac{1}{p + 1} \right)^{\frac{1}{p}} A(|\psi'(c)|, |\psi'(d)|). \tag{5.6}$$

This inequality coincides with the inequality in [26].

In [30], Iscan gave a refinement of Holder integral inequality as follows.

Theorem 5.5 (Holder–Iscan integral inequality [30]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ψ and ϕ are real functions defined on the interval $[c, d]$ and if $|\psi|^q$ and $|\phi|^q$ are integrable functions on $[c, d]$, then*

$$\begin{aligned} & \int_c^d |\psi(x)\phi(x)| dx \\ & \leq \frac{1}{d-c} \left\{ \left(\int_c^d (d-x)|\psi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_c^d (d-x)|\phi(x)|^q dx \right)^{\frac{1}{q}} \right\} \\ & \quad + \frac{1}{d-c} \left\{ \left(\int_c^d (x-c)|\psi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_c^d (x-c)|\phi(x)|^q dx \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{5.7}$$

A refinement of the power mean integral inequality as a different version of the Holder–Iscan integral inequality is given as follows.

Theorem 5.6 (Improved power-mean integral inequality [31]) *Let $q > 0$. If ψ and ϕ are real functions defined on the interval $[c, d]$ and if $|\psi|$, $|\psi||\phi|^q$ are integrable functions on $[c, d]$, then*

$$\begin{aligned} & \int_c^d |\psi(x)\phi(x)| dx \\ & \leq \frac{1}{d-c} \left\{ \left(\int_c^d (d-x)|\psi(x)|^p dx \right)^{1-\frac{1}{q}} \left(\int_c^d (d-x)|\phi(x)|^q dx \right)^{\frac{1}{q}} \right\} \\ & \quad + \frac{1}{d-c} \left\{ \left(\int_c^d (x-c)|\psi(x)|^p dx \right)^{1-\frac{1}{q}} \left(\int_c^d (x-c)|\phi(x)|^q dx \right)^{\frac{1}{q}} \right\} \end{aligned} \tag{5.8}$$

holds.

Theorem 5.7 *Let $\psi : I \rightarrow \mathbb{R}$ be a differentiable function on I^o , $c, d \in I^o$ with $c < d$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and assume that $\psi' \in L[c, d]$. If ψ' is an n -polynomial p -convex function on the interval $[c, d]$, then the following inequality holds for $t \in [0, 1]$:*

$$\begin{aligned} & \left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \\ & \leq \frac{d^p - c^p}{2p} (C_3(p))^{\frac{1}{p}} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|\psi'(c)|^q, |\psi'(d)|^q), \end{aligned} \tag{5.9}$$

where

$$C_3(p) = \int_0^1 \frac{|1-2t|^p}{|(tc^p + (1-t)d^p)^{1-\frac{1}{p}}|^p} dt.$$

Proof Using the definition of n -polynomial convexity and Lemma 5.1, we have

$$\begin{aligned} & \left| \psi'(tc^p + (1-t)d^p)^{\frac{1}{p}} \right| \\ & \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] |\psi'(c)| + \frac{1}{n} \sum_{s=1}^n [1 - (t)^s] |\psi'(d)|, \end{aligned} \tag{5.10}$$

which is an n -polynomial p -convex function of $|\psi'|^q$, we get

$$\begin{aligned}
 & \left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \\
 & \leq \frac{d^p - c^p}{2p} \int_0^1 |1 - 2t| (tc^p + (1-t)d^p)^{\frac{1}{p}-1} |\psi'(tc^p + (1-t)d^p)^{\frac{1}{p}}| dt \\
 & \leq \frac{d^p - c^p}{2p} \left(\int_0^1 \frac{|1 - 2t|^p}{|(tc^p + (1-t)d^p)^{1-\frac{1}{p}}|^p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |\psi'(tc^p + (1-t)d^p)|^q dt \right)^{\frac{1}{q}} \\
 & \leq \frac{d^p - c^p}{2p} (C_3(p))^{\frac{1}{p}} \left(\frac{|\psi'(c)|^q}{n} \sum_{s=1}^n \int_0^1 [1 - (1-t)^s] dt + \frac{|\psi'(d)|^q}{n} \sum_{s=1}^n \int_0^1 [1 - t^s] dt \right)^{\frac{1}{q}} \\
 & = \frac{d^p - c^p}{2p} (C_3(p))^{\frac{1}{p}} \left(|\psi'(c)|^q \frac{1}{n} \sum_{s=1}^n \frac{s}{s+1} + |\psi'(d)|^q \frac{1}{n} \sum_{s=1}^n \frac{s}{s+1} \right)^{\frac{1}{q}} \\
 & \quad \times \frac{d^p - c^p}{2p} (C_3(p))^{\frac{1}{p}} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|\psi'(c)|^q, |\psi'(d)|^q), \tag{5.11}
 \end{aligned}$$

where

$$\int_0^1 [1 - (1-t)^s] dt = \int_0^1 [1 - t^s] dt = \frac{s}{s+1},$$

and A is arithmetic mean. This completes the proof of the theorem. □

Remark 5.8 1. For $p = 1$, we have [24, Theorem 6].

Corollary 5.9 *If we take $n = 1$ and $p = 1$ in inequality (4.1), we get the following inequality:*

$$\left| \frac{\psi(c) + \psi(d)}{2} - \frac{1}{d - c} \int_c^d f(x) dx \right| \leq \frac{d - c}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} A^{\frac{1}{q}} (|\psi'(c)|^q, |\psi'(d)|^q). \tag{5.12}$$

This inequality coincides with the inequality in [26].

Theorem 5.10 *Let $\psi : I \rightarrow \mathbb{R}$ be a differentiable function on I^o , $c, d \in I^o$ with $c < d$, $q > 1$, and assume that $|\psi'|^q$ is an n -polynomial p -convex function on the interval $[c, d]$, then the following inequality holds for $t \in [0, 1]$:*

$$\begin{aligned}
 & \left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \\
 & \leq \frac{d^p - c^p}{2p} (C_4(p))^{1-\frac{1}{q}} \left[\left(\frac{|\psi'(c)|^q}{n} C_1(p) + \frac{|\psi'(d)|^q}{n} C_2(p) \right) \right]^{\frac{1}{q}}. \tag{5.13}
 \end{aligned}$$

Proof The definition of n -polynomial convexity and Lemma 5.1 yields the following:

$$\begin{aligned}
 & |\psi'(tc^p + (1-t)d^p)^{\frac{1}{p}}| \\
 & \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] |\psi'(c)| + \frac{1}{n} \sum_{s=1}^n [1 - t^s] |\psi'(d)|, \tag{5.14}
 \end{aligned}$$

$$\begin{aligned}
 & \left| \frac{\psi(c) + \psi(d)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{\psi(x)}{x^{1-p}} dx \right| \\
 & \leq \frac{d^p - c^p}{2p} \left(\int_0^1 |1 - 2t| |(tc^p + (1-t)d^p)^{\frac{1}{p}-1}| dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left(\int_0^1 |\psi'(tc^p + (1-t)d^p)^{\frac{1}{p}}| dt \right)^{\frac{1}{q}} \\
 & \leq \frac{d^p - c^p}{2p} (C_4(p))^{1-\frac{1}{q}} \left(\int_0^1 |1 - 2t| |(tc^p + (1-t)d^p)^{\frac{1}{p}-1}| \right. \\
 & \quad \times \left. \left[\frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] |\psi'(c)|^q \right]^{\frac{1}{q}} \right. \\
 & \quad + \frac{d^p - c^p}{2p} (C_4(p))^{1-\frac{1}{q}} \left(\int_0^1 |1 - 2t| |(tc^p + (1-t)d^p)^{\frac{1}{p}-1}| \right. \\
 & \quad \times \left. \left[\frac{1}{n} \sum_{s=1}^n [1 - t^s] |\psi'(d)|^q \right]^{\frac{1}{q}} \right) \\
 & = \frac{d^p - c^p}{2p} (C_4(p))^{1-\frac{1}{q}} \left[\left(\frac{|\psi'(c)|^q}{n} C_1(p) + \frac{|\psi'(d)|^q}{n} C_2(p) \right) \right]^{\frac{1}{q}}. \tag{5.15}
 \end{aligned}$$

This completes the proof of the theorem. □

Remark 5.11 1. For $p = 1$, we have [24, Theorem 7].

Corollary 5.12 *If we take $n = 1$ and $p = 1$ in (4.1), we get the following inequality:*

$$\left| \frac{\psi(c) + \psi(d)}{2} - \frac{1}{b-a} \int_c^d \psi(x) dx \right| \leq \frac{d-c}{4} A^{\frac{1}{q}} (|\psi'(c)|^q, |\psi'(d)|^q). \tag{5.16}$$

This inequality coincides with the inequality in [26] with $q = 1$.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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