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Zigzag strip bundles and highest weight crystals

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A B S T R A C T

Zigzag strip bundles are new combinatorial models realizing the crystals $B(\infty)$ for the quantum affine algebras $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$. In this paper, we give new realizations of the crystal bases $B(\lambda)$ for the irreducible highest weight modules $V(\lambda)$ over quantum affine algebras $U_q(\mathfrak{g})$ using zigzag strip bundles. Further, we discuss the connection between zigzag strip bundle realization, Nakajima monomial realization, and polyhedral realization of the crystals $B(\lambda)$.

Introduction

Since Kashiwara introduced the *crystal basis* theory over quantum groups $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}$ in [10,11], it has been a powerful combinatorial and geometric tool to investigate the structures of integrable modules over quantum groups and quantum groups themselves. In those papers, he proved the existence of the crystal bases $B(\lambda)$ of the irreducible highest weight representations $V(\lambda)$ for

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$U_q(\mathfrak{g})$ and the crystal bases $B(\infty)$ of the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$, and in the crystal basis theory, one of the most important problems is to realize crystal bases explicitly using several combinatorial or geometric objects.

In [2], Jimbo et al. constructed the crystal bases for the irreducible highest weight $U_q(A_n^{(1)})$-modules using *colored diagrams*. Motivated by the fact that the colored diagrams can be parameterized by certain paths arising from the theory of solvable lattice models, in [4,5], Kang et al. developed the theory of *perfect crystals* for general quantum affine algebras $U_q(\mathfrak{g})$, and gave realizations of crystal bases for the irreducible highest weight modules over $U_q(\mathfrak{g})$ in terms of *paths*. In [3], from perfect crystal theory, Kang devised combinatorial models for classical quantum affine algebras $U_q(\mathfrak{g})$, called the *Young walls*, and constructed the crystal bases of basic representations over $U_q(\mathfrak{g})$. In [9], Kang and Lee extended the combinatorics of Young walls to the higher level cases. In [16], the authors modified Young walls over $U_q(A_n^{(1)})$, i.e., colored diagrams, which they called *generalized Young walls*, and showed that the set of all generalized Young walls is isomorphic to the crystal basis $B(\infty)$ for $U_q(A_n^{(1)})$. Thus it is very natural to extend the combinatorics of generalized Young walls to other quantum affine algebras.

On the other hand, in [12], Kashiwara introduced the embedding of crystal $\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow B(\mathbf{i})$, where $\mathbf{i}$ is some infinite sequence from the index set of simple roots. In [23], Zelevinsky and Nakashima characterized the image of the embedding $\Psi_{\mathbf{i}}$, which is called the *polyhedral realization*. Also, in [22], Nakashima gave a polyhedral realization of the highest weight crystal $B(\lambda)$, and in [24,25], their idea was extended to the quantum generalized Kac–Moody algebras by the second author. However, their descriptions of the images of the Kashiwara embeddings contain many redundant linear inequalities, and so it is not easy to determine whether the elements belong to the images of the Kashiwara embeddings or not. In [13,21], Kashiwara and Nakajima independently gave a crystal structure on the set of so-called *Nakajima monomials*, and they showed that the connected component containing a maximal vector with a dominant integral weight λ is isomorphic to the crystal $B(\lambda)$. In [8], Kang and the authors modified the notion of Nakajima monomials by adding a new variable $\mathbf{1}$, and they showed that the connected component $C(\mathbf{1})$ containing $\mathbf{1}$ is isomorphic to the crystal $B(\infty)$. Moreover, in [1], Nakajima monomial theory was extended to the quantum generalized Kac–Moody algebras by Jeong, Kang and the authors. Thus it is natural to try characterizing the connected components containing maximal vectors, and we can find many articles dealing with this characterization. For example, see [1,6–8,14,15]. However, the characterization for general quantum affine algebras is still unknown. Indeed, Kashiwara embeddings and Nakajima monomials are closely related. More precisely, in some cases, we know explicit crystal isomorphisms between images of Kashiwara embeddings and the connected components of Nakajima monomials containing maximal vectors. Thus more understanding of Kashiwara embeddings gives more understanding of Nakajima monomials, and vice versa.

Recently, in [18,19], while trying to extend the theory of generalized Young walls to other quantum affine algebras, the authors introduced new combinatorial models called *zigzag strip bundles* for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$, $U_q(D_{n+1}^{(2)})$, $U_q(C_n^{(1)})$, $U_q(A_{2n-1}^{(2)})$, and

$U_q(A_{2n}^{(2)})$, and showed that the set $\mathcal{S}(\infty)$ of all zigzag strip bundles is isomorphic to the crystal $B(\infty)$. Like generalized Young walls, the zigzag strip bundles consist of colored blocks with various shapes that are built on a given board. The rules and patterns for building zigzag strip bundles and the action of Kashiwara operators are given explicitly using combinatorics of zigzag strip bundles. Main difference between Young walls (or generalized Young walls) and zigzag strip bundles is that Young walls and generalized Young walls are based on perfect crystal theory, but zigzag strip bundles are based on the theories of Nakajima monomials and Kashiwara embeddings. Now, for a dominant integral weight $\lambda \in P^+$, let $R_\lambda = \{r_\lambda\}$ be the crystal with the maps defined by

$$\mathrm{wt}(r_\lambda) = \lambda, \qquad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda\rangle, \qquad \varphi_i(r_\lambda) = 0, \qquad \tilde{e}_i r_\lambda = \tilde{f}_i r_\lambda = 0 \quad \text{for all } i \in I.$$

Then the highest weight crystal $B(\lambda)$ lies in $B(\infty) \otimes R_\lambda$ as a connected component containing $u_\infty \otimes r_\lambda$, where u_∞ is the highest weight vector in $B(\infty)$. Thus it is a very natural and interesting problem to construct $B(\lambda)$ using zigzag strip bundles.

In this paper, we devote ourselves to the realizations of the crystal bases $B(\lambda)$ of the irreducible highest weight modules $V(\lambda)$ over $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$, $D_{n+1}^{(2)}$, $C_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$) in terms of zigzag strip bundles. That is, we give characterizations of the connected components $\mathcal{S}(\lambda)$ of $\mathcal{S}(\infty) \otimes R_\lambda$ containing $\emptyset \otimes r_\lambda$. Here $\emptyset$ is the highest weight vector in $\mathcal{S}(\infty)$. Indeed, even though Young walls are very good combinatorial objects to realize the crystals $B(\lambda)$, it is not easy to treat Young walls in the higher level case. One advantage of zigzag strip bundles is that compared with Young walls, it is more easy to treat in the higher level case. Another advantage of zigzag strip bundles is in the connection with Nakajima monomials and Kashiwara embeddings.

We expect that we can extend the theory of zigzag strip bundles of highest weight type to other quantum affine algebras. Also, we believe that the combinatorics of zigzag strip bundles can be applied to the construction of highest weight crystals $B(\lambda)$ for classical finite simple Lie algebras. Recently, Lee, Lombardo and Salisbury gave a combinatorial description of the Casselman–Shalika formula in type A using Young tableaux [20]. We think that the highest weight zigzag strip bundles for $U_q(\mathfrak{g})$ can be used to give a combinatorial description of the affine Casselman–Shalika formula.

1. Kashiwara embeddings and Nakajima monomials

1.1. Crystals

Let $I = \{0, 1, \ldots, n\}$ be an index set and let $A = (a_{ij})_{i,j \in I}$ be a *Cartan matrix* of affine type. Let $P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$ be the *dual weight lattice*, and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ be the *Cartan subalgebra*. The *affine weight lattice* is defined to be $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$, and the *simple roots* α_i and the *fundamental weights* Λ_i ($i \in I$) are defined by

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d) = \delta_{0,i}, \qquad \Lambda_i(h_j) = \delta_{i,j}, \qquad \Lambda_i(d) = 0 \quad (i, j \in I).$$

We denote by $\Pi = \{\alpha_i \mid i \in I\}$ and $\Pi^\vee = \{h_i \mid i \in I\}$. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called an *affine Cartan datum*. We denote by $U_q(\mathfrak{g})$ the quantum affine algebra associated with the Cartan datum. We also denote by $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ the subalgebras of $U_q(\mathfrak{g})$ generated by the e_i's and the f_i's, respectively. A $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps $\mathrm{wt}: B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ $(i \in I)$ such that for all $i \in I$ and $b \in B$,

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(ii) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq 0$, and $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq 0$,
(iii) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \neq 0$,
(iv) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \neq 0$,
(v) $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $b, b' \in B$,
(vi) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

For instance, the crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ with $\lambda(P^\vee) \subset \mathbb{Z}_{\geq 0}$ and the crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$ are $U_q(\mathfrak{g})$-crystals. Also, for each $i \in I$, $B_i = \{b_i(n) \mid n \in \mathbb{Z}\}$ is a crystal with the maps defined by $\mathrm{wt}(b_i(n)) = n\alpha_i$, and

$$\varepsilon_i\big(b_i(n)\big) = -n, \qquad \varphi_i\big(b_i(n)\big) = n, \qquad \tilde{e}_i b_i(n) = b_i(n+1), \qquad \tilde{f}_i b_i(n) = b_i(n-1),$$
$$\varepsilon_j\big(b_i(n)\big) = \varphi_j\big(b_i(n)\big) = -\infty, \qquad \tilde{e}_j b_i(n) = \tilde{f}_j b_i(n) = 0 \quad \text{if } j \neq i.$$

The crystal B_i is called an *elementary crystal*. Moreover, for given crystals, we construct another crystals using tensor product. More precisely, let B and B' be crystals. Then their tensor product $B \otimes B' = \{b \otimes b' \mid b \in B,\ b' \in B'\}$ is a crystal with the maps $\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ given by $\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$, and

$$\varepsilon_i\big(b \otimes b'\big) = \max\big\{\varepsilon_i(b), \varepsilon_i\big(b'\big) - \big\langle h_i, \mathrm{wt}(b)\big\rangle\big\},$$
$$\varphi_i\big(b \otimes b'\big) = \max\big\{\varphi_i\big(b'\big), \varphi_i(b) + \big\langle h_i, \mathrm{wt}\big(b'\big)\big\rangle\big\},$$
$$\tilde{e}_i\big(b \otimes b'\big) = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geqslant \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \qquad \tilde{f}_i\big(b \otimes b'\big) = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leqslant \varepsilon_i(b'). \end{cases}$$

1.2. Kashiwara embeddings

Let $\mathbf{i} = (i_1, i_2, \ldots)$ be an infinite sequence of indices in I such that every $i \in I$ appears infinitely many times, and let

$$B(\mathbf{i}) = \big\{\big(b_{i_k}(-x_k)\big)_{k=1}^{\infty} \in \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_1} \bigm| x_k \in \mathbb{Z}_{\geq 0},\ x_k = 0 \text{ for all } k \gg 0\big\}.$$

For $\mathbf{b} = \cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in B(\mathbf{i})$ and $k \geq 1$, we set

$$\sigma_k(\mathbf{b}) = x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j, \quad \text{and} \quad \sigma^{(i)}(\mathbf{b}) = \max\big\{\sigma_k(\mathbf{b}) \bigm| i_k = i\big\}.$$

Let $m_f = \min\{k \mid i_k = i,\ \sigma_k(\mathbf{b}) = \sigma^{(i)}(\mathbf{b})\}$, and $m_e = \max\{k \mid i_k = i,\ \sigma_k(\mathbf{b}) = \sigma^{(i)}(\mathbf{b})\}$. Now, we define

$$\tilde{f}_i\mathbf{b} = \cdots \otimes \tilde{f}_i b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1),$$

where $k = m_f$, and

$$\tilde{e}_i\mathbf{b} = \begin{cases} \cdots \otimes \tilde{e}_i b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) & \text{if } \sigma^{(i)}(\mathbf{b}) > 0 \text{ and } k = m_e, \\ 0 & \text{otherwise.} \end{cases}$$

We also define

$$\mathrm{wt}(\mathbf{b}) = -\sum_{j=1}^{\infty} x_j \alpha_{i_j}, \qquad \varepsilon_i(\mathbf{b}) = \sigma^{(i)}(\mathbf{b}), \qquad \varphi_i(\mathbf{b}) = \langle h_i, \mathrm{wt}(\mathbf{b})\rangle + \varepsilon_i(\mathbf{b}).$$

Then $B(\mathbf{i})$ is a $U_q(\mathfrak{g})$-crystal. The following is the Kashiwara embedding theorem.

Theorem 1.1. *(See [12,23].) Let* $\mathbf{i}$ *be an infinite sequence such that every* $i \in I$ *appears infinitely many times. Then there exists a unique strict embedding of crystals*

$$\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow B(\mathbf{i}) \quad \textit{such that} \quad u_\infty \mapsto \cdots b_{i_k}(0) \otimes \cdots \otimes b_{i_1}(0),$$

where u_∞ *is the highest weight vector in* $B(\infty)$.

1.3. Nakajima monomials

Let $\mathcal{M}$ be the set of all monomials of commuting variables $Y_i(n)$ $(i \in I, n \in \mathbb{Z})$ and $\mathbf{1}$ of the form $M = \mathbf{1} \cdot \prod_{i\in I, n\geq 0} Y_i(n)^{y_i(n)}$ such that $y_i(n) \in \mathbb{Z}$ and $y_i(n) = 0$ for all but finitely many n's. The monomials in $\mathcal{M}$ are called the *Nakajima monomials*. For a Nakajima monomial $M \in \mathcal{M}$, we define $\mathrm{wt}(M) = \sum_{i\in I}(\sum_{n\geq 0} y_i(n))\Lambda_i$, and

$$\varphi_i(M) = \max\left\{ \sum_{0\leq k\leq n} y_i(k) \;\middle|\; n \geq 0 \right\}, \qquad \varepsilon_i(M) = \varphi_i(M) - \langle h_i, \mathrm{wt}(M)\rangle.$$

Choose a set $C = (c_{ij})_{i\neq j}$ of nonnegative integers such that $c_{ij} + c_{ji} = 1$, and for each $i \in I$, $n \in \mathbb{Z}_{\geq 0}$, define

$$A_i(n) = Y_i(n)Y_i(n+1)\prod_{j\neq i} Y_j(n + c_{ji})^{a_{ji}}.$$

Set

$$n_f = \min\Big\{n \geq 0 \;\Big|\; \varphi_i(M) = \sum_{0\leq k\leq n} y_i(k)\Big\},$$
$$n_e = \max\Big\{n \geq 0 \;\Big|\; \varphi_i(M) = \sum_{0\leq k\leq n} y_i(k)\Big\},$$

and we define

$$\tilde{f}_i M = A_i(n_f)^{-1} M, \qquad \tilde{e}_i M = \begin{cases} 0 & \text{if } \varepsilon_i(M) = 0, \\ A_i(n_e) M & \text{if } \varepsilon_i(M) > 0. \end{cases}$$

Then $\mathcal{M}$ becomes a $U_q(\mathfrak{g})$-crystal. Moreover, we have

Theorem 1.2. *(See [13].) The connected component $C(\mathbf{1})$ of $\mathcal{M}$ containing $\mathbf{1}$ is isomorphic to the $U_q(\mathfrak{g})$-crystal $B(\infty)$.*

Now, we denote by $\mathcal{M}(\infty)$ the connected component $C(\mathbf{1})$ containing $\mathbf{1}$ in $\mathcal{M}$ when we choose the set $C = (c_{ij})_{i\neq j}$ of as follows.

$$c_{ij} = 0 \quad \text{if } i > j, \quad \text{and} \quad c_{ij} = 1 \quad \text{if } i < j.$$

Let $\mathbf{i} = (0, 1\ldots, n, 0, 1, \ldots, n, 0, \ldots)$, and let $\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow B(\mathbf{i})$ be the crystal embedding given in Theorem 1.1. For a monomial

$$M = \mathbf{1} \cdot \prod_{i\in I, k\geq 0} A_i(k)^{-a_i(k)} \in \mathcal{M}(\infty),$$

we now define a map $\phi : \mathcal{M}(\infty) \to \operatorname{Im}\Psi_{\mathbf{i}}$ by

$$\begin{aligned}\phi(M) = \cdots \otimes b_0\big(-a_0(2)\big) \otimes b_n\big(-a_n(1)\big) \otimes \cdots \otimes b_1\big(-a_1(1)\big) \otimes b_0\big(-a_0(1)\big) \\ \otimes b_n\big(-a_n(0)\big) \otimes \cdots \otimes b_1\big(-a_1(0)\big) \otimes b_0\big(-a_0(0)\big).\end{aligned}$$

Then we have

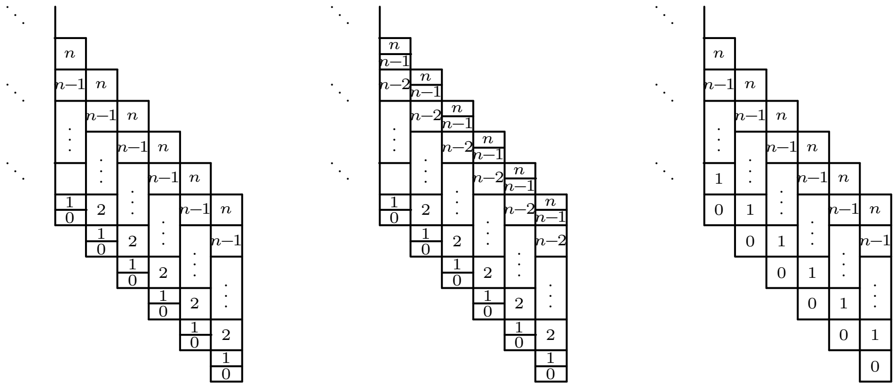
Theorem 1.3. *(See [15].) The map $\phi : \mathcal{M}(\infty) \to \operatorname{Im}\Psi_{\mathbf{i}}$ is a $U_q(\mathfrak{g})$-crystal isomorphism.*

2. Zigzag strip bundles

In [18,19], from the theories of Nakajima monomials and crystal embeddings, the authors introduced the zigzag strip bundles for the quantum affine Lie algebras $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$, and showed that the set of all zigzag strip bundles for $U_q(\mathfrak{g})$ realizes the crystal $B(\infty)$ over $U_q(\mathfrak{g})$. In this section, we give new exposition of zigzag strip bundles. The difference between this new exposition and old one is that the board $\mathcal{B}$ in new exposition is just the subboard $\overline{\mathcal{B}}$ in old exposition. That is, the board $\mathcal{B}$ in old exposition is obtained by adding up the board $\mathcal{B}$ in new exposition on the top of each $\mathcal{B}$ repeatedly. Finally, zigzag strip bundles in the new exposition are obtained from those in old version by moving zigzag strips in the subboard $\mathcal{B}_t$ $(t \geq 2)$ to $\mathcal{B}_1$.

2.1. Boards and blocks for zigzag strip bundles

For each quantum affine algebra $U_q(\mathfrak{g})$, we fix boards $\mathcal{B}$ with coloring as follows:

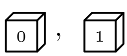


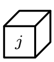
$\mathfrak{g} = B_n^{(1)}, A_{2n-1}^{(2)}$ $\qquad$ $\mathfrak{g} = D_n^{(1)}$ $\qquad$ $\mathfrak{g} = D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$

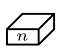
We see that the board $\mathcal{B}$ consists of squares with colorings, which we call *sites* of $\mathcal{B}$, and we regard the board $\mathcal{B}$ as the set $\mathbb{N} \times \mathbb{N}$ by identifying the jth site from the bottom of the ith column from the right with $(i, j) \in \mathbb{N} \times \mathbb{N}$.

The zigzag strip bundles for $U_q(\mathfrak{g})$ are built of colored blocks of three different shapes on the board $\mathcal{B}$ according to the colors of sites of $\mathcal{B}$:

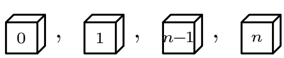
(i) $\mathfrak{g} = B_n^{(1)}$;

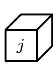
 : unit width, unit height, half-unit thickness,

 $(j = 2, \ldots, n-1)$: unit width, unit height, unit thickness,

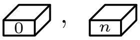
 : unit width, half-unit height, unit thickness.

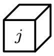
(ii) $\mathfrak{g} = D_n^{(1)}$;

 : unit width, unit height, half-unit thickness,

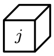
 $(j = 2, \ldots, n-2)$: unit width, unit height, unit thickness.

(iii) $\mathfrak{g} = D_{n+1}^{(2)}$;

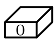
 : unit width, half-unit height, unit thickness,

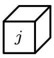
 $(j = 1, \ldots, n-1)$: unit width, unit height, unit thickness.

(iv) $\mathfrak{g} = C_n^{(1)}$;

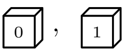
 $(j = 0, 1, \ldots, n)$: unit width, unit height, unit thickness

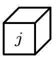
(v) $\mathfrak{g} = A_{2n}^{(2)}$;

 : unit width, half-unit height, unit thickness

 $(j = 1, \ldots, n)$: unit width, unit height, unit thickness

(vi) $\mathfrak{g} = A_{2n-1}^{(2)}$;

 : unit width, unit height, half-unit thickness

 $(j = 2, \ldots, n)$: unit width, unit height, unit thickness

With these colored blocks, we will pile up them on the board $\mathcal{B}$ according to some rules. For convenience, we will use the notation $\boxed{j^k}$ for the stack of k j-colored blocks $(j = 0, 1, \ldots, n)$. Also, we say an i-colored site (resp. a site with coloring 0 and 1, an i-colored block) an i-site (resp. a (0/1)-site, an i-block).

2.2. Zigzag strips

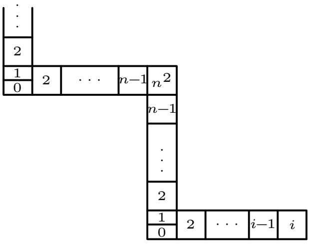
Let J be the subset of I as follows.

(i) $\mathfrak{g} = B_n^{(1)}, A_{2n-1}^{(2)}$: $I - \{0, 1, n\}$
(ii) $\mathfrak{g} = D_n^{(1)}$: $I - \{0, 1, n-1, n\}$
(iii) $\mathfrak{g} = D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$: $I - \{0, n\}$

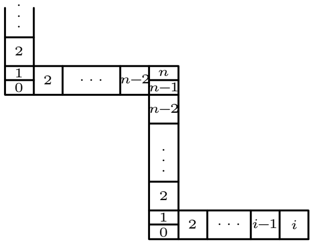
Definition 2.1. (a) We define a *zigzag* 0 (resp. 1)-*strip* for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(A_{2n-1}^{(2)})$ by the pile $\boxed{0}$ (resp. $\boxed{1}$) of only one 0-block (resp. 1-block) stacked on the (0/1)-site of the rightmost column of $\mathcal{B}$. Also, we define a *zigzag* 0-*strip* for $U_q(D_{n+1}^{(2)})$, $U_q(C_n^{(1)})$ or $U_q(A_{2n}^{(2)})$ by the pile consisting of only one 0-block stacked on the 0-site of the rightmost column of $\mathcal{B}$.

(b) For each $i \in J$, we define a *zigzag i-strip* for $U_q(\mathfrak{g})$ by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

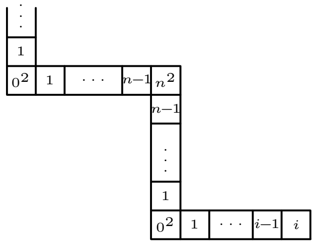
(i) It is obtained by stacking colored blocks starting from the i-site of the first column from the right in the pattern given below.



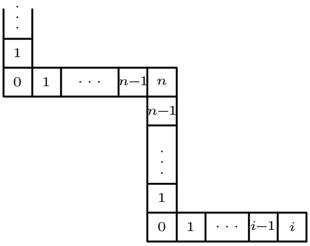
$\mathfrak{g} = B_n^{(1)}$



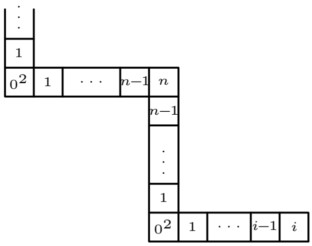
$\mathfrak{g} = D_n^{(1)}$



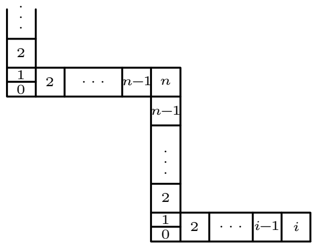
$\mathfrak{g} = D_{n+1}^{(2)}$



$\mathfrak{g} = C_n^{(1)}$



$\mathfrak{g} = A_{2n}^{(2)}$

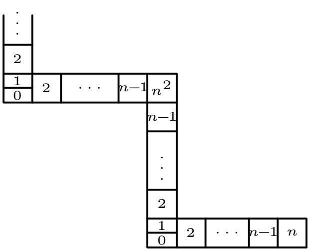


$\mathfrak{g} = A_{2n-1}^{(2)}$

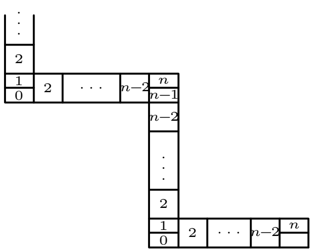
(ii) The volume, the height, and the thickness of stacked blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

(c) For each $k \in \mathbb{Z}_{>0}$, we define a kth *zigzag n-strip* for $U_q(\mathfrak{g})$ by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

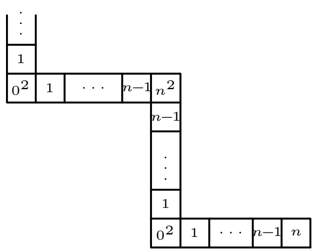
(i) It is obtained by stacking colored blocks starting from the n-site of the kth column from the right in the pattern given below.



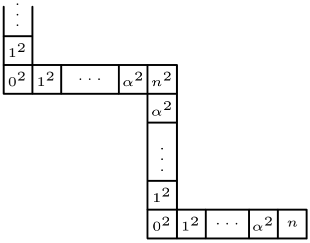
$\mathfrak{g} = B_n^{(1)}$



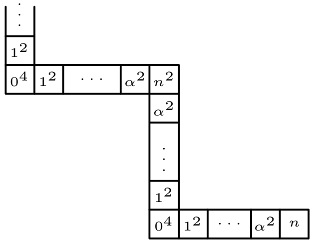
$\mathfrak{g} = D_n^{(1)}$



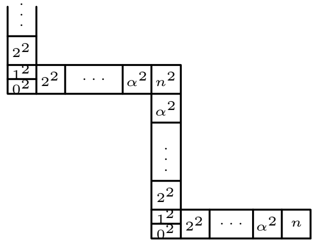
$\mathfrak{g} = D_{n+1}^{(2)}$



$\mathfrak{g} = C_n^{(1)}$



$\mathfrak{g} = A_{2n}^{(2)}$



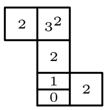
$\mathfrak{g} = A_{2n-1}^{(2)}$

Here, $\alpha = n - 1$.

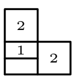
(ii) Except for the rightmost site of the pile, the volume, the height, and the thickness of blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

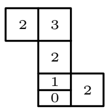
Also, a kth *zigzag* $(n-1)$-*strip* for $U_q(D_n^{(1)})$ is defined in a similar way. The only difference with a kth n-strip is the starting block. That is, the starting block of a kth zigzag $(n-1)$-strip is an $(n-1)$-block.

Example 2.2. (a) The following pile

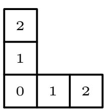


is a zigzag 2-strip for $U_q(B_3^{(1)})$. On the other hand, the following piles are not zigzag 2-strip for $U_q(B_3^{(1)})$.





(b) The following pile



is not a zigzag 2-strip for $U_q(A_4^{(2)})$ because in the leftmost column, the volume of the 1-blocks is bigger than that of the 0-blocks.

2.3. Zigzag strip bundles

Let J_1 be the subset of I as follows.

(i) $\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$: $I - \{n\}$ (ii) $\mathfrak{g} = D_n^{(1)}$: $I - \{n-1, n\}$

For a pile S over $U_q(\mathfrak{g})$ of stacked blocks on the board $\mathcal{B}$ which is decomposed into zigzag strips, we can arrange them in such a way that for each $i \in J_1$,

$$S_i^1 \supset \cdots \supset S_i^{t_i} \neq \emptyset, \tag{2.1}$$

and for each $k \geq 1$,

$$S_{n,k}^1 \supset \cdots \supset S_{n,k}^{\alpha_k} \neq \emptyset, \quad S_{n-1,k}^1 \supset \cdots \supset S_{n-1,k}^{\beta_k} \neq \emptyset. \tag{2.2}$$

Here, S_i^p $(p = 1, \ldots, t_i)$, $S_{n,k}^r$ $(r = 1, \ldots, \alpha_k)$ and $S_{n-1,k}^s$ $(s = 1, \ldots, \beta_k)$ are zigzag i-strips, the kth zigzag n-strips, and the kth zigzag $(n-1)$-strips, respectively. Of course, the kth zigzag $(n-1)$-strips exist only in the type of $U_q(D_n^{(1)})$. Also, for each $k \geq 1$, $r = 1, \ldots, \alpha_k$, and $s = 1, \ldots, \beta_k$, let $s_{n,k}^r$ and $s_{n-1,k}^s$ be the numbers of the blocks in $S_{n,k}^r$ and $S_{n-1,k}^s$, respectively.

Definition 2.3. A pile S of stacked blocks on the board $\mathcal{B}$ is called a *zigzag strip bundle* for $U_q(\mathfrak{g})$ $(\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)})$ if there is a decomposition of S into zigzag strips satisfying the following conditions: According to (2.1) and (2.2),

(i) for each $k \geq 1$,

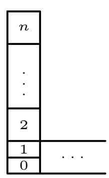
$$\begin{aligned} &\alpha_k \geq \alpha_{k+1} && && \text{if } \mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}, \\ &\alpha_k \geq \beta_{k+1}, && \beta_k \geq \alpha_{k+1} && \text{if } \mathfrak{g} = D_n^{(1)}, \end{aligned}$$

(ii) in (2.1) and (2.2), equalities hold only if zigzag strips consist of blocks of unit height.

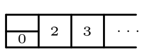
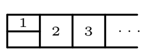
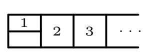
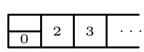
(iii) for each $k \geq 1$, $r = 1, \ldots, \alpha_{k+1}$, and $s = 1, \ldots, \beta_{k+1}$,

$$\begin{aligned} &s_{n,k}^r \geq s_{n,k+1}^r && && \text{if } \mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}, \\ &s_{n,k}^r \geq s_{n-1,k+1}^r, && s_{n-1,k}^s \geq s_{n,k+1}^s && \text{if } \mathfrak{g} = D_n^{(1)}. \end{aligned} \tag{2.3}$$

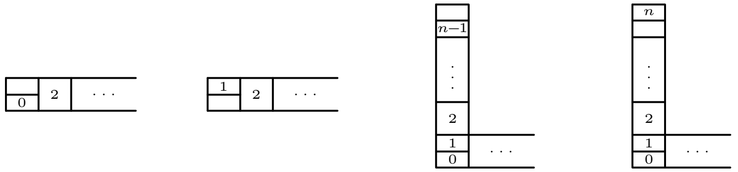
Here, equalities hold if and only if
(iii-a) when $\mathfrak{g} = B_n^{(1)}$,

$S_{n,k}^r = S_{n,k+1}^r =$  ,

or

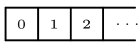
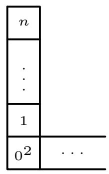
$(S_{n,k}^r, S_{n,k+1}^r) = ($  ,  $)$ or $($  ,  $)$.

(iii-b) when $\mathfrak{g} = D_n^{(1)}$, $S^s_{n-1,k}$ and $S^s_{n,k+1}$ (resp. $S^r_{n,k}$ and $S^r_{n-1,k+1}$) are one of the following forms, and the last blocks, the blocks stacked on top of the leftmost column, in $S^s_{n-1,k}$ and $S^s_{n,k+1}$ (resp. $S^r_{n,k}$ and $S^r_{n-1,k+1}$) are different:

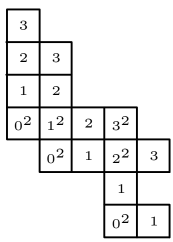


Here, in right two zigzag strips, the numbers of blocks are bigger than 1,

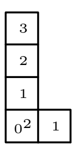
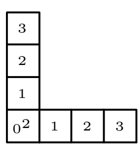
(iii-c) when $\mathfrak{g} = D_{n+1}^{(2)}$,

$S^r_{n,k} = S^r_{n,k+1} =$  or 

Example 2.4. Let

$S =$ 

be a pile for $U_q(D_4^{(2)})$. Then it is decomposed into the following zigzag strips:

$S^1_1 =$  $\quad S^1_{3,1} = S^1_{3,2} =$ 

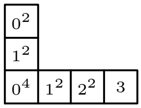
Therefore, S is a zigzag strip bundle for $U_q(D_4^{(2)})$.

Now, consider the $C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$-types.

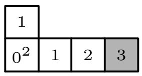
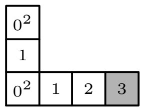
Definition 2.5. Let S be a kth zigzag n-strip for $U_q(\mathfrak{g})$, where $\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$. Now cut the n-block of rightmost site in S in half with a saw. Then S is divided into two strips, upper strip S^- and lower strip S_-, which are the strips consisting of the half n-block and the blocks of S stacked on slots of height > 1, and consisting of the half

n-block and the blocks of S stacked on slots of height < 1, respectively. We say these two strips a kth *upper half n-strip* and a kth *lower half n-strip* of S, respectively. Also, they are simply called kth half n-strips.

Example 2.6. Let

$S =$ 

be a 3-strip for $U_q(A_6^{(2)})$. Then the upper half strip S^- and lower half strip S_- of S are as follows.

$S^- =$  $S_- =$ 

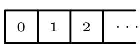
Here, the shaded parts mean the half 3-blocks.

Definition 2.7. A pile S of stacked blocks on the board $\mathcal{B}$ is called a *zigzag strip bundle* for $U_q(\mathfrak{g})$ ($\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$) if there is a decomposition of S into zigzag strips satisfying the following conditions: According to (2.1) and (2.2),

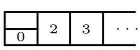
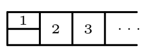
(i) for each $k \geq 1$, $\alpha_k \geq \alpha_{k+1}$,
(ii) for each $k \geq 1$, $r = 1, \ldots, \alpha_{k+1}$, if we let $(s^r_{n,k})^-$ and $(s^r_{n,k})_-$ be the number of blocks in the upper half n-strip $(S^r_{n,k})^-$ and the lower half n-strip $(S^r_{n,k})_-$ of $S^r_{n,k}$, respectively, then

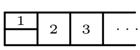
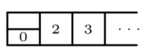
$$\left(s^r_{n,k}\right)^- \geq \left(s^r_{n,k+1}\right)^- \quad \text{and} \quad \left(s^r_{n,k}\right)_- \geq \left(s^r_{n,k+1}\right)_-. \tag{2.4}$$

Here, the equality $(s^r_{n,k})^- = (s^r_{n,k+1})^-$ holds if and only if

$\left(S^r_{n,k}\right)^- = \left(S^r_{n,k+1}\right)^- =$  for $\mathfrak{g} = A_{2n}^{(2)}$,

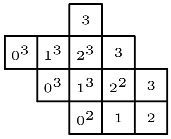
or

$\left(\left(S^r_{n,k}\right)^-, \left(S^r_{n,k+1}\right)^-\right) = \Big($  ,  $\Big)$,

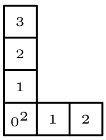
$\Big($  ,  $\Big)$ for $\mathfrak{g} = A_{2n-1}^{(2)}$.

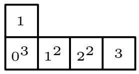
Similarly, the equality $(s^r_{n,k})_- = (s^r_{n,k+1})_-$ also holds under the same conditions.

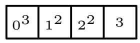
Example 2.8. Let

$S =$ 

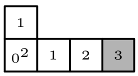
be a pile for $U_q(A_6^{(2)})$. Then it is decomposed into the following zigzag strips:

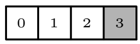
$S_2^1 =$ 

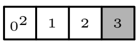
$S_{3,1}^1 =$ 

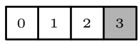
$S_{3,2}^1 =$ 

Moreover,

$(S_{3,1}^1)_- =$ 

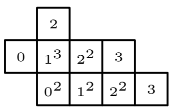
$(S_{3,1}^1)^- =$ 

$(S_{3,2}^1)_- =$ 

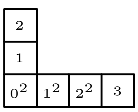
$(S_{3,2}^1)^- =$ 

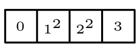
Therefore, S is a zigzag strip bundle for $U_q(A_6^{(2)})$.

On the other hand, the following $U_q(A_6^{(2)})$-pile

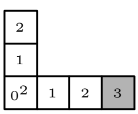
$S =$ 

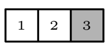
is decomposed as follows.

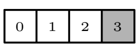
$S_{3,1}^1 =$ 

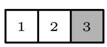
$S_{3,2}^1 =$ 

Also, we have

$(S_{3,1}^1)_- =$ 

$(S_{3,1}^1)^- =$ 

$(S_{3,2}^1)_- =$ 

$(S_{3,2}^1)^- =$ 

Thus S is not a zigzag strip bundle for $U_q(A_6^{(2)})$.

2.4. Crystal structure on the set of zigzag strip bundles

In this section, we give a crystal structure on the set of all zigzag strip bundles for $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$.

Definition 2.9. Let S be a zigzag strip bundle for $U_q(\mathfrak{g})$, and let S be decomposed into zigzag strips; $S = \{S_j\}_{j=1}^N$. A site P of the board $\mathcal{B}$ is said to be *k-times i-removable* and *l-times i-admissible* with respect to $\{S_j\}_{j=1}^N$ if one can have another zigzag strip bundles in $\mathcal{B}$ from $\{S_j\}_{j=1}^N$ by removing k i-blocks on P, and stacking l i-blocks on P, respectively.

Fix $i \in I$ and let a given zigzag strip bundle S be decomposed into zigzag strips; $S = \{S_j\}_{j=1}^N$. Let $P_1, P_2, \ldots$ be the all i-removable or i-admissible sites with respect to $\{S_j\}_{j=1}^N$ from west to east and from south to north. To each site P_α ($\alpha \geq 1$), we assign its i-signature $sgn_i(P_\alpha)$ as

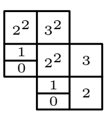
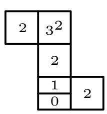
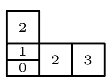
$$\underbrace{- - - \cdots -}_{p\text{-times}}\underbrace{+ + + \cdots +}_{q\text{-times}}$$

if P_α is p-times i-removable and q-times i-admissible. From the sequence $(sgn_i(P_1), sgn_i(P_2), \ldots)$ of $+$'s and $-$'s, cancel out every $(+,-)$-pair to obtain a sequence of $-$'s followed by $+$'s, reading from left to right. This sequence is called the *i-signature* of S with respect to $\{S_j\}_{j=1}^N$. Moreover, an i-signature of S is independent of decompositions into zigzag strips of S (See [18]). Thus, we may say the i-signature of S for a zigzag strip bundle S. Let $\mathcal{S}(\infty)$ be the set of all zigzag strip bundles, and let $S \in \mathcal{S}(\infty)$ be a zigzag strip bundle. We define $\tilde{f}_i S$ to be the zigzag strip bundle obtained from S by stacking an i-block on the i-admissible site corresponding to the leftmost $+$ in the i-signature of S. We define $\tilde{e}_i S$ to be the zigzag strip bundle obtained from S by eliminating the i-block corresponding to the rightmost $-$ in the i-signature of S. If there is no $-$ in the i-signature of S, we define $\tilde{e}_i S = 0$. We also define $\mathrm{wt}(S) = -\sum_{i \in I} k_i \alpha_i$, where k_i is the number of i-blocks in S, and

$$\varepsilon_i(S) = \text{the number of } -\text{'s in the } i\text{-signature of } S,$$
$$\varphi_i(S) = \varepsilon_i(S) + \langle h_i, \mathrm{wt}(S) \rangle.$$

Then $(\mathcal{S}(\infty), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is an affine crystal.

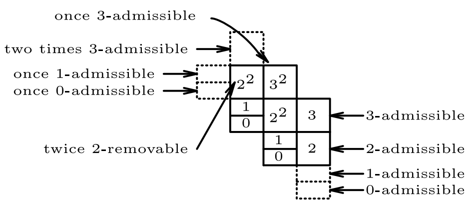
Example 2.10. Let

$S =$  $= S_2^1 \cup S_{3,1}^1 =$  $\cup$ 

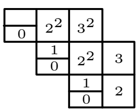
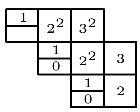
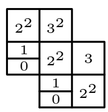
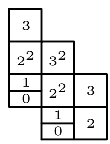
be a zigzag strip bundle lying on $\mathcal{B}_1$ for $U_q(B_3^{(1)})$. Then it is easy to see that

(i) the sites $(4,1)$, and $(3,2)$ are once 0-admissible and once 1-admissible, and once 2-removable from the strip S_2^1, respectively,
(ii) the sites $(3,3)$, and $(3,2)$ are twice 3-admissible, and once 2-removable from the strip $S_{3,1}^1$, respectively,
(iii) the site $(2,3)$ is once 3-admissible because if we stack a 3-block on $(2,3)$, then we have $S_{3,2}^1 = \boxed{3}$, and $S_{3,1}^1$ and $S_{3,2}^1$ satisfy the condition (iii-a) of Definition 2.3,
(iv) every site in the rightmost column of $\mathcal{B}$ is admissible.

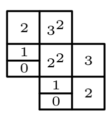
The following describes all the removable and admissible sites with respect to the above zigzag strips $\{S_2^1, S_{3,1}^1, S_{3,2}^1\}$.



Therefore, we have

$\tilde{f}_0 S =$  , $\quad \tilde{f}_1 S =$  , $\quad \tilde{f}_2 S =$  , $\quad \tilde{f}_3 S =$  ,

and

$\tilde{e}_0 S = \tilde{e}_1 S = \tilde{e}_3 S = 0, \qquad \tilde{e}_2 S =$  .

2.5. Realization of the crystal $B(\infty)$

Let $\mathcal{S}(\infty)$ be the set of all zigzag strip bundles for $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}$, $A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$). Let $\mathcal{M}(\infty)$ be the connected component containing $\mathbf{1}$ given in Section 1.3. For a zigzag strip bundle S, we now define a map $\varphi : \mathcal{S}(\infty) \to \mathcal{M}(\infty)$ by

$$\varphi(S) = \mathbf{1} \cdot \prod_{i \in I, k \geq 0} A_i(k)^{-a_i(k)},$$

where $a_i(k)$ is the number of i-blocks of S stacked on the $(k+1)$st column of $\mathcal{B}$. Then we have

Theorem 2.11. *(See [18,19].) For* $\mathfrak{g} = B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$ *or* $A_{2n-1}^{(2)}$*, the map* $\varphi : \mathcal{S}(\infty) \xrightarrow{\sim} \mathcal{M}(\infty)$ *is a* $U_q(\mathfrak{g})$*-crystal isomorphism sending* $\emptyset$ *to* $\mathbf{1}$*.*

Proof. For $S \in \mathcal{S}(\infty)$, in both new and old expositions, the numbers of i-blocks of S stacked on the $(k+1)$st column of $\mathcal{B}$ are the same. Thus it is clear. □

Let $\mathbf{i} = (0, 1, \ldots, n, 0, 1, \ldots, n, 0, \ldots)$ be an infinite sequence of indices in I. Let $\Psi_i : B(\infty) \hookrightarrow B(\mathbf{i})$ be the crystal embedding given in Theorem 1.1, and let $\phi : \mathcal{M}(\infty) \to \operatorname{Im} \Psi_i$ be the crystal isomorphism given in Theorem 1.3.

Corollary 2.12. *(See [18].) For* $\mathfrak{g} = B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$ *or* $A_{2n-1}^{(2)}$*, there is a* $U_q(\mathfrak{g})$*-crystal isomorphism*

$$\psi : \mathcal{S}(\infty) \xrightarrow{\varphi} \mathcal{M}(\infty) \xrightarrow{\phi} \operatorname{Im} \Psi_i \cong B(\infty)$$

sending $\emptyset$ *to* $\cdots \otimes b_0(0) \otimes b_n(0) \otimes \cdots \otimes b_1(0) \otimes b_0(0)$*.*

3. Realization of highest weight crystals

In this section, we give new realizations of the highest weight crystals $B(\lambda)$ ($\lambda \in P^+$) for $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$. More precisely, let $\mathcal{S}(\infty)$ be the set of all strip bundles for $U_q(\mathfrak{g})$, and for $\lambda \in P^+$, let $R_\lambda = \{r_\lambda\}$ be the crystal with the maps defined by

$$\operatorname{wt}(r_\lambda) = \lambda, \qquad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \qquad \varphi_i(r_\lambda) = 0, \qquad \tilde{e}_i r_\lambda = \tilde{f}_i r_\lambda = 0 \quad \text{for all } i \in I.$$

Then the connected component $C(\emptyset \otimes r_\lambda)$ of $\mathcal{S}(\infty) \otimes R_\lambda$ containing $\emptyset \otimes r_\lambda$ is a realization of the highest weight crystal $B(\lambda)$ over $U_q(\mathfrak{g})$. From now on, we devote ourselves to the characterization of $C(\emptyset \otimes r_\lambda)$.

3.1. Highest weight crystal structure on strip bundles

Let $\mathcal{S}(\lambda)$ be the subset of $\mathcal{S}(\infty)$ such that

$$C(\emptyset \otimes r_\lambda) = \mathcal{S}(\lambda) \otimes R_\lambda.$$

In order to give a characterization of $\mathcal{S}(\lambda)$, we give a new crystal structure on $\mathcal{S}(\lambda)$ which is different from the crystal structure as the subset of $\mathcal{S}(\infty)$. Since the methods are very similar, we only treat the $B_n^{(1)}$-type. Let S be a strip bundle of $\mathcal{S}(\lambda)$, and for each $i \in I$ let $s_i(1)$ be the number of i-blocks of S on the rightmost column. Let

$$\xi_i = \begin{cases} \langle h_i, \lambda\rangle - s_i(1) & \text{for } i = 0 \text{ or } 1, \\ \langle h_2, \lambda\rangle - s_2(1) + s_0(1) + s_1(1) & i = 2, \\ \langle h_n, \lambda\rangle - s_n(1) + 2s_{n-1}(1) & i = n, \\ \langle h_i, \lambda\rangle - s_i(1) + s_{i-1}(1) & \text{otherwise.} \end{cases} \tag{3.1}$$

Then the i-site of the rightmost column of $\mathcal{B}$ is ξ_i-times i-admissible. Also, the admissibility of the sites of other columns and the removability of the sites of any column are determined by the same way in the $\mathcal{S}(\infty)$. Moreover, we define the map $\mathrm{wt} : \mathcal{S}(\lambda) \to P$ by

$$\mathrm{wt}(S) = \lambda - \sum_{i\in I} k_i\alpha_i,$$

where k_i is the number of i-colored blocks in S. Finally, $\varepsilon_i, \varphi_i, \tilde{e}_i$ and $\tilde{f}_i$ are defined by the same way in the $\mathcal{S}(\infty)$. Then it is easy to see that $(\mathcal{S}(\lambda), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is a $U_q(\mathfrak{g})$-crystal. Moreover, we have

Theorem 3.1. *Let* $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ *or* $A_{2n-1}^{(2)}$. *The map* $\phi : C(\emptyset \otimes r_\lambda) \to \mathcal{S}(\lambda)$ *given by* $\phi(S \otimes r_\lambda) = S$ *is a* $U_q(\mathfrak{g})$*-crystal isomorphism.*

Proof. Since the argument is the same, we focus on the $B_n^{(1)}$-type. Now, it is tedious but is not difficult to see that as an element S of $\mathcal{S}(\infty)$, $\varphi_i(S)$ is determined by the number of $+$'s in the i-signature of S as follows.

$$\varphi_i(S) = \begin{cases} -s_i(1) + \#_{\geq 2}(S,+) & \text{for } i = 0 \text{ or } 1, \\ s_0(1) + s_1(1) - s_2(1) + \#_{\geq 2}(S,+) & i = 2, \\ 2s_{n-1}(1) - s_n(1) + \#_{\geq 2}(S,+) & i = n, \\ s_{i-1}(1) - s_i(1) + \#_{\geq 2}(S,+) & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\#_{\geq 2}(S,+)$ is the number of $+$'s corresponding to the i-admissible site of the kth ($k \geq 2$) column in the i-signature of S. Indeed, for a zigzag j-strip S having an i-removable site on only kth ($k \geq 2$) column,

$$\varepsilon_i(S) = \begin{cases} 2 & \text{if } i = n \text{ and } S \text{ has a twice removable } i\text{-site,} \\ 1 & \text{otherwise,} \end{cases}$$

and $\varphi_i(S)$ is given below.

(r-1) $j = 0$ or 1;

$$\varphi_i(S) = \begin{cases} -1 & \text{if } i = 0 \text{ or } 1, \\ 1 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(r-2) $j = 2, \ldots, n-2$;

$$\varphi_i(S) = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

(r-3) $j = n-1$;

$$\varphi_i(S) = \begin{cases} -1 & \text{if } i = n-1, \\ 3 & \text{if } i = n \text{ and once } i\text{-removable}, \\ 2 & \text{if } i = n \text{ and twice } i\text{-removable}, \\ 0 & \text{otherwise.} \end{cases}$$

(r-4) $j = n$;

$$\varphi_i(S) = \begin{cases} 1 & \text{if } i = n \text{ and once } i\text{-removable}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it satisfies (3.2). Similarly, we can check that a zigzag j-strip S having an i-admissible site on only kth ($k \geq 2$) column also satisfies (3.2).

Consider the zigzag strip bundles S consisting of two zigzag strips S_1 and S_2 having an i-removable site and an i-admissible site on only k_1th and k_2th ($k_1, k_2 \geq 2$) column, respectively. If the removable i-site is left of the admissible i-site, or they are the same sites, i.e., $k_1 \geq k_2$, then the i-signature of S is

$$\begin{cases} (--+) & \text{if } i = n \text{ and } S_1 \text{ has a twice removable } i\text{-site}, \\ (-+) & \text{otherwise.} \end{cases}$$

Thus, $\#_{\geq 2}(S, +) = \#_{\geq 2}(S_1, +) + \#_{\geq 2}(S_2, +)$, and so the right hand side of (3.2) for S is the sum of the right hand sides of (3.2) for S_1 and S_2. Also,

$$\varepsilon_i(S) = \varepsilon_i(S_1) + \varepsilon_i(S_2),$$

and since $\mathrm{wt}(S) = \mathrm{wt}(S_1) + \mathrm{wt}(S_2)$, we have

$$\begin{aligned} \varphi_i(S) &= \varepsilon_i(S) + \langle h_i, \mathrm{wt}(S) \rangle \\ &= \varepsilon_i(S_1) + \varepsilon_i(S_2) + \langle h_i, \mathrm{wt}(S_1) + \mathrm{wt}(S_2) \rangle = \varphi_i(S_1) + \varphi_i(S_2). \end{aligned}$$

Thus, S satisfies (3.2).

On the other hand, if the removable i-site is right of the admissible i-site, then the i-signature of S is

$$\begin{cases} (+--) = (-) & \text{if } i = n \text{ and } S_1 \text{ has a twice removable } i\text{-site}, \\ (+-) = (\cdot) & \text{otherwise.} \end{cases}$$

Thus, $\#_{\geq 2}(S,+) = \#_{\geq 2}(S_1,+) + \#_{\geq 2}(S_2,+) - 1$, and so the right hand side of (3.2) for S is the sum of the right hand sides of (3.2) for S_1 and S_2, and -1. Also,

$$\varepsilon_i(S) = 0 = \varepsilon_i(S_1) + \varepsilon_i(S_2) - 1,$$

and since $\mathrm{wt}(S) = \mathrm{wt}(S_1) + \mathrm{wt}(S_2)$, we have

$$\begin{aligned} \varphi_i(S) &= \varepsilon_i(S) + \langle h_i, \mathrm{wt}(S) \rangle \\ &= \varepsilon_i(S_1) + \varepsilon_i(S_2) - 1 + \langle h_i, \mathrm{wt}(S_1) + \mathrm{wt}(S_2) \rangle \\ &= \varphi_i(S_1) + \varphi_i(S_2) - 1. \end{aligned}$$

Thus, it also satisfies (3.2). By Similar method, we can check that (3.2) holds for any zigzag strip bundle S.

Finally, the tensor product rule of crystals and the definition of the crystal $\mathcal{S}(\lambda)$ complete the proof. □

Therefore, we devote ourselves to give a characterization of $\mathcal{S}(\lambda)$ with respect to the above crystal structure.

Theorem 3.2. (a) *For each $p \geq 1$, the set $\mathcal{S}(p\Lambda_n)$ for $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$) consists of all zigzag strip bundles S such that*

(i) *it consists of only n-strips,*
(ii) *it has at most p first n-strips.*

(b) *For each $p \geq 1$, the set $\mathcal{S}(p\Lambda_n)$ (resp. $\mathcal{S}(p\Lambda_{n-1})$) for $U_q(D_n^{(1)})$ consists of all zigzag strip bundles S satisfying the following:*

(i) *it consists of only $(n-1)$ or n-strips,*
(ii) *it has at most p first n-strips (resp. $(n-1)$-strips), and dose not have first $(n-1)$-strips (resp. n-strips).*

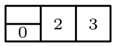
Proof. It is almost clear by the definition of Kashiwara operators on $\mathcal{S}(p\Lambda_n)$. □

Remark 3.3. By the conditions of zigzag strip bundles, for each $j \geq 1$, $S \in \mathcal{S}(p\Lambda_n)$ over $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$) has at most p jth n-strips. Also, if we let

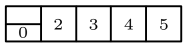
$$p_0 = \begin{cases} p & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}, \end{cases} \quad \text{and} \quad p_1 = \begin{cases} p & \text{if } j \text{ is odd}, \\ 0 & \text{if } j \text{ is even}, \end{cases}$$

every zigzag strip bundle $S \in \mathcal{S}(p\Lambda_n)$ over $U_q(D_n^{(1)})$ has at most p_0 jth $(n-1)$-strips and p_1 jth n-strips.

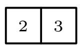
Before we describe the highest weight crystal $B(\Lambda_m)$ $(0 \le m < n)$ in terms of zigzag strip bundles, we take a closer look at j-strips. First, consider a zigzag j-strip S $(0 \le j < n)$ for $U_q(B_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$. Then if we regard S as only piles of blocks, it is contained in a zigzag n-strip, and we will denote it by $\overline{S}$. For example, if

$S =$ 

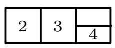
is a 3-strip for $U_q(B_5^{(1)})$, then

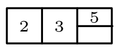
$\overline{S} =$  .

Second, consider a zigzag j-strip S $(0 \le j < n-1)$ for $U_q(D_n^{(1)})$. Then as only piles of blocks, it is contained in both a zigzag $(n-1)$-strip and n-strip, and we will denote them by $\overline{S}$. For example, if

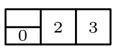
$S =$ 

is a 3-strip for $U_q(D_5^{(1)})$, then

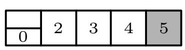
$\overline{S} =$ 

or  .

Finally, consider a zigzag j-strip S $(0 \le j < n)$ for $U_q(\mathfrak{g})$ $(\mathfrak{g} = C_n^{(1)}, A_{2n}^{(2)}$ or $A_{2n-1}^{(2)})$. If we regard S as only piles of blocks, it is contained in a zigzag half n-strip, and we will denote it by $\overline{S}$. For example, if

$S =$ 

is a 3-strip for $U_q(A_9^{(2)})$, then

$\overline{S} =$  .

3.2. Fundamental highest weight crystal $B(\Lambda_i)$ for $B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$

Definition 3.4. (a) The *zigzag strip bundles of type* 0 for $U_q(B_n^{(1)})$ (resp. $U_q(D_n^{(1)})$) are the zigzag strip bundles satisfying the following:

(i) There is no 1-strip, and there exist at most one 0-strip.
(ii) For each $j = 2, \dots, n-1$ (resp. $j = 2, \dots, n-2$), it has at most one j-strip and at most one first n-strip (resp. one of first $(n-1)$-strip and first n-strip).

(iii) If we let

$$S_0, S_2, \dots, S_{n-1} \text{ and } S_{n,1} \quad (\text{resp. } S_0, S_2, \dots, S_{n-2}, S_{n-1,1}, \text{ and } S_{n,1})$$

be the 0-strip, 2-strip, ..., $(n-1)$-strip and the first n-strip (resp. the 0-strip, 2-strip, ..., $(n-2)$-strip, the first $(n-1)$-strip and the first n-strip) in S which can be empty, then the pairs

$$(\overline{S_0}, \overline{S_2}), (\overline{S_j}, \overline{S_{j+1}}) \quad (2 \le j \le n-2), \quad \text{and} \quad (\overline{S_{n-1}}, S_{n,1})$$
$$\big(\text{resp. } (\overline{S_0}, \overline{S_2}), (\overline{S_j}, \overline{S_{j+1}}) \ (2 \le j \le n-3), \text{ and } (\overline{S_{n-2}}, S_{n-1,1}) \text{ or } (\overline{S_{n-2}}, S_{n,1})\big)$$

satisfy the condition (iii) in Definition 2.3.

(b) The *zigzag strip bundles of type* 1 for $U_q(B_n^{(1)})$ or $U_q(D_n^{(1)})$ are the zigzag strip bundles satisfying the conditions obtained from those of zigzag strip bundles of type 0 by changing the roles of 0 and 1.

(c) Set $m = 2, \dots, n$ over $U_q(B_n^{(1)})$, $0, 1, \dots, n$ over $U_q(D_{n+1}^{(2)})$, and $2, \dots, n-2$ over $U_q(D_n^{(1)})$. For each m, the *zigzag strip bundles of type* m for $U_q(B_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ (resp. $U_q(D_n^{(1)})$) are the zigzag strip bundles satisfying the following:

(i) There is no j-strips for all $j < m$.
(ii) For each $m \le j \le n-1$ (resp. $m \le j \le n-2$), it has at most one j-strip, and one first n-strip (resp. one of first $(n-1)$-strip and first n-strip).
(iii) If we let

$$S_m, S_{m+1}, \dots, S_{n-1} \text{ and } S_{n,1} \quad (\text{resp. } S_m, S_{m+1}, \dots, S_{n-2}, S_{n-1,1} \text{ and } S_{n,1})$$

be the m-strip, $(m+1)$-strip, ..., $(n-1)$-strip and the first n-strip (the m-strip, $(m+1)$-strip, ..., $(n-2)$-strip, the first $(n-1)$-strip and the first n-strip) in S, then the pairs

$$(\overline{S_j}, \overline{S_{j+1}}) \quad (m \le j \le n-2) \quad \text{and} \quad (\overline{S_{n-1}}, S_{n,1})$$
$$\big(\text{resp. } (\overline{S_j}, \overline{S_{j+1}}) \ (m \le j \le n-3), \text{ and } (\overline{S_{n-2}}, S_{n-1,1}) \text{ or } (\overline{S_{n-2}}, S_{n,1})\big)$$

satisfy the condition (iii) in Definition 2.3.

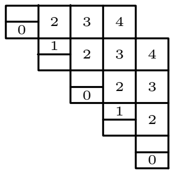
(d) We call the zigzag strip bundles in $\mathcal{S}(\Lambda_{n-1})$ and $\mathcal{S}(\Lambda_n)$ for $U_q(D_n^{(1)})$ the *zigzag strip bundles of type* $n-1$ and n, respectively.

Remark 3.5. (a) Because of the conditions (i) and (iii), the number of blocks in the j-strip $(j \ge 2)$ appearing in a zigzag strip bundle S of type 0 or 1 for $U_q(B_n^{(1)})$ (resp. $U_q(D_n^{(1)})$) is at most j (resp. $j-1$).

(b) A zigzag strip bundle S of type m ($m \in I$) is also a zigzag strip bundle. It means that the kth n-strips from S satisfy the condition (iii) in Definition 2.3.

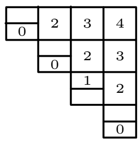
(c) It is clear that the set $\mathcal{S}(\Lambda_n)$ for $U_q(B_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ given in Theorem 3.2 is the set of all zigzag strip bundles of type n.

Example 3.6. Let

$S =$ 

be a zigzag strip bundle for $U_q(B_4^{(1)})$. Then it is a zigzag strip bundle of type 0.

On the other hand, the following zigzag strip bundle for $U_q(B_4^{(1)})$

$S' =$ 

is not a zigzag strip bundle of type 0 because $(\overline{S_3}, S_{4,1})$ does not satisfy the condition (iii) in Definition 2.3.

Theorem 3.7. (a) *For each $m = 0, 1, \ldots, n-1$, the set $\mathcal{S}(\Lambda_m)$ for $U_q(B_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ consists of all zigzag strip bundles S which can be decomposed into the pair $S = (S_h, S_v)$ of zigzag strip bundles of type m and n such that for each $k \geq 1$*

the existence of the nonempty kth n-strip in S_v guarantee
the existence of the nonempty kth n-strip in S_h.

(b) *For each $m = 0, 1, \ldots, n-2$, the set $\mathcal{S}(\Lambda_m)$ for $U_q(D_n^{(1)})$ consists of all zigzag strip bundles S which can be decomposed into the pair $S = (S_h, S_v)$ of zigzag strip bundles of type m, and $n-1$ or n such that for each $k \geq 1$*

the existence of the nonempty kth $(n-1)$-strip (resp. n-strip) in S_v guarantee
the existence of the nonempty kth n-strip (resp. $(n-1)$-strip) in S_h.

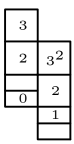
Remark 3.8. The zigzag strip bundles S_h and S_v of type m and n are obtained from the decomposition of S by choosing any zigzag strips. One method to construct S_h and S_v is as follows: First, all i-strip ($i \neq n$) should be used to make S_h. By the definition of zigzag strip bundles of type m and n, there are at most two nonempty kth n-strip. The kth n strip of larger size among these two n-strips is used to make S_v, and the other is used to make S_h. If only one nonzero kth n-strip exists, then it is used to construct S_h.

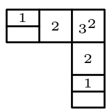
Proof. Since the proof is similar, we only treat the set $\mathcal{S}(\Lambda_m)$ $(1 < m < n)$ for $B_n^{(1)}$-type. Let S be a zigzag strip bundle in $\mathcal{S}(\Lambda_m)$ which can be decomposed as $S = (S_h, S_v)$, and assume that $\tilde{f}_i S$ is obtained from S by stacking an i-block on an i-site P. Then $\tilde{f}_i S = (S_h \leftarrow \boxed{i}\,, S_v)$ or $\tilde{f}_i S = (S_h, S_v \leftarrow \boxed{i}\,)$. Here, $S_h \leftarrow \boxed{i}$ and $S_v \leftarrow \boxed{i}$ are the piles obtained from S_h and S_v by stacking i-block on P, respectively. By the definition of zigzag strip bundles of type m and n, it is almost clear that $S_h \leftarrow \boxed{i}$ and $S_v \leftarrow \boxed{i}$ are also zigzag strip bundles of type m and n. Indeed, if $S_h \leftarrow \boxed{i}$ (resp. $S_v \leftarrow \boxed{i}\,$) is not a zigzag strip bundle of type m (resp. n), $\tilde{f}_i S \neq 0$ implies that P is at least once i-admissible from S_v (resp. S_h), and $S_v \leftarrow \boxed{i}$ (resp. $S_h \leftarrow \boxed{i}\,$) is a zigzag strip bundle of type n (resp. m). It means that $\tilde{f}_i S = (S_h, S_v \leftarrow \boxed{i}\,)$ (resp. $\tilde{f}_i S = (S_h \leftarrow \boxed{i}\,, S_v)$). Thus $\tilde{f}_i S$ can be decomposed as zigzag strip bundles of type m and n.

Now, it is clear that $\tilde{f}_i S = (S_h \leftarrow \boxed{i}\,, S_v)$ satisfies the condition given in Theorem 3.7 (a), and so consider the case when $\tilde{f}_i S = (S_h, S_v \leftarrow \boxed{i}\,)$. In the case when $i \neq n$, clearly $\tilde{f}_i S = (S_h, S_v \leftarrow \boxed{i}\,)$ satisfies the condition given in Theorem 3.7 (a). In the case when $i = n$, suppose that $S_v \leftarrow \boxed{i}$ has a new jth n-strip and there is no jth n-strip in S_h. Then by the definition of Kashiwara operator $\tilde{f}_i$ and the condition given in Theorem 3.7 (a), both S_h and S_v have $(j-1)$st n-strips. In particular, the length of the $(j-1)$st n-strip in S_v is at least 2. Moreover, if the length of the $(j-1)$st n-strip in S_h is 1, i.e., it consists of only n-block, $\tilde{f}_n S$ cannot be the zigzag strip bundle obtained from S stacking an n-block on P, so the length of the $(j-1)$st n-strip in S_h is also at least 2. It means that $\tilde{f}_i S$ can be decomposed $S = (S_h \leftarrow \boxed{i}\,, S_v)$. Thus, in any case, $\tilde{f}_i S$ satisfies the condition given in Theorem 3.7 (a). Therefore, $\mathcal{S}(\Lambda_m)$ is closed under $\tilde{f}_i$. Similarly, we can show that $\mathcal{S}(\Lambda_m)$ is closed under $\tilde{e}_i$.

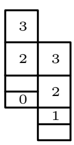
Now, assume that the leftmost top block B of S is an i-block on an i-site P_i. Then clearly P_i is once i-removable, and so $\tilde{e}_i S \neq 0$, which completes the proof. □

Example 3.9. The following zigzag strip bundles for $U_q(B_3^{(1)})$

$S =$ 

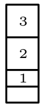
and $S' =$ 

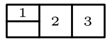
belong to $\mathcal{S}(\Lambda_1)$. Here, in S,

$S_h =$ 

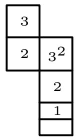
$S_v = \boxed{3}$

and in S',

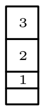
$S'_h =$ 

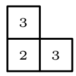
$S'_v =$ 

On the other hand, consider the following zigzag strip bundle

$S'' =$ 

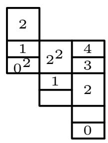
In this case,

$S''_h =$ 

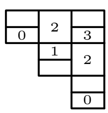
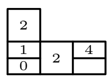
$S''_v =$ 

Because there is a second 3-strip in S''_v but there is no second 3-strip in S''_h, S'' does not belong to $\mathcal{S}(\Lambda_1)$.

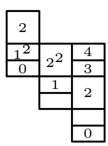
Example 3.10. Let

$S =$ 

be a zigzag strip bundle for $U_q(D_4^{(1)})$. Then it is decomposed into the pair $S = (S_h, S_v)$ given below.

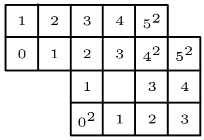
$S_h =$ 
$S_v =$ 

Thus S belongs to $\mathcal{S}(\Lambda_0)$. But, it is easy to see that the following zigzag strip bundle

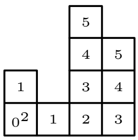
$S' =$ 

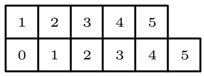
does not belong to $\mathcal{S}(\Lambda_0)$.

Example 3.11. Let

$S =$ 

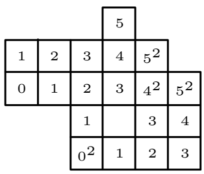
be a zigzag strip bundle for $U_q(D_6^{(2)})$. Then it can be decomposed into the pair $S = (S_h, S_v)$ of strip bundles of type 3 and 5 given below.

$S_h =$ 

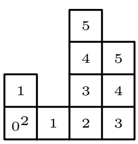
$S_v =$ 

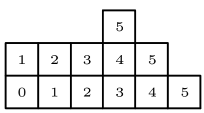
Therefore, it is easy to see that S belongs to $\mathcal{S}(\Lambda_3)$ for $U_q(D_6^{(2)})$.

On the other hand, consider the following zigzag strip bundle for $U_q(D_6^{(2)})$.

$S' =$ 

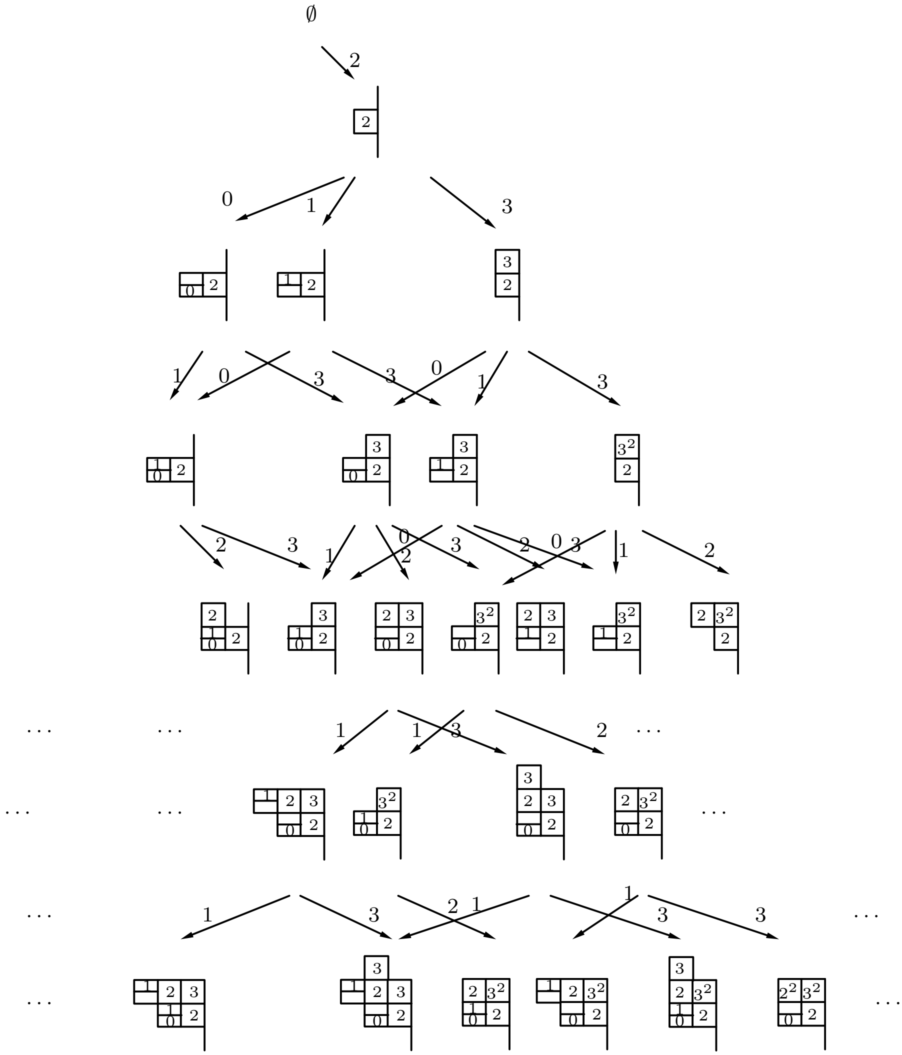
Then it should be decomposed into the pair $S' = (S'_h, S'_v)$ of strip bundles of type 3 and 5 given below.

$S'_h =$ 

$S'_v =$ 

Because there is a third 5-strip in S'_v but there is no third 5-strip in S'_h, S' does not belong to $\mathcal{S}(\Lambda_3)$ for $U_q(D_6^{(2)})$.

Example 3.12. The following is the top part of $\mathcal{S}(\Lambda_2)$ over $U_q(B_3^{(1)})$.



3.3. Fundamental highest weight crystal $B(\Lambda_i)$ for $C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$-types

Definition 3.13. (a) The *half zigzag strip bundles of type* 0 (resp. *type* 1) for $U_q(A_{2n-1}^{(2)})$ are the piles consisting of zigzag strips and half n-strips satisfying the following:

(i) There is no 1-strip (resp. 0-strip), and there exist at most one 0-strip (resp. 1-strip).

(ii) For each $2 \le j < n$ and $k \ge 1$, it has at most one j-strip, and one nonempty kth half n-strip.

(iii) If we let S_0 (resp. S_1), $S_2, \dots, S_{n-1}$ and $\overline{S_{n,k}}$ be the 0-strip (resp. 1-strip), 2-strip, $\dots$, $(n-1)$-strip, and kth half n-strip in S which can be empty, then the pairs

$(\overline{S_0}, \overline{S_2})$ (resp. $(\overline{S_1}, \overline{S_2})$), $(\overline{S_j}, \overline{S_{j+1}})$ $(2 \le j \le n-2)$, $(\overline{S_{n-1}}, \overline{S_{n,1}})$, and $(\overline{S_{n,k}}, \overline{S_{n,k+1}})$ satisfy the condition (ii) in Definition 2.7.

(b) For each $m = 1, \ldots, n$ for $U_q(C_n^{(1)})$, $0, 1, \ldots, n$ for $U_q(A_{2n}^{(2)})$, and $2, \ldots, n$ for $U_q(A_{2n-1}^{(2)})$, the *half zigzag strip bundles of type m* for $U_q(C_n^{(1)})$, $U_q(A_{2n}^{(2)})$ or $U_q(A_{2n-1}^{(2)})$ are the piles consisting of zigzag strips and half n-strips satisfying the following:

(i) There is no j-strips for all $j < m$.
(ii) For each $m \le j < n$ and $k \ge 1$, it has at most one j-strip, and one kth half n-strip.
(iii) If we let $S_m, S_{m+1}, \ldots, S_{n-1}$ and $\overline{S_{n,k}}$ $(k \ge 1)$ be the m-strip, $(m+1)$-strip, $\ldots$, $(n-1)$-strip, and kth half n-strip in S which can be empty, then the pairs $(\overline{S_j}, \overline{S_{j+1}})$ $(m \le j \le n-2)$, $(\overline{S_{n-1}}, \overline{S_{n,1}})$, and $(\overline{S_{n,k}}, \overline{S_{n,k+1}})$ satisfy the condition (ii) in Definition 2.7.

Remark 3.14. Clearly, the half zigzag strip bundles are not zigzag strip bundles because a half n-strip is not a zigzag strip.

Let $S \in \mathcal{S}(\infty)$ be a zigzag strip bundle for $U_q(C_n^{(1)})$, $U_q(A_{2n}^{(2)})$ or $U_q(A_{2n-1}^{(2)})$. For each $k \ge 1$ and $i \in I - \{n\}$, we denote by S_i and $S_{n,k}$ the zigzag i-strip and the kth zigzag n-strip, respectively. Also, we denote by $(S_{n,k})^-$ and $(S_{n,k})_-$ the upper half n-strip and the lower half n-strip of $S_{n,k}$, respectively.

Theorem 3.15. *For each $m = 1, \ldots, n-1$ (resp. $m = 0, 1, \ldots, n-1$), the set $\mathcal{S}(\Lambda_m)$ for $U_q(C_n^{(1)})$ (resp. $U_q(A_{2n-1}^{(2)})$ or $U_q(A_{2n-1}^{(2)})$) consists of all zigzag strip bundles S which can be decomposed into the pair $S = (S_h, S_v)$ of half strip bundles of type m and n using S_i, $(S_{n,k})^-$ and $(S_{n,k})_-$ such that*

the existence of the nonempty kth half n-strip in S_v guarantee
the existence of the nonempty kth half n-strip in S_h.

Proof. Since it is proved by the same argument in the proof of Theorem 3.7, we omit it. □

Remark 3.16. The set $\mathcal{S}(\Lambda_n)$ for $U_q(C_n^{(1)})$, $U_q(A_{2n-1}^{(2)})$ or $U_q(A_{2n-1}^{(2)})$ given in Theorem 3.2 is rewritten as follows: The set $\mathcal{S}(\Lambda_n)$ consists of all zigzag strip bundles S which can be decomposed into the pair $S = (S_h, S_v)$ of half strip bundles of type n and n such that the existence of the nonempty kth half n-strip in S_v guarantee the existence of the nonempty kth half n-strip in S_h.

Example 3.17. Consider the following zigzag strip bundle for $U_q(C_3^{(1)})$.

$S =$

	2^2	3	
0	1^2	2^2	3
	0	1	2

Then

$$S_2 = \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 0 & 1 & 2 \\ \hline \end{array} \qquad S_{3,1} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2^2 & 3 \\ \hline \end{array} \qquad S_{3,2} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}$$

and so,

$$(S_{3,1})_- = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & \colorbox{lightgray}{3} \\ \hline \end{array} \qquad (S_{3,1})^- = \begin{array}{|c|c|} \hline 2 & \colorbox{lightgray}{3} \\ \hline \end{array} \qquad (S_{3,2})_- = \begin{array}{|c|c|} \hline 2 & \colorbox{lightgray}{3} \\ \hline \end{array} \qquad (S_{3,2})^- = \begin{array}{|c|} \hline \colorbox{lightgray}{3} \\ \hline \end{array}$$

Therefore, S can be divided as

$$S_h = S_2 \cup (S_{3,1})_- \cup (S_{3,2})_- = \begin{array}{cccc} & 2^2 & \colorbox{lightgray}{3} & \\ 0 & 1^2 & 2 & \colorbox{lightgray}{3} \\ & 0 & 1 & 2 \end{array} \qquad S_v = (S_{3,1})^- \cup (S_{3,2})^- = \begin{array}{cc} \colorbox{lightgray}{3} & \\ 2 & \colorbox{lightgray}{3} \end{array}$$

and so S belongs to $\mathcal{S}(\Lambda_2)$.

On the other hand, consider the following zigzag strip bundle for $U_q(C_3^{(1)})$.

$$S = \begin{array}{cccc} 1^2 & 2 & 3 & \\ 0 & 1 & 2^2 & 3 \\ & & 1 & 2 \end{array}$$

Then

$$S_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \qquad S_{3,1} = \begin{array}{cccc} 1 & & & \\ 0 & 1 & 2^2 & 3 \end{array} \qquad S_{3,2} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

and so,

$$(S_{3,1})_- = \begin{array}{cccc} 1 & & & \\ 0 & 1 & 2 & \colorbox{lightgray}{3} \end{array} \qquad (S_{3,1})^- = \begin{array}{|c|c|} \hline 2 & \colorbox{lightgray}{3} \\ \hline \end{array} \qquad (S_{3,2})_- = \begin{array}{|c|c|c|} \hline 1 & 2 & \colorbox{lightgray}{3} \\ \hline \end{array} \qquad (S_{3,2})^- = \begin{array}{|c|} \hline \colorbox{lightgray}{3} \\ \hline \end{array}$$

Therefore, S can be divided as

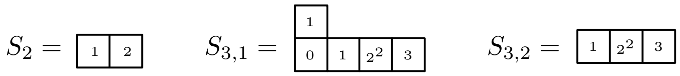
$$S_h = S_2 \cup (S_{3,1})^- \cup (S_{3,2})^- = \begin{array}{cc} \colorbox{lightgray}{3} & \\ 2 & \colorbox{lightgray}{3} \\ 1 & 2 \end{array} \qquad S_v = (S_{3,1})_- \cup (S_{3,2})_- = \begin{array}{cccc} 1^2 & 2 & \colorbox{lightgray}{3} & \\ 0 & 1 & 2 & \colorbox{lightgray}{3} \end{array}$$

and so S belongs to $\mathcal{S}(\Lambda_2)$.

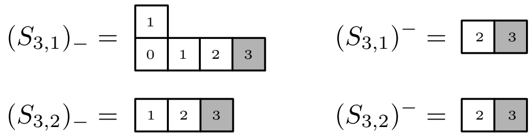
Example 3.18. Consider the following zigzag strip bundle for $U_q(C_3^{(1)})$.

$$S = \begin{array}{cccc} 1^2 & 2^2 & 3 & \\ 0 & 1 & 2^2 & 3 \\ & & 1 & 2 \end{array}$$

Then

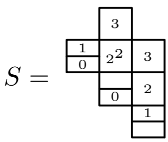


and so,

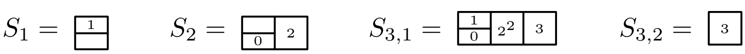


Therefore, we cannot construct the half strip bundle S_h of type 2, and so S does not belong to $\mathcal{S}(\Lambda_2)$.

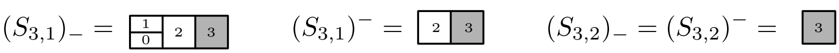
Example 3.19. Let



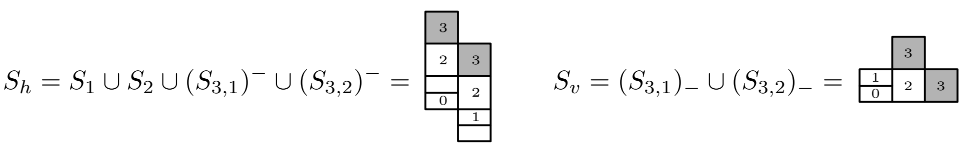
be a zigzag strip bundle for $U_q(A_5^{(2)})$. Then



and so,



Therefore, S can be decomposed into the pair $S = (S_h, S_v)$ given below.



Thus S belongs to $\mathcal{S}(\Lambda_1)$.

3.4. General highest weight crystal $B(\lambda)$

The proof of Theorem 3.7 give us more results about highest weight crystals. In this section, we deal with general highest weight crystals $B(\lambda)$ using zigzag strip bundles. Set

$$\begin{cases} 0 \le l \le m < n & \text{for } U_q(B_n^{(1)}),\ U_q(D_{n+1}^{(2)}), \\ 0 \le l \le m < n-1 & \text{for } U_q(D_n^{(1)}), \\ 0 \le l \le m \le n & \text{for } U_q(A_{2n}^{(2)}),\ U_q(A_{2n-1}^{(2)}) \\ 0 < l \le m \le n & \text{for } U_q(C_n^{(1)}). \end{cases}$$

Theorem 3.20. (a) *The set $\mathcal{S}(\Lambda_l+\Lambda_m)$ for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ consists of all zigzag strip bundles S which can be decomposed into the quadruple $S=(S_h^l, S_h^m, S_v^l, S_v^m)$ of zigzag strip bundles of type (l,m,n,n) such that the pairs $S^l=(S_h^l, S_v^l)$ and $S^m=(S_h^m, S_v^m)$ belong to $\mathcal{S}(\Lambda_l)$ and $\mathcal{S}(\Lambda_m)$, respectively. Here, for $U_q(D_n^{(1)})$, not only type (l,m,n,n), but also types $(l,m,n-1,n-1)$, $(l,m,n-1,n)$ and $(l,m,n,n-1)$ are possible.*

(b) *The set $\mathcal{S}(\Lambda_l+\Lambda_n)$ for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ consists of all zigzag strip bundles S which can be decomposed into the triple $S=(S_h^l, S^n, S_v^l)$ of zigzag strip bundles of type (l,n,n) such that $S^l=(S_h^l, S_v^l)$ and S^n belong to $\mathcal{S}(\Lambda_l)$ and $\mathcal{S}(\Lambda_n)$, respectively.*

(c) *The set $\mathcal{S}(\Lambda_l+\Lambda_n)$ (resp. $\mathcal{S}(\Lambda_l+\Lambda_{n-1})$) for $U_q(D_n^{(1)})$ consists of all zigzag strip bundles S which can be decomposed into the triple $S=(S_h^l, S^n, S_v^l)$ (resp. $S=(S_h^l, S^{n-1}, S_v^l)$) of zigzag strip bundles of type (l,n,n) or $(l,n,n-1)$ (resp. $(l,n-1,n)$ or $(l,n-1,n-1)$) such that $S^l=(S_h^l, S_v^l)$ and S^n (resp. S^{n-1}) belong to $\mathcal{S}(\Lambda_l)$ and $\mathcal{S}(\Lambda_n)$ (resp. $\mathcal{S}(\Lambda_{n-1})$), respectively.*

(d) *The set $\mathcal{S}(\Lambda_l+\Lambda_m)$ for $U_q(C_n^{(1)})$, $U_q(A_{2n}^{(2)})$ or $U_q(A_{2n-1}^{(2)})$ consists of all zigzag strip bundles S which can be decomposed into the quadruple $S=(S_h^l, S_h^m, S_v^l, S_v^m)$ of half zigzag strip bundles of type l, m, n and n such that $S^l=(S_h^l, S_v^l)$ and $S^m=(S_h^m, S_v^m)$ belong to $\mathcal{S}(\Lambda_l)$ and $\mathcal{S}(\Lambda_m)$, respectively.*

Proof. The proofs of (a), (b), (c) and (d) are very similar, and they are obtained by the same argument in the proof of Theorem 3.7. Now, we only focus on the type of $\mathcal{S}(\Lambda_l+\Lambda_m)$ for $U_q(B_n^{(1)})$. Let S be a zigzag strip bundle in $\mathcal{S}(\Lambda_l+\Lambda_m)$ which can be decomposed as $S=(S_h^l, S_v^l, S_h^m, S_v^m)$, and assume that $\tilde{f}_i S$ is obtained from S by stacking an i-block on an i-site P. Then $\tilde{f}_i S$ is one of the following:

$$\big(S_h^l \leftarrow \boxed{i}\ , S_v^l, S_h^m, S_v^m\big) \qquad \big(S_h^l, S_v^l \leftarrow \boxed{i}\ , S_h^m, S_v^m\big)$$
$$\big(S_h^l, S_v^l, S_h^m \leftarrow \boxed{i}\ , S_v^m\big) \qquad \big(S_h^l, S_v^l, S_h^m, S_v^m \leftarrow \boxed{i}\,\big)$$

In the case when $\tilde{f}_i S=(S_h^l \leftarrow \boxed{i}\ , S_v^l, S_h^m, S_v^m)$, if $S_h^l \leftarrow \boxed{i}$ is not a zigzag strip bundle of type l, by the definition of Kashiwara operator $\tilde{f}_i$, the site P is at least once i-admissible from one of S_v^l, S_h^m, S_v^m, say S_v^l, then $\tilde{f}_i S$ can be decomposed as $(S_h^l, S_v^l \leftarrow \boxed{i}\ , S_h^m, S_v^m)$, and $S_v^l \leftarrow \boxed{i}$ is a zigzag strip bundle of type n. Similarly, we can prove in any case $\tilde{f}_i S$ belongs to $\mathcal{S}(\Lambda_l+\Lambda_m)$. □

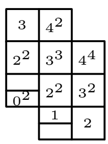
Remark 3.21. (a) The statement of Theorem 3.20 is very simple because there is no condition on the relationship among S^l and S^m. However, the decomposition of $S \in \mathcal{S}(\Lambda_l+\Lambda_m)$

into the quadruple $S = (S_h^l, S_h^m, S_v^l, S_v^m)$ is not unique. That is, the existence of various decompositions makes no relationship between S^l and S^m. Nevertheless, Theorem 3.20 says that every element S in $\mathcal{S}(\Lambda_l + \Lambda_m)$ can be obtained from an element S^l in $\mathcal{S}(\Lambda_l)$ by adding an element S^m in $\mathcal{S}(\Lambda_m)$. More precisely, S can be obtained from S^l by stacking $s_i(k)$ i-blocks to the kth column from the right. Here, $s_i(k)$ is the number of i-blocks on the kth column of S^m. In other words, one of the important implications of Theorem 3.20 is that the highest weight crystals of higher level can be obtained by using highest weight crystals of lower level.

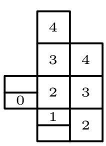
(b) In Theorem 3.20, decomposition of $S \in \mathcal{S}(\Lambda_l + \Lambda_m)$ into the quadruple $S = (S_h^l, S_h^m, S_v^l, S_v^m)$ is not unique, and there are various decompositions. Now, we give a standard decomposition of S, and any other decomposition can be changed to this standard decomposition. For S_β^α ($\alpha = l, m$, $\beta = h, v$) and $p \geq 1$, let $(S_\beta^\alpha)_{\geq p}$ be the subpile of S_β^α consisting of blocks on the qth ($q \geq p$) rows from the bottom. Then the standard decomposition $S = (S_h^l, S_h^m, S_v^l, S_v^m)$ of S is a decomposition satisfying the following conditions:

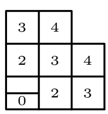
(i) If $S_h^l - (S_h^l)_{\geq p} + (S_h^m)_{\geq p}$ is still a zigzag strip bundle of type l, then $(S_h^m)_{\geq p} \subset (S_h^l)_{\geq p}$.
(ii) The pairs $(S_h^l,\, S_v^m)$ and $(S_h^m,\, S_v^l)$ satisfy the condition (i).
(iii) If $S_v^l - (S_v^l)_{\geq p} + (S_v^m)_{\geq p}$ is still a zigzag strip bundle of type n, and S_h^l together with it still belongs to $\mathcal{S}(\Lambda_l)$, $(S_v^m)_{\geq p} \subset (S_v^l)_{\geq p}$.

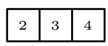
For instance, the zigzag strip bundle for $U_q(B_4^{(1)})$

$S =$ 

is decomposed into

$S_h^2 =$ 

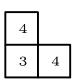
, $S_h^3 =$ 

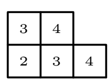
, $S_v^2 =$ 

, $S_v^3 =$ 

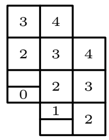
.

However, we can see that

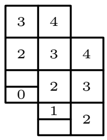
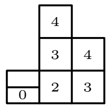
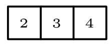
$(S_h^2)_{\geq 4} =$ 

, $(S_h^3)_{\geq 4} =$ 

and

$$S_h^2 - (S_h^2)_{\geq 4} + (S_h^3)_{\geq 4} =$$ 

is still a zigzag strip bundle of type 2. Thus it is not the standard decomposition of S. Moreover, we can easily check that the following is the standard decomposition of S.

$S_h^2 =$ , $S_h^3 =$ , $S_v^2 =$ , $S_v^3 =$ .

By the same argument given in the proof of Theorem 3.20, we have more general result. Let $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_t}$, where $0 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq n$ except for $U_q(C_n^{(1)})$, and $1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq n$ for $U_q(C_n^{(1)})$.

Theorem 3.22. *The set $\mathcal{S}(\lambda)$ for $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$, or $A_{2n-1}^{(2)}$ consists of all zigzag strip bundles S which can be decomposed into $S = (S^{i_1}, S^{i_2}, \ldots, S^{i_t})$ of zigzag strip bundles such that S^{i_j} $(j = 1, \ldots, t)$ belong to $\mathcal{S}(\Lambda_{i_j})$.*

3.5. Connection with Nakajima monomials and crystal embeddings

Let $R_\lambda = \{r_\lambda\}$ $(\lambda \in P)$ be the crystal given in section 3.1, and let $\Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda$ be the strict embedding of crystals. Then, by Theorem 1.1, there exists the unique strict embedding of crystals

$$\Psi_{\mathbf{i}}^\lambda : B(\lambda) \overset{\Omega_\lambda}{\hookrightarrow} B(\infty) \otimes R_\lambda \overset{\Psi_{\mathbf{i}} \otimes \mathrm{id}}{\hookrightarrow} B(\mathbf{i}) \otimes R_\lambda,$$

sending the highest weight vector u_λ to $(\ldots, 0, \ldots, 0) \otimes r_\lambda$.

Let $S(\lambda)$ be the set of zigzag strip bundles given in Sections 3.1, 3.2, and 3.3, and let $\mathcal{M}(\lambda) \otimes R_\lambda$ be the connected component of $\mathcal{M}(\infty) \otimes R_\lambda$ containing $\mathbf{1} \otimes r_\lambda$. By Theorem 3.7, Theorem 3.15 and Corollary 2.12, it is clear that the map $\Theta : \mathcal{S}(\lambda) \otimes R_\lambda \to \mathcal{M}(\lambda) \otimes R_\lambda$ given by

$$\Theta(S \otimes r_\lambda) = \mathbf{1} \cdot \prod_{i \in I, k \geq 0} A_i(k)^{-a_i(k)} \otimes r_\lambda, \tag{3.3}$$

where $a_i(k)$ is the number of i-blocks of S stacked on the $(k+1)$st column of $\mathcal{B}$, is a $U_q(\mathfrak{g})$-crystal isomorphism. Moreover, there is a $U_q(\mathfrak{g})$-crystal isomorphism

$$\Delta : \mathcal{S}(\lambda) \otimes R_\lambda \xrightarrow{\Theta} \mathcal{M}(\lambda) \otimes R_\lambda \subset \mathcal{M}(\infty) \otimes R_\lambda \xrightarrow{\phi \otimes \mathrm{id}} \mathrm{Im}\, \Psi_{\mathbf{i}}^\lambda \subset B(\mathbf{i}) \otimes R_\lambda$$

sending $\emptyset \otimes r_\lambda$ to $\cdots \otimes b_0(0) \otimes b_n(0) \otimes \cdots \otimes b_1(0) \otimes b_0(0) \otimes r_\lambda$.

Example 3.23. Let $S \in \mathcal{S}(\Lambda_1)$ be the zigzag strip bundle over $U_q(B_3^{(1)})$ given in Example 3.9. Then

$$\Theta(S \otimes r_{\Lambda_1}) = \mathbf{1} \cdot A_1(0)^{-1}A_2(0)^{-1}A_3(0)^{-2}A_0(1)^{-1}A_2(1)^{-1}A_3(1)^{-1} \otimes r_{\Lambda_1},$$

and

$$\begin{aligned}\Delta(S \otimes r_{\Lambda_1}) = \cdots \otimes b_1(0) \otimes b_0(0) \otimes b_3(-1) \otimes b_2(-1) \otimes b_1(0) \otimes b_0(-1) \\ \otimes b_3(-2) \otimes b_2(-1) \otimes b_1(-1) \otimes b_0(0) \otimes r_{\Lambda_1}.\end{aligned}$$

As we have already seen in Remark 3.21 (a), every element in $\mathcal{S}(\Lambda_l + \Lambda_m)$ can be obtained from an element in $\mathcal{S}(\Lambda_l)$ by adding an element in $\mathcal{S}(\Lambda_m)$. Thus any monomial in $\mathcal{M}(\Lambda_l + \Lambda_m)$ is obtained from a monomial in $\mathcal{M}(\Lambda_l)$ by multiplying a monomial in $\mathcal{M}(\Lambda_m)$. Also, if we identify $\cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in B(\mathbf{i})$ with the infinite sequence $(\ldots, x_k, \ldots, x_1)$ of nonnegative integers, and if we define $(\ldots, x_k, \ldots, x_1) + (\ldots, y_k, \ldots, y_1) = (\ldots, x_k + y_k, \ldots, x_1 + y_1)$, then $\operatorname{Im}\Psi_{\mathbf{i}}^{\Lambda_l+\Lambda_m}$ is the set consisting of the addition of an element in $\operatorname{Im}\Psi_{\mathbf{i}}^{\Lambda_l}$ and an element in $\operatorname{Im}\Psi_{\mathbf{i}}^{\Lambda_m}$. Moreover, by Theorem 3.22, we have

Corollary 3.24. *Let $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_t}$, where $0 \le i_1 \le i_2 \le \cdots \le i_t \le n$ except for $U_q(C_n^{(1)})$, and $1 \le i_1 \le i_2 \le \cdots \le i_t \le n$ for $U_q(C_n^{(1)})$. Then the sets $\mathcal{M}(\lambda)$ and $\operatorname{Im}\Psi_{\mathbf{i}}^{\lambda}$ are as follows.*

$$\begin{aligned}\mathcal{M}(\lambda) &= \left\{M_{i_1}M_{i_2}\cdots M_{i_t} \;\middle|\; M_{i_j} \in \mathcal{M}(\Lambda_{i_j}) \text{ for all } j = 1, \ldots, t\right\}, \\ \operatorname{Im}\Psi_{\mathbf{i}}^{\lambda} &= \left\{\mathbf{b}_{i_1} + \mathbf{b}_{i_2} + \cdots + \mathbf{b}_{i_t} \;\middle|\; \mathbf{b}_{i_j} \in \operatorname{Im}\Psi_{\mathbf{i}}^{\Lambda_{i_j}} \text{ for all } j = 1, \ldots, t\right\}.\end{aligned}$$

Remark 3.25. (a) For $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_t}$, we have shown that the map $\Theta : \mathcal{S}(\lambda) \otimes R_\lambda \to \mathcal{M}(\lambda) \otimes R_\lambda$ given by (3.3) is a crystal isomorphism. Here, $\mathcal{M}(\lambda) \otimes R_\lambda$ is the connected component of $\mathcal{M}(\infty) \otimes R_\lambda$ containing $\mathbf{1} \otimes r_\lambda$. Now, let $\mathfrak{M}(\lambda)$ be the connected component in the set $\mathfrak{M}$ of Nakajima monomials of highest weight type containing $Y_{i_1}(0) \cdots Y_{i_t}(0)$ (see [13,17]). Then it is also a realization of the highest weight crystal $B(\lambda)$. If we define a map $\overline{\Theta} : \mathcal{S}(\lambda) \otimes R_\lambda \to \mathfrak{M}(\lambda)$ by

$$\overline{\Theta}(S \otimes r_\lambda) = Y_{i_1}(0) \cdots Y_{i_t}(0) \cdot \prod_{i \in I, k \ge 0} A_i(k)^{-a_i(k)},$$

where $a_i(k)$ is the number of i-blocks of S stacked on the $(k+1)$st column of $\mathcal{B}$, we can verify that $\overline{\Theta}$ is also a crystal isomorphism.

(b) The set $\mathcal{S}(\Lambda_0)$ for $U_q(C_n^{(1)})$ has a very different feature from other fundamental highest weight crystal $\mathcal{S}(\Lambda_i)$. When we approach $\mathcal{S}(\Lambda_0)$ by similar method, we meet zigzag strip bundles of type $0, 1, n, n$ and it seems that they have several relation. We expect this part will also be solved someday.

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References

[1] K. Jeong, S.-J. Kang, J.-A. Kim, D.-U. Shin, Crystals and Nakajima monomials for quantum generalized Kac–Moody algebras, J. Algebra 319 (2008) 3732–3751.
[2] M. Jimbo, K.C. Misra, T. Miwa, M. Okado, Combinatorics of representation of $U_q(\widehat{sl}(n))$ at $q = 0$, Comm. Math. Phys. 136 (1991) 543–566.
[3] S.-J. Kang, Crystal bases for quantum affine algebras and combinatorics of Young walls, Proc. Lond. Math. Soc. 86 (2003) 29–69.
[4] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Affine crystals and vertex models, Internat. J. Modern Phys. A Suppl. 1A (1992) 449–484.
[5] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (1992) 499–607.
[6] S.-J. Kang, J.-A. Kim, D.-U. Shin, Monomial realization of crystal bases for special linear Lie algebras, J. Algebra 274 (2004) 629–642.
[7] S.-J. Kang, J.-A. Kim, D.-U. Shin, Crystal bases for quantum classical algebras and Nakajima's monomials, Publ. Res. Inst. Math. Sci. 40 (2004) 758–791.
[8] S.-J. Kang, J.-A. Kim, D.-U. Shin, Modified Nakajima monomials and the crystal $B(\infty)$, J. Algebra 308 (2007) 524–535.
[9] S.-J. Kang, H. Lee, Crystal bases for quantum affine algebras and Young walls, J. Algebra 322 (2009) 1979–1999.
[10] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1990) 249–260.
[11] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991) 465–516.
[12] M. Kashiwara, The crystal base and Littlemann's refined Demazure character formula, Duke Math. J. 71 (1993) 839–858.
[13] M. Kashiwara, Realizations of Crystals, Contemp. Math., vol. 325, Amer. Math. Soc., 2003, pp. 133–139.
[14] J.-A. Kim, Monomial realization of crystal graphs for $U_q(A_n^{(1)})$, Math. Ann. 332 (2005) 17–35.
[15] J.-A. Kim, D.-U. Shin, Monomial realization of crystal bases $B(\infty)$ for the quantum finite algebras, Algebr. Represent. Theory 11 (2008) 93–105.
[16] J.-A. Kim, D.-U. Shin, Generalized Young walls and crystal bases for quantum affine algebra of type A, Proc. Amer. Math. Soc. 138 (2010) 3877–3889.
[17] J.-A. Kim, D.-U. Shin, Nakajima monomials, Young walls and Kashiwara embedding for $U_q(A_n^{(1)})$, J. Algebra 330 (2011) 234–250.
[18] J.-A. Kim, D.-U. Shin, Zigzag strip bundles and crystals, J. Combin. Theory Ser. A 120 (2013) 1087–1115.
[19] J.-A. Kim, D.-U. Shin, Zigzag strip bundles and the crystal $B(\infty)$ for quantum affine algebras, Comm. Algebra (2014), in press.
[20] K.-H. Lee, P. Lombardo, B. Salisbury, Combinatorics of the Casselman–Shalika formula in type A, Proc. Amer. Math. Soc. 142 (2014) 2291–2301.
[21] H. Nakajima, t-Analogs of q-Characters of Quantum Affine Algebras of Type A_n, D_n, Contemp. Math., vol. 325, Amer. Math. Soc., 2003, pp. 141–160.
[22] T. Nakashima, Polyhedral realizations of crystal bases for integrable highest weight modules, J. Algebra 219 (1999) 571–597.

[23] T. Nakashima, A. Zelevinsky, Polyhedral realization of crystal bases for quantized Kac–Moody algebras, Adv. Math. 131 (1997) 253–278.
[24] D.-U. Shin, Polyhedral realization of crystal bases for generalized Kac–Moody algebras, J. Lond. Math. Soc. 77 (2008) 273–286.
[25] D.-U. Shin, Polyhedral realization of the highest weight crystals for generalized Kac–Moody algebras, Trans. Amer. Math. Soc. 360 (2008) 6371–6387.