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Non-Archimedean and random HUR-approximation of a Cauchy-Jensen additive mapping

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Abstract

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam-Rassias approximation (briefly, HUR-approximation) of a Cauchy-Jensen additive (briefly, CJA) functional equation in various normed spaces.

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1 Introduction

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?* If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In 1941, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. Aoki [3] proved a generalization of Hyers' theorem for additive mappings and Rassias [4] proved a generalization of Hyers' theorem for linear mappings.

Theorem 1.1 (ThM Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists, for all $x \in E$, and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p,$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

This new concept is known as a the Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [5] by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [8] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The readers are referred to [9–29] and references therein for detailed information on stability of functional equations.

In 1897, Hensel [30] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [31–35]).

Definition 1.1 By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|: \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold: (a) $|r| = 0$ if and only if $r = 0$; (b) $|rs| = |r||s|$; (c) $|r + s| \leq \max\{|r|, |s|\}$.

Clearly, by (b), $|1| = |-1| = 1$ and so, by induction, it follows from (c) that $|n| \leq 1$, for all $n \geq 1$.

Definition 1.2 Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$.

(1) A function $\|\cdot\|: X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions: (a) $\|x\| = 0$ if and only if $x = 0$, for all $x \in X$; (b) $\|rx\| = |r|\|x\|$, for all $r \in \mathbb{K}$ and $x \in X$; (c) the strong triangle inequality (ultra-metric) holds, that is, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in X$.
(2) The space $(X, \|\cdot\|)$ is called a *non-Archimedean normed space* (briefly, NAN-space).

Note that $\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\}$, for all $m, n \in \mathbb{N}$ with $n > m$.

Definition 1.3 Let $(X, \|\cdot\|)$ be a non-Archimedean normed space.

(a) A sequence $\{x_n\}$ is a *Cauchy sequence* in X if $\{x_{n+1} - x_n\}$ converges to zero in X.
(b) The non-Archimedean normed space $(X, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence in X is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists a positive integer n such that $x < ny$.

Example 1.1 Fix a prime number p. For any nonzero rational number x, there exists a unique positive integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are positive integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is a locally compact field.

In Section 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [36]. Throughout this paper, let $\triangle^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0$, $F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \triangle^+ : l^-F(-\infty) = 1\}$, where $l^-f(x) = \lim_{t\to x^-} f(t)$, is a subset of $\triangle^+$. The set $\triangle^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

We can easily show that the maximal element in $\triangle^+$ is the distribution function $H_0(t)$.

Definition 1.4 A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions: (a) T is commutative and associative; (b) T is continuous; (c) $T(x,1) = x$, for all $x \in [0,1]$; (d) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$, for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T_P(x,y) = xy$, $T_{\max}(x,y) = \max\{a+b-1,0\}$, $T_M(x,y) = \min(a,b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$, for all $n \geq 2$. $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$.

Definition 1.5 A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:

(a) $\mu_x(t) = H_0(t)$, for all $t > 0$ if and only if $x = 0$;
(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$, for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
(c) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$, for all $x,y \in X$ and $t,s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where $\mu_u(t) = \frac{t}{t+\|u\|}$, for all $t > 0$ and T_M is the minimum t-norm. This space X is called the *induced random normed space*.

If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every RN-space (X, μ, T) is a metrizable linear topological space with the topology τ (called the *μ-topology* or the *(ε, δ)-topology*, where $\varepsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\varepsilon, \lambda)\}$ of neighborhoods of θ, where

$$U(\varepsilon, \lambda) = \{x \in X : \mu_x(\varepsilon) > 1 - \lambda\}.$$

Definition 1.6 Let (X, μ, T) be an RN-space.

(a) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$, for all $t > 0$.
(b) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$, for all $t > 0$.
(c) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 1.2 *If (X, μ, T) is RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

Definition 1.7 Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(c) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Theorem 1.3 ([37, 38]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$, for all nonnegative integers n, or there exists a positive integer n_0 such that*

(a) $d(J^n x, J^{n+1}x) < \infty$, *for all $n_0 \geq n_0$;*
(b) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;*
(d) $d(y, y^*) \leq \frac{d(y,Jy)}{1-L}$, *for all $y \in Y$.*

In this paper, using the fixed point and direct methods, we prove the HUR-approximation of the following CJA functional equation:

$$2f\left(\frac{x+y+z}{2}\right) = f(x) + f(y) + f(z) \tag{1.1}$$

in various normed spaces.

2 NAN-stability

In this section, we deal with the stability problem for the Cauchy-Jensen additive functional equation (1.1) in non-Archimedean normed spaces.

Theorem 2.1 *Let X be a non-Archimedean normed space and Y is a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{\alpha\varphi(x, y, z)}{|2|}, \tag{2.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z) \right\|_Y \leq \varphi(x, y, z), \tag{2.2}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $\Im : X \to Y$ such that

$$\left\| f(x) - \Im(x) \right\|_Y \leq \alpha\varphi(x, 2x, x)\bigl(|2| - |2|\alpha\bigr)^{-1}, \tag{2.3}$$

for all $x \in X$.

Proof Putting $y = 2x$ and $z = x$ in (2.1), we get $\|f(2x) - 2f(x)\|_Y \leq \varphi(x, 2x, x)$, for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_Y \leq |2|^{-1}\alpha\varphi(x, 2x, x), \tag{2.4}$$

for all $x \in X$. Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S:

$$d(g, h) = \inf_{\mu \in (0,+\infty)} \left\|g(x) - h(x)\right\|_Y \leq \mu\varphi(x, 2x, x),$$

for all $x \in X$, where, as usual, $\inf\phi = +\infty$. It is easy to show that (S, d) is complete (see [39]). Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := 2g(\frac{x}{2})$, for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\|_Y \leq \varepsilon\varphi(x, 2x, x)$, for all $x \in X$. Hence

$$\left\|Jg(x) - Jh(x)\right\|_Y = \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\|_Y \leq \alpha \cdot \varepsilon\varphi(x, 2x, x),$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$, for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq |2|^{-1}\alpha$. By Theorem 1.3, there exists a mapping $\Im : X \to Y$ satisfying the following:

(1) $\Im$ is a fixed point of J, *i.e.*,

$$\Im(x) = 2\Im\left(\frac{x}{2}\right), \tag{2.5}$$

for all $x \in X$. The mapping $\Im$ is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that $\Im$ is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - \Im(x)\|_Y \leq \mu\varphi(x, 2x, x)$, for all $x \in X$;

(2) $d(J^n f, \Im) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \Im(x), \tag{2.6}$$

for all $x \in X$;

(3) $d(f, \Im) \leq \frac{d(f,Jf)}{1-\alpha}$, which implies the inequality $d(f, \Im) \leq \alpha(|2| - |2|\alpha)^{-1}$. This implies that the inequalities (2.3) holds. It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\|2\Im\left(\frac{x+y+z}{2}\right) - \Im(x) - \Im(y) - \Im(z)\right\|_Y \\
&\quad = \lim_{n\to\infty} \frac{\|2f(\frac{x+y+z}{2^{n+1}}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}) - f(\frac{z}{2^n})\|_Y}{|2|^{-n}} \\
&\quad \leq \lim_{n\to\infty} \alpha^n \varphi(x, y, z) = 0,
\end{aligned}$$

for all $x, y, z \in X$. So $2\Im(\frac{x+y+z}{2}) = \Im(x) + \Im(y) + \Im(z)$, for all $x, y, z \in X$. Hence $\Im : X \to Y$ is an CJA mapping and we get the desired results. □

Corollary 2.1 *Let θ be a positive real number and r is a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\left\| 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z) \right\|_Y \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right), \tag{2.7}$$

for all $x, y, z \in X$. Then there exists a unique CJA mapping $\Im : X \to Y$ such that

$$\left\|f(x) - \Im(x)\right\|_Y \leq |2|\theta\left(2 + |2|^r\right)\left(|2|^{r+1} - |2|^2\right)^{-1}\|x\|^r,$$

for all $x \in X$.

Proof The proof follows from Theorem 2.1 by taking $\varphi(x, y, z) = (\|x\|^r + \|y\|^r + \|z\|^r)$, for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{1-r}$ and we get the desired result. □

Theorem 2.2 *Let X be a non-Archimedean normed space and Y is a complete non-Archimedean space. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with $\varphi(x, y, z) \leq |2|\alpha\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$, for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying* (2.2). *Then there exists a unique CJA mapping $\Im : X \to Y$ such that*

$$\left\|f(x) - \Im(x)\right\|_Y \leq \varphi(x, 2x, x)\left(|2| - |2|\alpha\right)^{-1}, \tag{2.8}$$

for all $x \in X$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g(2x)}{2}$, for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\|_Y \leq \varepsilon\varphi(x, 2x, x)$, for all $x \in X$. Hence

$$\left\|Jg(x) - Jh(x)\right\|_Y = \left\|\frac{g(2x)}{2} - \frac{h(2x)}{2}\right\|_Y \leq \frac{|2|\alpha \cdot \varepsilon\varphi(x, 2x, x)}{|2|},$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$, for all $g, h \in S$. It follows from (2.4) that $d(f, Jf) \leq |2|^{-1}$. By Theorem 1.3, there exists a mapping $\Im : X \to Y$ satisfying the following:

(1) $\Im$ is a fixed point of J, *i.e.*,

$$\frac{\Im(2x)}{2} = \Im(x), \tag{2.9}$$

for all $x \in X$. The mapping $\Im$ is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that $\Im$ is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - \Im(x)\|_Y \leq \mu\varphi(x, 2x, x)$, for all $x \in X$;

(2) $d(J^n f, \Im) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \Im(x)$, for all $x \in X$;

(3) $d(f, \Im) \leq \frac{d(f,Jf)}{1-\alpha}$, which implies the inequality $d(f, \Im) \leq (|2| - |2|\alpha)^{-1}$. This implies that the inequalities (2.8) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2 *Let θ be a positive real number and r is a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (2.7). *Then there exists a unique CJA mapping $\Im : X \to Y$ such that*

$$\|f(x) - \Im(x)\|_Y \leq \theta(2 + |2|^r)(|2| - |2|^r)^{-1}\|x\|^r,$$

for all $x \in X$.

Proof The proof follows from Theorem 2.2 by taking $\varphi(x,y,z) = (\|x\|^r + \|y\|^r + \|z\|^r)$, for all $x, y, z \in X$. Then we can choose $\alpha = |2|^{r-1}$ and we get the desired result. □

Theorem 2.3 *Let G be an additive semigroup and X is a non-Archimedean Banach space. Assume that $\lambda : G^3 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0, \tag{2.10}$$

for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit

$$£(x) = \lim_{n\to\infty} \max_{0\leq k<n} |2|^k \lambda\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) \tag{2.11}$$

exists and $f : G \to X$ be a mapping satisfying

$$\left\|2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right\|_X \leq \lambda(x,y,z). \tag{2.12}$$

Then the limit $\Im(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists, for all $x \in G$, and defines an CJA mapping $\Im : G \to X$ such that

$$\|f(x) - \Im(x)\| \leq £(x). \tag{2.13}$$

Moreover, if $\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} |2|^k \lambda(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}) = 0$ then $\Im$ is the unique CJA mapping satisfying (2.13).

Proof Putting $y = 2x$ and $z = x$ in (2.12), we get

$$\|f(2x) - 2f(x)\|_Y \leq \lambda(x, 2x, x), \tag{2.14}$$

for all $x \in G$. Replacing x by $\frac{x}{2^{n+1}}$ in (2.14), we obtain

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\| \leq |2|^n \lambda\left(\frac{x}{2^{n+1}}, \frac{x}{2^n}, \frac{x}{2^{n+1}}\right). \tag{2.15}$$

Thus, it follows from (2.10) and (2.15) that the sequence $\{2^n f(\frac{x}{2^n})\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, it follows that $\{2^n f(\frac{x}{2^n})\}_{n\geq 1}$ is convergent. Set $\Im(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. By induction on n, one can show that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \leq \max_{0\leq k<n} |2|^k \lambda\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right), \tag{2.16}$$

for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (2.16) and using (2.11), one obtains (2.13). By (2.10) and (2.12), we get

$$\begin{aligned}
&\left\| 2\Im\left(\frac{x+y+z}{2}\right) - \Im(x) - \Im(y) - \Im(z)\right\| \\
&\quad = \lim_{n\to\infty} |2|^n \left\| 2f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&\quad \leq \lim_{n\to\infty} |2|^n \lambda\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0,
\end{aligned}$$

for all $x, y, z \in X$. So

$$\Im\left(\frac{x+y+z}{2}\right) = \Im(x) + \Im(y) + \Im(z). \tag{2.17}$$

Letting $x = y = z = 0$ in (2.17), we get $\Im(0) = 0$. Letting $z = x + y$ in (2.17), we get $\Im(x + y) = \Im(x) + \Im(y)$, for all $x, y \in X$. Hence the mapping $\Im : X \to Y$ is Cauchy additive.

To prove the uniqueness property of $\Im$, let $\Re$ be another mapping satisfying (2.13). Then we have

$$\begin{aligned}
\left\|\Im(x) - \Re(x)\right\|_X &= \lim_{n\to\infty} |2|^n \left\|\Im\left(\frac{x}{2^n}\right) - \Re\left(\frac{x}{2^n}\right)\right\|_X \\
&\leq \lim_{k\to\infty} |2|^n \max\left\{\left\|\Im\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_X, \left\|f\left(\frac{x}{2^n}\right) - \Re\left(\frac{x}{2^n}\right)\right\|_X\right\} \\
&\leq \lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} |2|^k \lambda\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) = 0,
\end{aligned}$$

for all $x \in G$. Therefore, $\Im = \Re$. This completes the proof. □

Corollary 2.3 *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying $\xi(\frac{t}{|2|}) \leq \xi(\frac{1}{|2|})\xi(t)$, $\xi(\frac{1}{|2|}) < \frac{1}{|2|}$, for all $t \geq 0$. Assume that $\kappa > 0$ and $f : G \to X$ be a mapping such that*

$$\left\| 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right\|_Y \leq \kappa\bigl(\xi(|x|) + \xi(|y|) + \xi(|z|)\bigr),$$

for all $x, y, z \in G$. Then there exists a unique CJA mapping $\Im : G \to X$ such that

$$\left\| f(x) - \Im(x)\right\| \leq |2|^{-1}\bigl(2 + |2|\bigr)\xi\bigl(|x|\bigr).$$

Proof If we define $\lambda : G^3 \to [0, \infty)$ by $\lambda(x, y, z) := \kappa(\xi(|x|) + \xi(|y|) + \xi(|z|))$, then we have $\lim_{n\to\infty} |2|^n \lambda(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$, for all $x, y, z \in G$. On the other hand, it follows that $£(x) = |2|^{-1}(2 + |2|)\xi(|x|)$ exists, for all $x \in G$. Also, we have

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} |2|^k \lambda\left(\frac{x}{2^{k+1}}, \frac{x}{2^k}, \frac{x}{2^{k+1}}\right) = \lim_{j\to\infty} |2|^j \lambda\left(\frac{x}{2^{j+1}}, \frac{x}{2^j}, \frac{x}{2^{j+1}}\right) = 0.$$

Thus, applying Theorem 2.3, we have the conclusion. This completes the proof. □

Theorem 2.4 *Let G be an additive semigroup and X is a non-Archimedean Banach space. Assume that $\lambda : G^3 \to [0, +\infty)$ be a function such that $\lim_{n\to\infty} \frac{\lambda(2^n x, 2^n y, 2^n z)}{2^n} = 0$, for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit*

$$\pounds(x) = \lim_{n\to\infty} \max_{0\leq k<n} \frac{\lambda(2^k x, 2^{k+1}x, 2^k x)}{|2|^k} \tag{2.18}$$

exists and $f : G \to X$ be a mapping satisfying (2.12). *Then the limit $\Im(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists, for all $x \in G$, and*

$$\left\|f(x) - \Im(x)\right\| \leq \frac{\pounds(x)}{|2|}, \tag{2.19}$$

for all $x \in G$. Moreover, if $\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} \frac{\lambda(2^k x, 2^{k+1}x, 2^k x)}{|2|^k} = 0$, then $\Im$ is the unique CJA mapping satisfying (2.19).

Proof It follows from (2.14) that

$$\left\|f(x) - \frac{f(2x)}{2}\right\|_X \leq \frac{\lambda(x, 2x, x)}{|2|}, \tag{2.20}$$

for all $x \in G$. Replacing x by $2^n x$ in (2.20), we obtain

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right\|_X \leq \frac{\lambda(2^n x, 2^{n+1}x, 2^n x)}{|2|^{n+1}}. \tag{2.21}$$

Thus it follows from (2.21) that the sequence $\{\frac{f(2^n x)}{2^n}\}_{n\geq 1}$ is convergent. Set $\Im(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$. On the other hand, it follows from (2.21) that

$$\begin{aligned}\left\|\frac{f(2^p x)}{2^p} - \frac{f(2^q x)}{2^q}\right\| &= \left\|\sum_{k=p}^{q-1} \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\| \leq \max_{p\leq k<q}\left\{\left\|\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\|\right\} \\ &\leq \frac{1}{|2|} \max_{p\leq k<q} \frac{\lambda(2^k x, 2^{k+1}x, 2^k x)}{|2|^k},\end{aligned}$$

for all $x \in G$ and $p, q \geq 0$ with $q > p \geq 0$. Letting $p = 0$, taking $q \to \infty$ in the last inequality and using (2.18), we obtain (2.19).

The rest of the proof is similar to the proof of Theorem 2.3. This completes the proof. □

Corollary 2.4 *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying $\xi(|2|t) \leq \xi(|2|)\xi(t)$, $\xi(|2|) < |2|$, for all $t \geq 0$. Let $\kappa > 0$ and $f : G \to X$ be a mapping satisfying*

$$\left\|2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right\| \leq \kappa\left(\xi\left(|x|\right) \cdot \xi\left(|y|\right) \cdot \xi\left(|z|\right)\right),$$

for all $x, y, z \in G$. Then there exists a unique CJA mapping $\Im : G \to X$ such that

$$\left\|f(x) - \Im(x)\right\| \leq \kappa\xi\left(|x|\right)^3.$$

Proof If we define $\lambda : G^3 \to [0,\infty)$ by $\lambda(x,y,z) := \kappa(\xi(|x|)\cdot\xi(|y|)\cdot\xi(|z|))$ and apply Theorem 2.4, then we get the conclusion. $\square$

3 RNS-stability

In this section, using the fixed point and direct methods, we prove the HUR-approximation of the functional equation (1.1) in random normed spaces.

Theorem 3.1 *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\varphi : X^3 \to Z$ be a function such that there exists $0 < \alpha < \frac{1}{2}$ such that*

$$\mu'_{\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2})}(t) \ge \mu'_{\varphi(x,y,z)}\left(\frac{t}{\alpha}\right), \tag{3.1}$$

for all $x,y,z \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})}(\frac{t}{2^n}) = 1$, for all $x,y,z \in X$ and $t > 0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f : X \to Y$ be a mapping such that

$$\mu_{2f(\frac{x+y+z}{2})-f(x)-f(y)-f(z)}(t) \ge \mu'_{\varphi(x,y,z)}(t), \tag{3.2}$$

for all $x,y,z \in X$ and $t > 0$. Then the limit $\Im(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists, for all $x \in X$, and defines a unique CJA mapping $\Im : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \ge \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right), \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof Putting $y = 2x$ and $z = x$ in (3.2), we see that

$$\mu_{f(2x)-2f(x)}(t) \ge \mu'_{\varphi(x,2x,x)}(t). \tag{3.4}$$

Replacing x by $\frac{x}{2}$ in (3.4), we obtain

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \ge \mu'_{\varphi(\frac{x}{2},x,\frac{x}{2})}(t) \ge \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\alpha}\right), \tag{3.5}$$

for all $x \in X$. Replacing x by $\frac{x}{2^n}$ in (3.5) and using (3.1), we obtain

$$\mu_{2^{n+1}f(\frac{x}{2^{n+1}})-2^n f(\frac{x}{2^n})}(t) \ge \mu'_{\varphi(\frac{x}{2^{n+1}},\frac{x}{2^n},\frac{x}{2^{n+1}})}\left(\frac{t}{2^n}\right) \ge \mu'_{\varphi(x,2x,x)}\left(\frac{t}{2^n\alpha^{n+1}}\right)$$

and so

$$\begin{aligned}
\mu_{2^n f(\frac{x}{2^n})-f(x)}\left(\sum_{k=0}^{n-1} 2^k\alpha^{k+1}t\right) &= \mu_{\sum_{k=0}^{n-1} 2^{k+1}f(\frac{x}{2^{k+1}})-2^k f(\frac{x}{2^k})}\left(\sum_{k=0}^{n-1} 2^k\alpha^{k+1}t\right) \\
&\ge T_{k=0}^{n-1}\left(\mu_{2^{k+1}f(\frac{x}{2^{k+1}})-2^k f(\frac{x}{2^k})}\left(2^k\alpha^{k+1}t\right)\right) \\
&\ge T_{k=0}^{n-1}\left(\mu'_{\varphi(x,2x,x)}(t)\right) = \mu'_{\varphi(x,2x,x)}(t).
\end{aligned}$$

This implies that

$$\mu_{2^n f(\frac{x}{2^n})-f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right). \tag{3.6}$$

Replacing x by $\frac{x}{2^p}$ in (3.6), we obtain

$$\mu_{2^{n+p} f(\frac{x}{2^{n+p}})-2^p f(\frac{x}{2^p})}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=p}^{n+p-1} 2^k \alpha^{k+1}}\right). \tag{3.7}$$

Since $\lim_{p,n\to\infty} \mu'_{\varphi(x,2x,x)}(\frac{t}{\sum_{k=p}^{n+p-1} 2^k \alpha^{k+1}}) = 1$, it follows that $\{2^n f(\frac{x}{2^n})\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RN-space $(Y, \mu, \min)$ and so there exists a point $\Im(x) \in Y$ such that $\lim_{n\to\infty} 2^n f(\frac{x}{2^n}) = \Im(x)$. Fix $x \in X$ and put $p = 0$ in (3.7) and so, for any $\varepsilon > 0$,

$$\mu_{\Im(x)-f(x)}(t+\varepsilon) \geq T\left(\mu_{\Im(x)-2^n f(\frac{x}{2^n})}(\varepsilon), \mu'_{\varphi(x,2x,x)}\left(\frac{t}{\sum_{k=0}^{n-1} 2^k \alpha^{k+1}}\right)\right). \tag{3.8}$$

Taking $n \to \infty$ in (3.8), we get $\mu_{\Im(x)-f(x)}(t+\varepsilon) \geq \mu'_{\varphi(x,2x,x)}(\frac{(1-2\alpha)t}{\alpha})$. Since ε is arbitrary, by taking $\varepsilon \to 0$ in the previous inequality, we get

$$\mu_{\Im(x)-f(x)}(t) \geq \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{\alpha}\right).$$

Replacing x, y and z by $\frac{x}{2^n}$, $\frac{y}{2^n}$ and $\frac{z}{2^n}$ in (3.2), respectively, we get

$$\mu_{2^{n+1} f(\frac{x+y+z}{2^{n+1}})-2^n f(\frac{x}{2^n})-2^n f(\frac{y}{2^n})-2^n f(\frac{z}{2^n})}(t) \geq \mu'_{\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})}\left(\frac{t}{2^n}\right),$$

for all $x, y, z \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\varphi(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})}(\frac{t}{2^n}) = 1$, we conclude that $\Im$ satisfies (1.1). On the other hand

$$2\Im\left(\frac{x}{2}\right) - \Im(x) = \lim_{n\to\infty} 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = 0.$$

This implies that $\Im : X \to Y$ is an CJA mapping. To prove the uniqueness of the CJA mapping $\Im$, assume that there exists another CJA mapping $\Re : X \to Y$ which satisfies (3.3). Then we have

$$\begin{aligned}
\mu_{\Im(x)-\Re(x)}(t) &= \lim_{n\to\infty} \mu_{2^n \Im(\frac{x}{2^n})-2^n \Re(\frac{x}{2^n})}(t) \\
&\geq \lim_{n\to\infty} \min\left\{\mu_{2^n \Im(\frac{x}{2^n})-2^n f(\frac{x}{2^n})}\left(\frac{t}{2}\right), \mu_{2^n f(\frac{x}{2^n})-2^n \Re(\frac{x}{2^n})}\left(\frac{t}{2}\right)\right\} \\
&\geq \lim_{n\to\infty} \mu'_{\varphi(\frac{x}{2^n},\frac{2x}{2^n},\frac{x}{2^n})}\left(\frac{(1-2\alpha)t}{2^n}\right) \geq \lim_{n\to\infty} \mu'_{\varphi(x,2x,x)}\left(\frac{(1-2\alpha)t}{2^n \alpha^n}\right).
\end{aligned}$$

Since $\lim_{n\to\infty} \mu'_{\varphi(x,2x,x)}(\frac{(1-2\alpha)t}{2^n \alpha^n}) = 1$. Therefore, we have $\mu_{\Im(x)-\Re(x)}(t) = 1$, for all $t > 0$, and so $\Im(x) = \Re(x)$. This completes the proof. □

Corollary 3.1 *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r be a positive real number with $r>1$, $z_0\in Z$ and $f:X\to Y$ be a mapping satisfying*

$$\mu_{2f(\frac{x+y+z}{2})-f(x)-f(y)-f(z)}(t)\geq\mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t),\tag{3.9}$$

for all $x,y\in X$ and $t>0$. Then the limit $\Im(x)=\lim_{n\to\infty}2^nf(\frac{x}{2^n})$ exists, for all $x\in X$, and defines a unique CJA mapping $\Im:X\to Y$ such that

$$\mu_{f(x)-\Im(x)}(t)\geq\mu'_{\|x\|^pz_0}\left(\frac{(2^r-2)t}{2^r+2}\right),$$

for all $x\in X$ and $t>0$.

Proof Let $\alpha=2^{-r}$ and $\varphi:X^3\to Z$ be a mapping defined by $\varphi(x,y,z)=(\|x\|^r+\|y\|^r+\|z\|^r)z_0$. Then, from Theorem 3.1, the conclusion follows. □

Theorem 3.2 *Let X be a real linear space, $(Z,\mu',\min)$ be an RN-space and $\varphi:X^3\to Z$ be a function such that there exists $0<\alpha<2$ such that $\mu'_{\varphi(2x,2y,2z)}(t)\geq\mu'_{\alpha\varphi(x,y,z)}(t)$, for all $x\in X$ and $t>0$, and*

$$\lim_{n\to\infty}\mu'_{\varphi(2^nx,2^ny,2^nz)}\bigl(2^nt\bigr)=1,$$

for all $x,y,z\in X$ and $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f:X\to Y$ be a mapping satisfying (3.2). *Then the limit $\Im(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ exists, for all $x\in X$, and defines a unique CJA mapping $\Im:X\to Y$ such that*

$$\mu_{f(x)-\Im(x)}(t)\geq\mu'_{\varphi(x,2x,x)}\bigl((2-\alpha)t\bigr),\tag{3.10}$$

for all $x\in X$ and $t>0$.

Proof It follows from (3.4) that

$$\mu_{\frac{f(2x)}{2}-f(x)}(t)\geq\mu'_{\varphi(x,2x,x)}(2t).\tag{3.11}$$

Replacing x by 2^nx in (3.11), we obtain

$$\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}}(t)\geq\mu'_{\varphi(2^nx,2^{n+1}x,2^nx)}\bigl(2^{n+1}t\bigr)\geq\mu_{\varphi(x,2x,x)}\left(\frac{2^{n+1}t}{\alpha^n}\right).$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.2 *Let X be a real normed linear space, $(Z,\mu',\min)$ be an RN-space and $(Y,\mu,\min)$ be a complete RN-space. Let r be a positive real number with $0<r<1$, $z_0\in Z$ and $f:X\to Y$ be a mapping satisfying* (3.9). *Then the limit $\Im(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ exists, for all $x\in X$, and defines a unique CJA mapping $\Im:X\to Y$ such that*

$$\mu_{f(x)-\Im(x)}(t)\geq\mu'_{\|x\|^pz_0}\left(\frac{(2-2^r)t}{2^r+2}\right),$$

for all $x\in X$ and $t>0$.

Proof Let $\alpha = 2^r$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x,y,z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, from Theorem 3.2, the conclusion follows. $\square$

Theorem 3.3 *Let X be a linear space, (Y,μ,T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y,z}$) such that there exists $0 < \alpha < \frac{1}{2}$ such that*

$$\Phi_{2x,2y,2z}(t) \le \Phi_{x,y,z}(\alpha t), \tag{3.12}$$

for all $x,y,z \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying

$$\mu_{2f(\frac{x+y+z}{2})f(x)-f(y)-f(z)}(t) \ge \Phi_{x,y,z}(t), \tag{3.13}$$

for all $x,y,z \in X$ and $t > 0$. Then, for all $x \in X$, $\Im(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists and $\Im : X \to Y$ is a unique CJA mapping such that

$$\mu_{f(x)-\Im(x)}(t) \ge \Phi_{x,2x,x}\left(\frac{(1-2\alpha)t}{\alpha}\right), \tag{3.14}$$

for all $x \in X$ and $t > 0$.

Proof Putting $y = 2x$ and $z = x$ in (3.13), we have

$$\mu_{2f(\frac{x}{2})-f(x)}(t) \ge \Phi_{x,2x,x}\left(\frac{t}{\alpha}\right), \tag{3.15}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf_{u\in(0,\infty)} \left\{\mu_{g(x)-h(x)}(ut) \ge \Phi_{x,2x,x}(t), \forall x \in X, t > 0\right\}, \tag{3.16}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see [39], Lemma 2.1). Now, we consider a linear mapping $J : (S,d) \to (S,d)$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right), \tag{3.17}$$

for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant 2α. In fact, let $g,h \in S$ be such that $d(g,h) < \varepsilon$. Then we have $\mu_{g(x)-h(x)}(\varepsilon t) \ge \Phi_{x,2x,x}(t)$, for all $x \in X$ and $t > 0$, and so

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(2\alpha\varepsilon t) &= \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(2\alpha\varepsilon t) = \mu_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha\varepsilon t) \\ &\ge \Phi_{\frac{x}{2},x,\frac{x}{2}}(\alpha t) \\ &\ge \Phi_{x,2x,x}(t), \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) < \varepsilon$ implies that $d(Jg,Jh) < 2\alpha\varepsilon$. This means that $d(Jg,Jh) \le 2\alpha d(g,h)$, for all $g,h \in S$. It follows from (3.15) that $d(f,Jf) \le \alpha$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $\Im$ is a fixed point of J, that is,

$$\Im\left(\frac{x}{2}\right) = \frac{1}{2}\Im(x), \tag{3.18}$$

for all $x \in X$. The mapping $\Im$ is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that $\Im$ is a unique mapping satisfying (3.18) such that there exists $u \in (0,\infty)$ satisfying $\mu_{f(x)-\Im(x)}(ut) \geq \Phi_{x,2x,x}(t)$, for all $x \in X$ and $t > 0$.

(2) $d(J^n f, \Im) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} 2^n f(\frac{x}{2^n}) = \Im(x)$, for all $x \in X$.

(3) $d(f,\Im) \leq \frac{d(f,Jf)}{1-2\alpha}$ with $f \in \Omega$, which implies the inequality $d(f,\Im) \leq \frac{\alpha}{1-2\alpha}$ and so

$$\mu_{f(x)-\Im(x)}(t) \geq \Phi_{x,2x,x}\left(\frac{(1-2\alpha)t}{\alpha}\right),$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.14) holds. On the other hand

$$\mu_{2^{n+1}f(\frac{x+y+z}{2^{n+1}})-2^n f(\frac{x}{2^n})-2^n f(\frac{y}{2^n})-2^n f(\frac{z}{2^n})}(t) \geq \Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}\left(\frac{t}{2^n}\right),$$

for all $x,y,z \in X$, $t > 0$ and $n \geq 1$. By (3.12), we know that $\Phi_{\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}}(\frac{t}{2^n}) \geq \Phi_{x,y,z}(\frac{t}{(2\alpha)^n})$. Since $\lim_{n\to\infty}\Phi_{x,y,z}(\frac{t}{(2\alpha)^n}) = 1$, for all $x,y,z \in X$ and $t > 0$, we have $\mu_{2\Im(\frac{x+y+z}{2})-\Im(x)-\Im(y)-\Im(z)}(t) = 1$, for all $x,y,z \in X$ and $t > 0$. Thus the mapping $\Im : X \to Y$ satisfying (1.1). Furthermore

$$\begin{aligned}
\Im(2x) - 2\Im(x) &= \lim_{n\to\infty} 2^n f\left(\frac{x}{2^{n-1}}\right) - 2\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \\
&= 2\left[\lim_{n\to\infty} 2^{n-1} f\left(\frac{x}{2^{n-1}}\right) - \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)\right] \\
&= 0.
\end{aligned}$$

This completes the proof. □

Corollary 3.3 *Let X be a real normed space, $\theta \geq 0$ and r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{2f(\frac{x+y+z}{2})-f(x)-f(y)-f(z)}(t) \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}, \tag{3.19}$$

for all $x,y,z \in X$ and $t > 0$. Then $\Im(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists, for all $x \in X$, and $\Im : X \to Y$ is a unique CJA mapping such that

$$\mu_{f(x)-\Im(x)}(t) \geq \frac{(2^r-2)t}{(2^r-2)t + (2^r+2)\theta\|x\|^r},$$

for all $x \in X$ and $t > 0$.

Proof The proof follows from Theorem 3.3 if we take $\Phi_{x,y,z}(t) = \frac{t}{t+\theta(\|x\|^r+\|y\|^r+\|z\|^r)}$, for all $x,y,z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^{-r}$, then we get the desired result. □

Theorem 3.4 *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^3 to D^+ ($\Phi(x,y,z)$ is denoted by $\Phi_{x,y,z}$) such that for some $0 < \alpha < 2$, $\Phi_{\frac{x}{2},\frac{y}{2},\frac{z}{2}}(t) \leq \Phi_{x,y,z}(\alpha t)$, for all $x, y, z \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying* (3.13). *Then the limit $\Im(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists, for all $x \in X$, and $\Im : X \to Y$ is a unique CJA mapping such that*

$$\mu_{f(x)-\Im(x)}(t) \geq \Phi_{x,2x,x}\bigl((2-\alpha)t\bigr), \tag{3.20}$$

for all $x \in X$ and $t > 0$.

Proof Putting $y = 2x$ and $z = x$ in (3.13), we have

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq \Phi_{x,2x,x}(2t), \tag{3.21}$$

for all $x \in X$ and $t > 0$. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1. Now, we consider a linear mapping $J : (S, d) \to (S, d)$ such that $Jh(x) := \frac{1}{2}h(2x)$, for all $x \in X$. It follows from (3.21) that $d(f, Jf) \leq \frac{1}{2}$. By Theorem 1.3, there exists a mapping $\Im : X \to Y$ satisfying the following:

(1) $\Im$ is a fixed point of J, that is,

$$\Im(2x) = 2\Im(x), \tag{3.22}$$

for all $x \in X$. The mapping $\Im$ is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that $\Im$ is a unique mapping satisfying (3.22) such that there exists $u \in (0, \infty)$ satisfying $\mu_{f(x)-\Im(x)}(ut) \geq \Phi_{x,2x,x}(t)$, for all $x \in X$ and $t > 0$.

(2) $d(J^n f, \Im) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \Im(x),$$

for all $x \in X$.

(3) $d(f, \Im) \leq \frac{d(f,Jf)}{1-\frac{\alpha}{2}}$ with $f \in \Omega$, which implies the inequality $\mu_{f(x)-\Im(x)}(\frac{t}{2-\alpha}) \geq \Phi_{x,2x,x}(t)$, for all $x \in X$ and $t > 0$. This implies that the inequality (3.20) holds. The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.4 *Let X be a real normed space, $\theta \geq 0$ and r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping satisfying* (3.19). *Then the limit $\Im(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists, for all $x \in X$, and $\Im : X \to Y$ is a unique CJA mapping such that*

$$\mu_{f(x)-\Im(x)}(t) \geq \frac{(2-2^r)t}{(2-2^r)t + (2^r+2)\theta\|x\|^r},$$

for all $x \in X$ and $t > 0$.

Proof The proof follows from Theorem 3.4 if we take $\Phi_{x,y,z}(t) = \frac{t}{t+\theta(\|x\|^r+\|y\|^r+\|z\|^r)}$, for all $x, y, z \in X$ and $t > 0$. In fact, if we choose $\alpha = 2^r$, then we get the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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