Boundary conditions for SU(2) Yang-Mills on AdS_4

Dileep P. Jatkar^{a1} and Jae-Hyuk Oh^{a2}

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, India^a

Abstract

We consider SU(2) Yang-Mills theory on AdS_4 by imposing various boundary conditions, which correspond to non-trivial deformations of its boundary CFT. We obtain classical solutions of Yang-Mills fields up to the first subleading order correction by using small amplitude expansion of the gauge field without considering gravitational back reaction. We also consider SU(2) Yang-Mills instanton solution in AdS_4 bulk, and propose a boundary action. It turns out that the boundary theory is the Chern Simons theory with a non-local deformation which has the form similar to the Wilson line. In the limit of the deformation parameter $\rho \to \infty$, this non-local deformation is suppressed and the boundary theory becomes pure Chern Simons. For large but finite values of ρ , this non-local deformation can be treated perturbatively within the Chern-Simon theory.

¹e-mail:dileep@hri.res.in

²e-mail:jack.jaehyuk.oh@gmail.com

Contents

1	Introduction	1
2	$SU(2)$ Yang-Mills on AdS_4 and its Solution	4
3	Boundary Conditions and the Effective Action3.1Boundary Deformations in the First Order in ε 3.2Boundary Deformation in the Second Order in ε	8 11 15
4	"Approximate" Electric-Magnetic Duality in SU(2) Yang-Mills in AdS_4	19
5	Yang-Mills Instanton	21
6	Conclusion	23
\mathbf{A}	Gauge Transformation	25
в	Evaluation of the Second order Bulk Solution	25
\mathbf{C}	Bulk Solutions in the Position Space	27
D	Yang-Mills Instantons in Radial Gauge	27

1 Introduction

"Alternate quantization" was first studied by Breitenlohner and Freedman[1] in the context of compactifications of supergravity theories to anti-de Sitter space. In the wake of developments in AdS/CFT correspondence, there has been a renewed interest in it. Klebanov and Witten[2] first discussed it in AdS/CFT, providing interesting boundary conformal field theories generated by the alternate scheme[3, 4]. In a certain window of the conformal dimensions of the boundary composite operators corresponding the bulk excitations in AdS_{d+1} , there are two possible quantization schemes for their boundary CFTs.

For massive scalar fields in AdS_{d+1} , in the window of their masses as $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$, there are two possible quantization schemes, so two possible boundary conformal field theories, which are called Δ_+ -theory and Δ_- -theory, where $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$, which is conformal dimension of the boundary operator in each theory [2]. In this window, both CFTs are above the unitarity bound, $\Delta_{\pm} \geq \frac{d}{2} - 1$, so their two point correlators are positive definite in the position space. In Δ_+ -theory, near AdS boundary expansion of the bulk scalar fields is $\phi(r, x^{\mu})|_{r\to 0} = \phi_0(x^{\mu})r^{d-\Delta_+} + A(x^{\mu})r^{\Delta_+}$. It is well known that $A(x^{\mu})$ corresponds to certain composite operator in the CFT and $\phi_0(x^{\mu})$ corresponds to the external source coupled to it. In fact, Δ_+ -theory is not independent from Δ_- -theory. They are related to each other by Legendre transform. This Legendre transform switches the role of $A(x^{\mu})$ and $\phi_0(x^{\mu})$ in Δ_- -theory, because they are canonical conjugates of each other. In the dual gravity theory, two possible boundary CFTs can be obtained by imposing different boundary conditions. Boundary conditions for bulk fields and corresponding boundary terms in the AdS/CFT context has been studied by various authors in the past. For Dirac fields it was studied by Henneaux [5], whereas for Rarita-Schwinger fields it was analysed in [6]. This issue in case of inequivalent quantization was addressed by [3, 7] and in Lorentzian AdS/CFT case was dealt with in [8]. For Δ_+ -theory, the corresponding boundary condition is the Dirichlet boundary condition as $\delta\phi_0(x^{\mu}) = 0$. Dirichlet boundary condition is the usual boundary condition in AdS/CFT context. Since the Δ_+ -theory is unitary even when $m^2 \geq -\frac{d^2}{4} + 1$, Dirichlet boundary condition is always a possible boundary condition. The Δ_- theory can be obtained by imposing Neumann boundary condition, $\delta A(x^{\mu}) = 0$. This Neumann boundary condition is obtained by adding boundary term $S_{bdy} \sim \int \phi(x^{\mu})A(x^{\mu})d^dx$ at r = 0. Adding such a term in turn generates the same effect as performing the Legendre transform of the Δ_+ -theory, therefore such a boundary term takes the Δ_+ -theory to the Δ_- -theory.

The Neumann boundary condition can be generalized by deforming the boundary CFT by adding a general form of S_{bdy} . Such a boundary action can be an arbitrary function of $\phi_0(x^{\mu})$ and $A(x^{\mu})$. By adding the boundary action, one can obtain the "on-shell action" $I_{os} = S_{bulk} + S_{bdy}$, where S_{bulk} is the boundary contributions from the bulk action. The boundary condition is obtained by performing functional variation of the on-shell action and setting it to zero, $\delta I_{os} = 0$. This corresponds to saddle point of the on-shell action, which is the classical vacuum of the boundary theory.

An interesting example with general deformations is the conformally coupled scalar field theory in $AdS_4[9]$. In [9] authors consider a massless scalar field theory with $\lambda \phi^4$ and $\frac{1}{6}R\phi^2$ interactions, where ϕ is the scalar field, λ is the quartic self-coupling of the scalar field and Ris curvature scalar of AdS_4 . The boundary theory corresponding to the conformally coupled bulk scalar field contains a triple trace deformation term $S_{bdy} \sim \alpha \int d^3x \phi_0^3(x^{\mu})$, where α is a numerical parameter and ϕ_0 is the boundary value of the scalar field ϕ . Under the field redefinition $\phi_0 = \varphi^2$ and truncation up to the second order in small derivative expansion, the boundary on-shell action takes the canonical form with φ^6 coupling[10].

 Δ_{\pm} -theories for U(1) vector fields in AdS_4 are well-defined [11, 12, 13] in which $\Delta_+ = 2$ and $\Delta_- = 1$. The unitarity bound for the vector like local observables in *d*-dimensional CFTis $\Delta \geq d - 1[14]$, so $\Delta_+ = 2$ theory is unitary when d = 3. It also would imply that the $\Delta_- = 1$ theory does not satisfy the unitarity bound. One way to interpret the dual operator with conformal dimension '1' (the conjugate of source term from the bulk action) is as the U(1) vector gauge field in the boundary theory. Clearly there is an ambiguity in defining this operator due to non-invariance of it under gauge transformations but instead one interprets it as an observable which is not local[15]. The field strength constructed out of this gauge fields is in fact a local observable. It also resolves the apparent contradiction with the unitarity bound because the field strength has conformal dimension 2, which satisfies the unitarity bound. There are many possible boundary deformations which provide interesting on-shell actions. In [13], the authors consider "massive deformation" from which they derive the on-shell action to be the massive gauge field action. In the case that self-duality condition together with massive deformation, one obtain massive Chern Simons boundary on-shell action[16].

In this paper, we extend the discussion to non-abelian gauge theory in AdS_4 . Before summarizing our main result, let us briefly explain our motivations. The motivations are

three folds. First, Dirichlet and Neumann boundary conditions in abelian gauge field theory in the bulk correspond to free CFT. The most natural way to introduce interactions is to consider Yang-Mills theory. As will be shown, this will give non-trivial momentum dependent interactions in the boundary action for Dirichlet and Neumann boundary conditions. We will discuss generalizations of other deformations which in the abelian context were considered in [13]. Second, it is well-known that abelian gauge field theory action in 4-dimension is manifestly invariant under electric-magnetic duality, and it is also successfully embodied in AdS space[13]. We want to extend this duality to Yang-Mills. In [28], it is reported that when one retains cubic order interactions only, one can implement electric-magnetic duality in Yang-Mills system up to that order, even if that is not possible to construct with quartic interaction terms. In fact, this symmetry is not manifest electric-magnetic duality since it turns out that the variation of electric field is not proportional to magnetic field, but it is the most natural extension of the abelian duality. We will discuss how this symmetry is embodied in AdS space. Finally, Yang-Mills theory on $AdS_4(U(1))$ gauge theory too) should be the same with that in the half of 4-dimension flat space through Weyl scaling of the AdS_4 metric. Therefore, SU(2) Yang-Mills instanton solution in \mathbb{R}^4 can easily be adapted for AdS_4 as well. As $r \to 0$, Yang-Mills instanton has non-trivial boundary value whereas near Poincare horizon $r \to \infty$ it becomes a pure gauge solution. Therefore, Yang-Mills instanton definitely changes the boundary condition on the AdS_4 boundary and exploring implications of these boundary condition is interesting.

In our study, we develop boundary deformations with certain reasonable boundary conditions derived from perturbative and non-perturbative bulk solutions. We briefly list the results here. For perturbative approach, we solve bulk Yang-Mills equations of motion in power expansion order by order in small amplitudes of Yang-Mills fields. To retain the leading interactions in Yang-Mills coupling g, we obtain the bulk solutions up to first subleading order corrections in the small amplitudes. Up to this order, we can account for cubic interactions in the boundary on-shell actions(also its dual CFT actions). In the Dirichlet boundary condition case(in which case, the boundary source becomes $A_i^{a(0)}$, the boundary value of Yang-Mills fields), the boundary on-shell action I_{os}^D gives rise to the propagator which is proportional to absolute value of the 3-momenta, |q|, and exotic 3-momenta dependent cubic interactions as

$$\Delta_{ijk}^{D,abc}(q,l,p) \sim ig\epsilon^{abc}\delta^3(q+p+l)\frac{(l-q)_k\delta_{ij}+(p-l)_i\delta_{jk}+(q-p)_j\delta_{ik}}{2(|q|+|p|+|l|)},$$
(1.1)

where $\Delta_{ijk}^{D,abc}(q,l,p)$ are 3-point function on the boundary CFT a,b,c are SU(2) gauge indices and q_i , l_i and p_i are 3-momenta along the boundary directions with the boundary spacetime indices i, j, k = 1, 2, 3. For the Neumann boundary condition(the source becomes $-A_i^{a(1)}$, which is the boundary value of the canonical momentum of Yang-Mills fields), the propagator is proportional to $\frac{1}{|q|}$ and the 3-point function is given by

$$\Delta_{ijk}^{N,abc}(q,l,p) \sim \frac{\Delta_{ijk}^{D,abc}(q,l,p)}{|q||p||l|}.$$
(1.2)

The most interesting cases are massive and self-dual boundary conditions. The massive boundary condition is written as

$$(\sqrt{-\nabla^2} - m)A_i^{a(0)}(x^i) = 0, \qquad (1.3)$$

where m is a mass dimension 1 constant. Once one applies this boundary condition to the bulk fields, then the boundary theory becomes massive gauge field theory. In case of the self-dual boundary condition we do small r expansion of the self-duality condition,

$$F^a_{MN} = \frac{1}{2} \epsilon_{MNPQ} F^a_{PQ}, \qquad (1.4)$$

which is given by

$$A_i^{a(1)} = \frac{1}{2} \epsilon_{ijk} F_{jk}^a.$$
 (1.5)

Imposing these two different boundary conditions on the bulk fields, we get non-abelian massive Chern Simons gauge theory on the 3-dimensional boundary.

We also explore the non-perturbative solutions from the bulk theory. We consider Yang-Mills instanton with its winding number 1. It turns out that one of the possible boundary conditions that we can impose results in the boundary term consisting of the Chern Simon action with some non-local deformations. In this non-local deformation contains a line integration of the form

$$\sim e^{\int_0^z \epsilon^{ai} j A_i^{a(0)}(\tilde{z}) d\tilde{z}^j}.$$
(1.6)

This kind of non-local interactions never comes out from any perturbative deformations, which can be the genuine properties from the bulk instanton backgrounds.

This note is organized as follows. In Sec.2, the bulk equations of motion of Yang-Mills fields are solved and its perturbative solutions are obtained up to the first subleading order corrections in the small amplitudes of Yang-Mills fields. In Sec.3, we perform various boundary deformations of the on-shell actions and obtain some interesting boundary actions. In Sec.4, we explore implications of approximate electric-magnetic duality in the context of SU(2) Yang-Mills theory in AdS_4 . In Sec.5, we explore Yang-Mills instanton solutions and their boundary conditions. In the conclusion section we summarise our results and discuss their implications. Some technical details are presented in appendices.

2 SU(2) Yang-Mills on AdS_4 and its Solution

We start with the SU(2) Yang-Mills(Euclidean) action in AdS_4 space-time background as

$$S[A] = \frac{1}{4} \int d^4x \sqrt{G} F^a_{MN} F^{aMN}, \qquad (2.1)$$

where the space-time indices M, N run from 1 to 4 and the gauge indices a does from 1 to 3. The background metric, G is

$$ds^{2} = G_{MN} dx^{M} dx^{N} = \frac{dr^{2} + \sum_{i=1}^{3} dx^{i} dx^{i}}{r^{2}},$$
(2.2)

where we define the radial coordinate r as $r \equiv x^4$, the indices i, j. are defined boundary space-time coordinate, which run from 1 to 3. Yang-Mills field strength is given by

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M - g \epsilon^{abc} A^b_M A^c_N.$$
(2.3)

One interesting feature of this system is that the Weyl rescaling of background metric, $ds^2 \rightarrow r^2 ds^2$, maps the Yang-Mills theory in AdS_4 to that defined in 4-dimensional flat spacetime. The space-time is a half of \mathbb{R}^4 , because the radial coordinate in AdS space runs from 0 to ∞ . Therefore, the Yang-Mills action becomes

$$S[A] = \frac{1}{4} \int_{\mathbb{R}^4_+} d^4 x F^a_{MN} F^{aMN}, \qquad (2.4)$$

where the space-time indices are contracted with δ_{MN} and \mathbb{R}^4_+ denotes a half of the 4-dimensional flat space.

In this section, we evaluate bulk equations of motion and obtain their solutions with a power series expansion in small amplitude of Yang-Mills fields (This expansion is effectively the same as the Yang-Mills coupling g expansion). We would solve Yang-Mills equations up to the first sub-leading order corrections to take into account effects of interaction terms in Yang-Mills action. Up to this order, only cubic interactions terms participate. The equations of motion are given by

$$0 = \mathcal{D}_M F^a_{MN} = \partial_M F^a_{MN} + g \epsilon^{abc} F^b_{MN} A^c_M, \qquad (2.5)$$

where \mathcal{D}_M is the gauge covariant derivative. To evaluate the perturbative equations of motion, we expand Yang-Mills field as

$$A_M^a = \varepsilon \bar{A}_M^a + \varepsilon^2 \tilde{A}_M^a + O(\varepsilon^3).$$
(2.6)

where ε is a book keeping parameter for the expansion, which is a dimensionless small number. The equations of motion are evaluated for each order in ε as

First Order : 0 =
$$\partial_M (\partial_M \bar{A}^a_N - \partial_N \bar{A}^a_M),$$
 (2.7)

Second Order : 0 =
$$\partial_M (\partial_M \tilde{A}^a_N - \partial_N \tilde{A}^a_M) - g \epsilon^{abc} \left(\partial_M (\bar{A}^b_M \bar{A}^c_N) - (\partial_M \bar{A}^b_N - \partial_N \bar{A}^b_M) \bar{A}^c_M \right),$$
 (2.8)

and so on. We start with the first order equations in $O(\varepsilon)$ are given by

$$0 = \nabla^2 \bar{A}^a_r - \partial_r \partial_i \bar{A}^a_i, \tag{2.9}$$

$$0 = (\partial_r^2 + \nabla^2)\bar{A}_i^a - \partial_i(\partial_r\bar{A}_r^a + \partial_j\bar{A}_j^a), \qquad (2.10)$$

where we split the indices M, N.. into r and i, j.. and $\nabla^2 \equiv \sum_{j=1}^3 \partial_j \partial_j$. At this order, equations of motion are identical to 3 copies of U(1) gauge theory equations. Solutions to these equations has already been obtained in [13] (See Sec.2 and Appendix.B in it). We briefly list the leading order solution \bar{A}^a_M , in momentum space as

$$\bar{A}^{a}_{i,q}(r) = \bar{A}^{aT}_{i,q}(r) - iq_i\bar{\phi}^{a}_q(r), \quad \bar{A}^{a}_{r,q}(r) = \partial_r\bar{\phi}^{a}_q(r), \quad q_i\bar{A}^{aT}_{i,q}(r) = 0,$$
and
$$\bar{A}^{aT}_{i,q}(r) = \bar{A}^{aT(0)}_{i,q}\cosh(|q|r) + \frac{1}{|q|}\bar{A}^{aT(1)}_{i,q}\sinh(|q|r),$$
(2.11)

where q_i are three momenta along the boundary direction and the solution is obtained by using Fourier transform of the position space representation defined as

$$\Phi^{a}_{M}(x,r) = \int_{-\infty}^{\infty} d^{3}p e^{-ip_{i}x_{i}} \Phi^{a}_{Mp}(r), \qquad (2.12)$$

$$\Phi^{a}_{Mp}(r) = \frac{1}{(2\pi)^{3}} \int_{-\infty}^{\infty} d^{3}x e^{ip_{i}x_{i}} \Phi^{a}_{M}(x,r),$$

where Φ denotes any fields appearing in the bulk theory. $\bar{A}_{i,q}^{aT}$ is the transverse part of the gauge field, which is given by

$$\bar{A}_{i,q}^{aT} = P_{ij}(q)\bar{A}_{j,q}^{a}, \qquad (2.13)$$

where we define a projection operator,

$$P_{ij}(q) = \delta_{ij} - \frac{q_i q_j}{q^2}.$$
 (2.14)

If $\bar{A}^a_{i,q}(r)$ and $\bar{A}^a_{r,q}(r)$ are solutions then

$$\bar{A}_{i,q}^{a\prime}(r) = \bar{A}_{i,q}^{a}(r) - iq_i\bar{\phi}_q^{a}(r) \text{ and } \bar{A}_{r,q}^{a\prime}(r) = \partial_r\bar{\phi}_q^{a}(r)$$
 (2.15)

also solve the equations of motion. $\bar{\phi}^a$ is a gauge freedom which is not completely determined by equations of motion.

To proceed further we will use the radial gauge, namely $\bar{A}^a_{r,q}(r) = 0$. In the radial gauge, the residual gauge freedom is obtained by restricting the gauge parameter $\bar{\phi}^a_{r,q}(r)$ to be independent of r,

$$\bar{\phi}^a_q(r) \to \bar{\phi}^a_q. \tag{2.16}$$

Then by definition, $\bar{A}_{i,q}^{aT}(r)$ is gauge invariant under this residual gauge transformation. For the regularity of the solutions at the Poincare horizon, at $r = \infty$, we require that

$$\bar{A}_{ip}^{aT(0)} + \frac{1}{|p|} \bar{A}_{ip}^{aT(1)} = 0.$$
(2.17)

This removes the term proportional to $e^{|p|r}$ near the Poincare horizon. Using this regularity condition we can write the solution in the following form

$$\bar{A}_{i,p}^{aT}(r) = \bar{A}_{ip}^{aT(0)} e^{-|p|r}.$$
(2.18)

After having obtained leading order solution, we will now solve the equation second order in ε . The precise procedure is given in Appendix.B, here we briefly discuss the equations and their solutions. The equation for N = r and N = i from eq.(2.8) become

$$0 = \left(\nabla^2 \tilde{A}^a_r - \partial_r \partial_j \tilde{A}^a_j\right) - g \epsilon^{abc} \left(\partial_i (\bar{A}^b_i \bar{A}^c_r) - \bar{A}^c_i (\partial_i \bar{A}^b_r - \partial_r \bar{A}^b_i)\right), \qquad (2.19)$$

$$0 = (\partial_r^2 + \nabla^2) \tilde{A}_i^a - \partial_i (\partial_r \tilde{A}_r^a + \partial_j \tilde{A}_j^a) - g \epsilon^{abc} \left(\partial_r (\bar{A}_r^b \bar{A}_i^c) + \partial_j (\bar{A}_j^b \bar{A}_i^c) - (\partial_j \bar{A}_i^b - \partial_i \bar{A}_j^b) \bar{A}_i^c - (\partial_r \bar{A}_i^b - \partial_i \bar{A}_r^b) \bar{A}_r^c \right),$$

$$(2.20)$$

respectively.

The first order solution in ε , $\bar{A}^a_{i,q}$ appear in the form of source terms in the second order equations in ε , (2.19) and (2.20). Using this we divide up the second order solutions into the homogeneous part $\tilde{A}^a_{i,q(H)}$ and inhomogeneous part $\tilde{A}^a_{i,q(I)}$. The homogeneous solution has the same form with \bar{A}^a_M as in eq.(2.11). This is because the homogeneous equations are again linear and are identical to eq.(2.7). With the regularity condition in the interior, we get $\tilde{A}^a_{i,p(H)}(r) = \tilde{A}^{a(0)}_{i,p(H)}e^{-|p|r}$ and by imposing the radial gauge, the solution takes the form

$$\tilde{A}^{a}_{i,p(H)}(r) = \tilde{A}^{aT(0)}_{ip(H)} e^{-|p|r} - ip_i \tilde{\phi}^{a}_{p(H)}, \qquad (2.21)$$

where as argued previously, $\tilde{A}_{ip(H)}^{aT(0)}e^{-|p|r}$ is gauge invariant part of the solutions and $\tilde{\phi}_{p(H)}^{a}$ is gauge parameter, which also does not depend on r like $\bar{\phi}_{p}^{a}$ in the radial gauge.

With the projection operator, one can split the inhomogeneous part $\tilde{A}^a_{i,q(I)}$ into two pieces as

$$\tilde{A}^{a}_{i,q(I)} = \tilde{A}^{aT}_{i,q(I)} + \tilde{A}^{aL}_{i,q(I)}, \qquad (2.22)$$

where $\tilde{A}_{i,q(I)}^{aT} = P_{ij}(q)\tilde{A}_{j,q(I)}^{a}$ is the transverse part of the gauge field and $\tilde{A}_{i,q(I)}^{aL} = \frac{q_iq_j}{q^2}\tilde{A}_{j,q(I)}^{a}$ is the longitudinal part of the inhomogeneous solutions. The equations are also separated into the longitudinal and the transverse part, which are given by

$$-iq_{j}\partial_{r}\tilde{A}^{aL}_{j,q(I)}(r) = g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p|q-p| \left(\bar{A}^{cT}_{j,p}(r)\bar{A}^{bT}_{j,q-p}(r) - iq_{j}\bar{\phi}^{c}_{p}\partial_{r}A^{bT}_{j,q-p}(r)\right), \qquad (2.23)$$

$$(\partial_{r}^{2} - q^{2})\tilde{A}_{i,q(I)}^{aT} = g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p \bar{A}_{k,p}^{cT}(r) \alpha_{ijk}(p,q) \bar{A}_{j,q-p}^{bT}(r)$$

$$+ g\epsilon^{abc} P_{ij}(q) \int_{-\infty}^{\infty} d^{3}p((q-p)^{2} - q^{2}) \bar{\phi}_{p}^{b} \bar{A}_{j,q-p}^{cT}(r)$$

$$- \frac{i}{2}g\epsilon^{abc} q^{2} P_{ij}(q) \int_{-\infty}^{\infty} d^{3}p p_{j} \bar{\phi}_{p}^{b} \bar{\phi}_{q-p}^{c}$$

$$(2.24)$$

in the radial gauge, where

$$\alpha_{ijk}(p,q) \equiv \left(\frac{iq_i}{q^2}(q-p)^2 - i(q-p)_i\right)\delta_{jk} + iq_k\delta_{ij} - iq_j\delta_{ik}.$$
(2.25)

Solutions to these equations in the momentum space are given by

$$\tilde{A}_{i,q(I)}^{aL}(r) = -ig\epsilon^{abc}\frac{q_i}{q^2} \int_{-\infty}^{\infty} d^3p \bar{A}_{j,p}^{cT(0)} \bar{A}_{j,q-p}^{bT(0)} \frac{|q-p|}{|p|+|q-p|} e^{-(|p|+|q-p|)r}$$

$$- g\epsilon^{abc}\frac{q_iq_j}{q^2} \int_{-\infty}^{\infty} d^3p \bar{\phi}_p^c \bar{A}_{j,q-p}^{bT(0)} e^{-|q-p|r} + q_i f_q^a,$$

$$\tilde{A}_{i,q(I)}^{aT} = g\epsilon^{abc} \int_{-\infty}^{\infty} d^3p \bar{A}_{k,p}^{cT(0)} \bar{A}_{j,q-p}^{bT(0)} \frac{\alpha_{ijk}(p,q)}{(|p|+|q-p|)^2 - q^2} e^{-(|p|+|q-p|)r}$$

$$+ \frac{i}{2}g\epsilon^{abc} P_{ij}(q) \int_{-\infty}^{\infty} d^3p p_j \bar{\phi}_p^b \bar{\phi}_{q-p}^c + g\epsilon^{abc} P_{ij}(q) \int_{-\infty}^{\infty} d^3p \bar{\phi}_p^b \bar{A}_{j,q-p}^{cT(0)} e^{-|q-p|r},$$
(2.26)

where f_q^a is an integration constant. Therefore, the total solution up to $O(\varepsilon^2)$ is given by

$$A_{i,p}^{a}(r) = \varepsilon \bar{A}_{i,p}^{a}(r) + \varepsilon^{2} (\tilde{A}_{i,p(H)}^{a}(r) + \tilde{A}_{i,p(I)}^{aT}(r) + \tilde{A}_{i,p(I)}^{aL}(r)) + O(\varepsilon^{3}), \qquad (2.28)$$

under radial gauge. This total solution can be sorted out into gauge invariant parts and gauge parameter dependent parts. The transverse parts in $\bar{A}^a_{i,q}(r)$ and $\tilde{A}^a_{i,q(H)}(r)$ are gauge invariant. In eq.(2.26) and eq.(2.27), the first term in each equation is gauge invariant because it is comprised of $\bar{A}^{aT}_{i,q}(r)$ only.

One can choose the integration constant f_q^a as

$$f_{q}^{a} = \frac{i}{2}g\epsilon^{abc}\frac{q_{j}}{q^{2}}\int_{-\infty}^{\infty}d^{3}pp_{j}\bar{\phi}_{p}^{b}\bar{\phi}_{q-p}^{c} + f_{q}^{a\prime}$$
(2.29)

with another arbitrary function f'_q and it can absorbed into $\tilde{\phi}^a_{q(H)}$ by a redefinition

$$\tilde{\phi}^a_{q(H)} \to \tilde{\phi}^a_{q(H)} - i f^{a\prime}_q. \tag{2.30}$$

At this point it is worth pointing out that under such choice, gauge parameters dependent parts of the total solution(2.28) has exactly the same form as the gauge transformation(A.3) except the fact that the gauge parameters are r independent(See eq.(A.4) for the gauge transformation in $O(\epsilon)$ and eq.(A.5) in $O(\epsilon^2)$). Since the bulk action is manifestly gauge invariant under the residual gauge transformation, if we plug in this solution into the bulk action, the gauge parameter dependent parts drop out and the bulk on-shell action is written purely in terms of gauge invariant parts of the total solution.

It has already been noted in the past [11, 15, 17, 12] that imposition of the Neumann boundary condition on the AdS boundary, leads to an ambiguity in the computation of correlation functions of the dual operators. This ambiguity is associated with the residual gauge symmetry surviving at the boundary. However, we want to look at the boundary on-shell action, and this ambiguity appears as a total derivative term in the boundary action as long as the current coupled to the boundary value of the Yang-Mills field is covariantly conserved, $\mathcal{D}_i F_{ri}^a = 0.$

3 Boundary Conditions and the Effective Action

In the previous section, we have discussed the bulk solution in the radial gauge $A_r^a = 0$. In this section, we would like to discuss boundary deformations due to the bulk solutions that we obtained in the previous section.

Before we get into the detailed discussion, we briefly discuss bulk action. Up to equations of motion, the bulk action(2.4) can be written as

$$S[A] = \frac{1}{2} \int d^4x \left(\partial_M (A^a_N F^a_{MN}) + \frac{1}{2} g \epsilon^{abc} A^a_M F^b_{NM} A^c_N \right).$$
(3.1)

We do not need to add any counter terms [18, 19] since there are no divergences in the $r \to 0$ limit, which is manifest from the bulk solutions obtained in the previous section³. We are

³There is another way of adding counterterm action subtracting all the terms in the boundary action at any finite r slice as [20] which is indeed cut-off independent action.

interested in studying small r behaviour (equivalently behaviour near the AdS boundary). Both terms in the action contain radial derivatives and can be written as total derivative with respect to r which would result in boundary contribution. However, once we choose the radial gauge, the action becomes

$$S[A] = \frac{1}{2} \int d^4x \left(\partial_r (A^a_i F^a_{ri}) + \partial_j (A^a_i F^a_{ji}) + \frac{1}{2} g \epsilon^{abc} A^a_i F^b_{ji} A^c_j \right).$$
(3.2)

The second term in eq.(3.2) then becomes independent of r derivatives and the only place where r-derivative appears in the first term, and that too as total derivative. The third term also contributes to small r boundary even if it is not total derivative with respect to r. In general, it is non-trivial to extract its boundary contributions out but by using our perturbative solutions, we can evaluate those upto cubic order in small amplitude expansion(The precise expression will be given in Sec.3.2). As a result contribution of the bulk Yang-Mills action up to the bulk equations of motion to small r-boundary is given by

$$S_{bulk} \equiv \frac{1}{2} \int d^3x A^a_i(r, x) F^a_{ri}(r, x) + \frac{1}{4} \int d^3x drg \epsilon^{abc} A^a_i F^b_{ji} A^c_j.$$
(3.3)

From now on we will call eq.(3.3) the bulk action, although it is a contribution of bulk theory to the boundary action. We will mostly work in momentum space. Therefore, we perform a Fourier transform of bulk action(3.3) using eq.(2.12) and we define a new bulk action as

$$\hat{S}_{bulk} \equiv \frac{S_{bulk}}{(2\pi)^3},\tag{3.4}$$

where S_{bulk} is a momentum space expression of the bulk action. We define \hat{S}_{bulk} to remove $(2\pi)^3$ factor from S_{bulk} and \hat{S}_{bulk} will be mostly used for the construction of boundary action.

One can define the boundary value of bulk canonical momentum, $\partial_r A^a_{i,q}(r)$ of Yang-Mills field $A^{a(0)}_{i,q}(r)$ as

$$\hat{\Pi}^{a}_{i,q} \equiv \frac{\delta \hat{S}_{bulk}}{\delta A^{a(0)}_{i,q}} \tag{3.5}$$

The boundary on-shell action, I_{os} can be defined by choosing specific boundary conditions. To fix the boundary condition, we add the boundary action, S_{bdy} to the bulk action as

$$I_{os} = S_{bulk} + S_{bdy},\tag{3.6}$$

where we want S_{bdy} is composed of the boundary value of the gauge invariant part of the total solution (2.28) and that of its conjugate momentum only. Then, the on-shell action is a functional of A_i^a and its canonical momentum Π_i^a . After adding S_{bdy} , the generating functional for the boundary CFT will have two integration measures with $A_i^{a(0)}$ and Π_i^a as

$$Z[J] = e^{-W[J(A_i^{a(0)}, \Pi_i^a)]} = \int D[A_i^{a(0)}, \Pi_i^a] exp\left(-S_{bulk}(A_i^{a(0)}) - S_{bdy}(A_i^{a(0)}, \Pi_i^a)\right).$$
(3.7)

The generating functional, W[J] with a source J is defined as

$$W[J(A_i^a, \Pi_i^a)] \equiv I_{os}[A_i^a, \Pi_i^a], \qquad (3.8)$$

where the source J is again a non-trivial function of $A_i^{a(0)}$ and Π_i^a in general. The boundary conditions are given at the saddle point of the on-shell action:

$$\frac{\delta I_{os}[A_i^{a(0)}, \Pi_i^a]}{\delta A_i^{a(0)}} = 0 \text{ and } \frac{\delta I_{os}[A_i^{a(0)}, \Pi_i^a]}{\delta \Pi_i^a} = 0,$$
(3.9)

and in terms of the generating functional, which is given by

$$\frac{\delta W[J(A_i^{a(0)}, \Pi_i^a)]}{\delta J[A_i^{a(0)}, \Pi_i^a]} \frac{\delta J(A_i^{a(0)}, \Pi_i^a)}{\delta A_i^{a(0)}} = 0 \text{ and } \frac{\delta W[J(A_i^{a(0)}, \Pi_i^a)]}{\delta J[A_i^{a(0)}, \Pi_i^a]} \frac{\delta J(A_i^{a(0)}, \Pi_i^a)}{\delta \Pi_i^a} = 0.$$
(3.10)

This corresponds to the vacuum states of the boundary CFT. eq.(3.9) provides a relation between $A_i^{a(0)}$ and Π_i^a . Using this, one can re-write the on-shell action in terms of $A_i^{a(0)}$ as saddle point approximation.

The boundary effective action can be obtained by Legendre transform defined as

$$\Gamma[\sigma] = -\int J\sigma + W[J], \qquad (3.11)$$

where Γ is the boundary effective action and σ is the vacuum expectation value of certain boundary operators. From this relation, one gets

$$\sigma = \frac{\delta W[J]}{\delta J} \text{ and } J = -\frac{\delta \Gamma[\sigma]}{\delta \sigma}.$$
(3.12)

Now, let us suppose that for a certain boundary deformation, S_{bdy} , the effective action changes in the following way

$$\tilde{\Gamma}[\sigma] = \Gamma[\sigma] + \int d^d x f(\sigma(x)), \qquad (3.13)$$

where Γ denotes the effective action before the deformation and $\tilde{\Gamma}$ denotes that after the deformation. f is a function of the vacuum expectation value σ . The relation between f and S_{bdy} will become clear momentarily. Varying both sides of (3.13), one obtains the expression for the deformed source $\tilde{J} \equiv -\frac{\delta \tilde{\Gamma}[\sigma]}{\delta \sigma}$ as

$$\tilde{J} = J - \frac{df(\sigma)}{d\sigma}.$$
(3.14)

Finally, the deformed generating functional $\tilde{W}[\sigma] = \tilde{\Gamma}[\sigma] + \int \tilde{J}\sigma$ can be written as

$$\tilde{W}[\tilde{J}] = W[J] + \int d^d x \left(f(\sigma) - \sigma f(\sigma) \right).$$
(3.15)

It is now clear from the definition of the on-shell action(3.6) and (3.8) that

$$S_{bdy} = \int d^d x \left(f\left(\frac{\delta W[J]}{\delta J}\right) - \frac{\delta W[J]}{\delta J} f\left(\frac{\delta W[J]}{\delta J}\right) \right).$$
(3.16)

In next section, we use these relations to derive I_{os} , W[J] and $\Gamma[\sigma]$ for the various deformations from SU(2) Yang-Mills theory in AdS_4 . Before going on, we note that the effective action of Δ_+ theory has the same form as the on-shell action of Δ_- theory. In the case of $S_{bdy} = 0$, the only possible boundary condition is the Dirichlet boundary condition, which gives us the $\Delta_+ = 2$ theory. As we will see, to obtain the Neumann boundary condition, we will have to set $S_{bdy} = -\int d^d x \Pi_i^a A_i^{a(0)}$. Since Π_i^a is canonically conjugate of $A_i^{a(0)}$, adding this boundary term results in Legendre transform from Δ_+ theory to Δ_- theory. Imposition of the Neumann boundary condition therefore results in the $\Delta_- = 1$ theory. Thus we have argued that Legendre transform of the generating functional W[J] gives us the classical effective action $\Gamma[\sigma]$. Therefore, the effective action of Δ_+ theory should be the same with the on-shell action of $\Delta_$ theory.

3.1 Boundary Deformations in the First Order in ε

As a warm up, we start with bulk solutions with truncations up to $O(\varepsilon)$ and derive their onshell actions, generating functionals and boundary effective actions. Since, we are considering the non-abelian gauge theory case, we explicitly write the gauge group indices, however, up to $O(\varepsilon)$, the precess is almost the same with the abelian gauge theory on $AdS_4[13]$. The only difference is that we have 3 copies of them. Therefore, the genuine properties of the boundary effective action from SU(2) Yang-Mills on AdS_4 will appear from the second order in ε onwards, which would be discussed in the next subsection.

The bulk solution in the first order in ε in momentum space would be expanded near AdS boundary as

$$A_{i,q}^{a} = A_{i,q}^{a(0)} + rA_{i,q}^{a(1)} + O(r^{2}), \qquad (3.17)$$

where

$$A_{i,q}^{a(0)} = \varepsilon \bar{A}_{i,q}^{aT(0)} \text{ and } A_{i,q}^{a(1)} = \varepsilon \bar{A}_{i,q}^{a(1)} = -\varepsilon |q| \bar{A}_{i,q}^{aT(0)}.$$
 (3.18)

In eq.(3.18), we have used the regularity condition(2.17) for the last equality. As discussed in the last section, we only deal with the gauge invariant parts of the solutions. The bulk action up to the bulk EOM is given by

$$\hat{S}_{bulk} = \frac{1}{2} \varepsilon^2 \int d^3 p \bar{A}_{i,p}^{a(0)} \bar{A}_{i,-p}^{a(1)} = -\frac{1}{2} \varepsilon^2 \int d^3 p |p| \bar{A}_{i,p}^{a(0)} \bar{A}_{i,-p}^{a(0)}.$$
(3.19)

With this expression, one can find the canonical momentum of the boundary Yang-Mills field $A_{i,q}^{a(0)}$, which is given by

$$\hat{\Pi}_{i,q}^{a} = \frac{\delta S_{bulk}}{\delta A_{i,q}^{a(0)}} = \frac{\delta S_{bulk}}{\delta \varepsilon \bar{A}_{i,q}^{a(0)}} = -\varepsilon |q| \bar{A}_{i,-q}^{aT(0)} = -|q| A_{i,-q}^{a(0)} = A_{i,-q}^{a(1)}.$$
(3.20)

Variation of the bulk action with respect to the boundary field $A_{i,q}^{a(0)}$ is then given by

$$\delta \hat{S}_{bulk} = \int d^3 p |p| \delta A^{a(0)}_{i,p} A^{a(0)}_{i,-p} = \int d^3 p \delta A^{a(0)}_{i,p} \hat{\Pi}^a_{i,p}.$$
(3.21)

Dirichlet and Neumann Boundary Conditions: For the case that $\hat{S}_{bdy} = 0$, a possible boundary condition is the Dirichlet boundary condition, $\delta A_{i,q}^{a(0)} = 0$. In this case, the onshell action(also the generating functional) is the same as \hat{S}_{bulk} and the source J and the corresponding vacuum expectation value, σ in the generating functional are

$$J_D = A_{i,q}^{a(0)} \text{ and } \sigma_D \equiv \frac{\delta W[J_D]}{\delta J_D} = \frac{\delta \hat{S}_{bulk}}{\delta A_{i,q}^{a(0)}} = \hat{\Pi}_{i,q}^a = -|q| A_{i,-q}^{a(0)},$$
(3.22)

respectively, where the subscript D denotes "Dirichlet". The boundary effective action can be obtained by Legendre transform defined in eq.(3.11). We apply the Legendre transform for the Dirichlet case, then the effective action is given by

$$\Gamma^{D}[\hat{\Pi}^{a}_{i,q}] = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^{3}p}{|p|} \hat{\Pi}^{a}_{i,p} \hat{\Pi}^{a}_{i,-p}.$$
(3.23)

Neumann boundary condition can be obtained by considering that $\hat{S}_{bdy}^N = -\int d^3p A_{i,p}^{a(0)} \hat{\Pi}_{i,p}^a$, where the superscript N denotes "Neumann". To find out stationary points, we vary $I_{os}^N[A_{i,q}^{a(0)}, \Pi_{i,q}^a]$ as

$$\delta I_{os}^{N}[A_{i,q}^{a(0)},\hat{\Pi}_{i,q}^{a}] = \int d^{3}p \delta A_{i,p}^{a(0)} \Pi_{i,p}^{a} + \delta S_{bdy} = -\int d^{3}p A_{i,p}^{a(0)} \delta \hat{\Pi}_{i,p}^{a} = 0, \qquad (3.24)$$

so we get Neumann boundary condition: $\delta \hat{\Pi}_{i,q}^a = 0$. For Neumann case, the role of the source J and the vacuum expectation value σ are interchanged with respect to the Dirichlet case. This is because adding $\hat{S}_{bdy}^N = \int J\sigma$ is effectively performing Legendre transform of \hat{S}_{bulk} . As a result, the boundary effective action is obtained from Legendre transformation of eq.(3.23):

$$\Gamma^{N}[A_{i,p}^{a(0)}] = -\frac{1}{2} \int d^{3}p |p| A_{i,p}^{a(0)} A_{i,-p}^{a(0)}, \quad J_{N} = \hat{\Pi}_{i,q}^{a} \quad \text{and} \quad \sigma_{N} = A_{i,p}^{a(0)}.$$
(3.25)

Massive Deformation: One can also discuss generalized Neumann boundary conditions, for example, the Massive Deformation. At the first order in ε , the massive deformation leads to a boundary condition given by

$$\bar{A}_{ip}^{a(0)T} + \frac{1}{m}\bar{A}_{ip}^{a(1)} = 0.$$
(3.26)

To obtain this boundary condition, we introduce the boundary action

$$\hat{S}_{bdy}^{M} = -\int d^{3}p \left(A_{i,p}^{a(0)} \hat{\Pi}_{i,p}^{a} + \frac{1}{2m} \hat{\Pi}_{i,p}^{a} \hat{\Pi}_{i,-p}^{a} \right).$$
(3.27)

By varying the on-shell action with above boundary action, we end up with

$$\delta I_{os}^{M}[A_{i,q}^{a(0)},\Pi_{i,q}^{a}] = -\int d^{3}p \delta A_{i,p}^{a(0)} \left(|p|A_{i,-p}^{a(0)T} + \hat{\Pi}_{i,p}^{a} \right) - \int d^{3}p \delta \Pi_{i,-p}^{a} \left(A_{i,-p}^{aT(0)} + \frac{1}{m} \Pi_{i,p}^{a} \right) = 0,$$
(3.28)

where the first integral gives the regularity condition. Rather than imposing Neumann boundary condition for the second integration, if we set the quantity inside the parenthesis to zero, then the canonical momentum becomes

$$\hat{\Pi}^{a}_{i,-q} = -mA^{a(0)T}_{i,q}.$$
(3.29)

For the consistency with the regularity condition (2.17), it is demanded that |p| = m. Therefore, the boundary field $A_{i,q}^{a(0)}$ becomes on-shell and massive under such a condition.

We rewrite the on-shell action I_{os}^{M} with replacing every $\prod_{i,q}^{a}$ by $A_{i,q}^{a(0)}$ using eq.(3.29) as

$$I_{os}^{M}[A_{i,q}^{a(0)}] = -\frac{1}{2} \int d^{3}p\left(|p|-m\right) A_{i,p}^{a(0)T} A_{i,-p}^{a(0)T}.$$
(3.30)

The fact that this procedure is justified can be seen by varying $I_{os}^{M}[A_{i,q}^{a(0)}]$ with respect to $A_{i,q}^{a(0)}$ and noticing that it produces the correct boundary condition

$$\frac{\delta I_{os}^{M}[A_{i,q}^{a(0)}]}{\delta A_{i,q}^{a(0)}} = -\left(|p|-m\right)A_{i,-p}^{a(0)} = 0.$$
(3.31)

The final step for the massive deformation case is to obtain the dual CFT (or effective) action. Unfortunately, one cannot easily figure out what is the deformed source J in above expression and therefore cannot perform Legendre transform either. However, there is another way to deal with this situation where one writes down an expected form of the dual CFT action. Let us consider the following form:

$$\Gamma^{M}[A_{i,q}^{a(0)}] = \frac{1}{2} \int d^{3}p \alpha(p) A_{i,p}^{a(0)} A_{i,-p}^{a(0)}, \qquad (3.32)$$

where α is an arbitrary momentum dependent function and we assume that vacuum expectation value σ is still $A_{i,q}^{a(0)}$ under any deformation[21]. Using this for the effective action, one can derive the expression of the source

$$J_M[A_{i,q}^{a(0)}] = -\frac{\delta\Gamma^M[J(A_{i,q}^{a(0)})]}{\delta A_{i,q}^{a(0)}} = -\alpha(q)A_{i,-q}^{a(0)}.$$
(3.33)

We can then use this source term J to perform inverse Legendre transform from Γ to obtain the generating functional W using eq.(3.11). We then demand that this inverse transformation reproduce the correct generating functional W, which imposes a constraint on α , and also determines expression of the source term,

$$\alpha = (|p| - m) , J_M = -(|p| - m)A_{i,-q}^{a(0)} \text{ and } \Gamma^M = \frac{1}{2}\int d^3p(|p| - m)A_{i,p}^{a(0)}A_{i,-p}^{a(0)}.$$
(3.34)

The generating functional W is usually expressed as the functional of source J_M , which is done by using eq.(3.34),

$$W^{M}[J_{i,q}^{M}] = \frac{1}{2} \int d^{3}p \frac{J_{i,p}^{M} J_{i,-p}^{M}}{|p| - m}.$$
(3.35)

Self-Dual Boundary Condition and Massive Deformation: The most interesting case is the self-dual boundary condition, together with the massive deformation. Self-duality condition in four dimension is given by

$$F^a_{MN} = \frac{1}{2} \epsilon_{MNPQ} F^a_{PQ}. \tag{3.36}$$

To study self-dual boundary condition, we expand Yang-Mills field near the AdS boundary, *i.e.*, around r = 0 as in eq.(3.17). Once we choose the index M = r in eq.(3.36), the boundary condition derived from it becomes

$$A_i^{a(1)} = \mathcal{D}_i A_r^{a(0)} + \frac{1}{2} \epsilon_{ijk} F_{jk}^{a(0)}, \qquad (3.37)$$

where $\mathcal{D}_i A_r^a = \partial_i A_r^a - g \epsilon^{abc} A_i^b A_r^c$. Since we have used the radial gauge $A_r^a = 0$ for our bulk solutions, $\mathcal{D}_i A_r^a = 0$ in eq.(3.37). Up to the leading order in ε , the self dual boundary condition is given by

$$A_{i}^{a(1)}(x) = \epsilon_{ijk}\partial_{j}A_{k}^{a(0)}(x), \text{ in momentum space } A_{i,q}^{a(1)} = \hat{\Pi}_{i,-q}^{a} = \epsilon_{ijk}(-iq_{j})A_{k,q}^{a(0)}.$$
 (3.38)

In addition to this, if we impose the on-shell condition, $(|p|-m)A_{i,p}^{a(0)} = 0$, it gives rise to massive deformation of the boundary on-shell action. That is, eq.(3.26) together with eq.(3.38), gives rise to the boundary condition

$$0 = mA_{i,p}^{a(0)} + \epsilon_{ijk}(-ip_j)A_{k,p}^{a(0)}.$$
(3.39)

This boundary condition can be incorporated in boundary on-shell action in the following way,

$$\hat{S}_{bdy}^{MS} = \int d^3p \left[\beta \left(A_{i,p}^{a(0)} \hat{\Pi}_{i,p}^a + \frac{1}{2m} \hat{\Pi}_{i,p}^a \hat{\Pi}_{i,-p}^a \right) - \frac{\beta + 1}{2} \epsilon_{ijk} A_{i,p}^{a(0)}(ip_j) A_{k,-p}^{a(0)} \right],$$
(3.40)

where β is a numerical parameter. Variation of the on-shell action, $I_{os}^{MS} = \hat{S}_{bulk} + \hat{S}_{bdy}^{MS}$, provides

$$\delta I_{os}^{MS}[A_{i,q}^{a(0)},\hat{\Pi}_{i,q}^{a}] = \int d^{3}p\beta\delta\hat{\Pi}_{i,p}^{a} \left(A_{i,p}^{a(0)} + \frac{1}{m}\hat{\Pi}_{i,-p}^{a}\right)$$

$$- \int d^{3}p\delta A_{i,p}^{a(0)} \left(|p|A_{i,-p}^{a(0)} - \beta\hat{\Pi}_{i,p}^{a} + (\beta+1)\epsilon_{ijk}(ip_{j})A_{k,-p}^{a(0)}\right).$$
(3.41)

The first line in above equation (3.41) can be set to zero by considering the massive deformation

$$A_{i,q}^{a(0)} + \frac{1}{m}\hat{\Pi}_{i,-q}^{a} = 0$$
(3.42)

rather than imposing the Neumann boundary condition, $\delta \Pi_{i,q}^a = 0$. For consistency with the regularity condition, we demand $(|p| - m)A_{i,p}^{a(0)} = 0$ and the massive deformation implies the canonical momentum is given by $\Pi_{i,-q}^a = -mA_{i,q}^{a(0)}$. For the second line in equation (3.41),

rather than imposing Dirichlet boundary condition $\delta A_{i,q}^{a(0)} = 0$, we equate the expression inside the parenthesis to zero. This choice corresponds to the self dual boundary condition in momentum space,

$$\hat{\Pi}_{i,q}^{a} = -|q| A_{i,-q}^{a(0)} = \epsilon_{ijk}(-iq_j) A_{k,-q}^{a(0)}.$$
(3.43)

This condition together with on-shell condition, is exactly the same with eq.(3.39). When we substitute this relation into $I_{os}^{MS}[A_{i,q}^{a(0)}, \hat{\Pi}_{i,q}^{a}]$ along with the regularity condition we can eliminate $\hat{\Pi}_{i,q}^{a}$ by expressing it in terms of $A_{i,-q}^{a(0)}$ to get

$$I_{os}^{MS}[A_{i,q}^{a(0)}] = -\frac{1}{2}(1+\beta) \int d^3p \left(m A_{i,p}^{a(0)T} A_{i,-p}^{a(0)T} + \epsilon_{ijk} A_{i,p}^{a(0)}(ip_j) A_{k,-p}^{a(0)} \right), \tag{3.44}$$

which is abelian massive Chern-Simons action [16, 22]. We also obtain the deformed source and dual CFT action by the same method in the previous discussion with massive deformation. They are given by

$$J_{MS} = -(1+\beta) \left(m A_{i,-q}^{a(0)} + \epsilon_{ijk}(iq_j) A_{k,-q}^{a(0)} \right), \qquad (3.45)$$

$$\Gamma^{MS}[A_{i,q}^{a(0)}] = \frac{1}{2}(1+\beta) \int d^3p \left(m A_{i,p}^{a(0)T} A_{i,-p}^{a(0)T} + \epsilon_{ijk} A_{i,p}^{a(0)}(ip_j) A_{k,-p}^{a(0)} \right).$$
(3.46)

3.2 Boundary Deformation in the Second Order in ε

A way of imposing boundary conditions for the second order solution in ε is in principle the same with previous discussion. There are some technical difficulties due to appearance of quadratic terms involving the first order solutions. However, this nonlinearity in the equation involves lower order solutions only, which are already derived using the small amplitude expansion. For evaluating the boundary on-shell action, we would like to choose a gauge for boundary gauge fields, $A_{i,q}^{a(0)}$, in fact, we will set $\phi_{i,q}^a = \varepsilon \overline{\phi}_{i,q}^a + \varepsilon^2 \widetilde{\phi}_{i,q}^a = 0$. Since the bulk action is manifestly gauge invariant, choosing a particular gauge is not a problem. With such choice of gauge degree of freedom, the boundary gauge field appearing on the boundary on-shell action will be effectively transverse. Therefore, in the following, we only deal with gauge parameter independent parts of the solutions for the construction of the boundary theory. We start with a general discussion of the solution(2.28). The near AdS boundary expansion is given by

$$A_{i,q}^{a}(r)|_{r\to 0} = A_{i,q}^{a(0)} + rA_{i,q}^{a(1)} + O(r^{2}), \qquad (3.47)$$

where

$$A_{i,-q}^{a(1)} = -|q|A_{i,-q}^{a(0)} - g\epsilon^{abc} \int_{-\infty}^{\infty} d^3p A_{k,p}^{c(0)} A_{j,-q-p}^{b(0)} \Delta_{ijk}(p,-q) + O(\varepsilon^3)$$
(3.48)

and

$$\Delta_{ijk}(p,q) = \frac{\alpha_{ijk}(p,q)}{|p| + |q-p| + |q|} - \frac{iq_i\delta_{jk}|q-p|(|q-p| + |p| - |q|)}{q^2(|p| + |q-p|)}.$$
(3.49)

 $A_{i,q}^{a(0)}$ is the boundary value of the full solution $A_{i,p}^{a}(r)$ defined in eq.(2.28)(See also eq.(2.27), eq.(2.26) and eq.(2.21)), which is given by

$$A_{i,q}^{a(0)} = \varepsilon \bar{A}_{i,p}^{aT(0)} + \varepsilon^2 \tilde{A}_{i,p(H)}^{aT(0)} - \varepsilon^2 i g \epsilon^{abc} \frac{q_i}{q^2} \int_{-\infty}^{\infty} d^3 p \bar{A}_{j,p}^{cT(0)} \bar{A}_{j,q-p}^{bT(0)} \frac{|q-p|}{|p|+|q-p|}$$
(3.50)

+
$$\varepsilon^2 g \epsilon^{abc} \int_{-\infty}^{\infty} d^3 p \bar{A}_{k,p}^{cT(0)} \bar{A}_{j,q-p}^{bT(0)} \frac{\alpha_{ijk}(p,q)}{(|p|+|q-p|)^2-q^2} + O(\varepsilon^3).$$

Now, we evaluate the bulk action(3.3) explicitly by substituting the bulk solution and keeping terms upto the leading interaction terms,

$$S_{bulk} \equiv \frac{1}{2} \int d^3q A^{a(0)}_{i,q} A^{a(1)}_{i,-q} + \frac{1}{4} \int d^3q d^3l d^3p drg \epsilon^{abc} A^a_{i,q} F^b_{ji,l} A^c_{j,p} \delta^3(q+l+p),$$
(3.51)

where gauge fields in the second term contains the first order solutions only, which means that

$$A_{i}^{a} = \varepsilon \bar{A}_{i,q}^{a(0)} e^{-|q|r} + O(\varepsilon^{2}) = A_{i,q}^{a(0)} e^{-|q|r} + O(\varepsilon^{2}).$$
(3.52)

Therefore, the second term in eq.(3.51) becomes

$$S_{bulk}^{2nd\ term} = \frac{1}{2} \int d^3q d^3l d^3p \, \varepsilon^3 \bar{A}_{i,q}^{a(0)T} \bar{A}_{j,l}^{b(0)T} \bar{A}_{k,p}^{c(0)T} \frac{i l_k \delta_{ij}}{|q| + |l| + |p|} e^{-(|q| + |l| + |p|)r} \delta^3(q + l + p) \Big|_{r=\infty}^{r=0} (3.53)$$
$$= \frac{1}{2} \int d^3q d^3l d^3p A_{i,q}^{a(0)} A_{j,l}^{b(0)} A_{k,p}^{c(0)} \frac{i l_k \delta_{ij}}{|q| + |l| + |p|} \delta^3(q + l + p) + O((A_{i,q}^{a(0)})^4)$$

With this, one can construct the bulk action as

$$\hat{S}_{bulk} = -\frac{1}{2} \int_{-\infty}^{\infty} d^3 q |q| A_{i,q}^{a(0)} A_{i,-q}^{a(0)} - \frac{1}{2} g \epsilon^{abc} \int_{-\infty}^{\infty} d^3 q d^3 p A_{i,q}^{a(0)} A_{j,-q-p}^{b(0)} \Delta_{ijk}(p,-q) (3.54)
+ \frac{1}{2} \int d^3 q d^3 l d^3 p A_{i,q}^{a(0)} A_{j,l}^{b(0)} A_{k,p}^{c(0)} \frac{i l_k \delta_{ij}}{|q| + |l| + |p|} \delta^3(q+l+p),$$

upto cubic interactions. Notice that Δ_{ijk} and $\frac{il_k \delta_{ij}}{|q|+|l|+|p|}$, in order to be non-vanishing, should be fully anti-symmetric in indices, i, j and k together with appropriate momentum exchange due to ϵ^{abc} . The second term is then written as

$$\hat{S}_{2nd \ term} = -\frac{1}{2} g \epsilon^{abc} \int_{-\infty}^{\infty} d^3 q d^3 p d^3 l A^{a(0)}_{i,q} A^{b(0)}_{j,l} A^{c(0)}_{k,p} \delta^3 (q+p+l) \tilde{\Delta}_{ijk}(q,l,p),$$
(3.55)

where

$$\tilde{\Delta}_{ijk}(q,l,p) = \tilde{\Delta}_{ijk}^T(q,l,p) + \tilde{\Delta}_{ijk}^L(q,l,p).$$
(3.56)

 $\tilde{\Delta}^T_{ijk}(q,l,p)$ and $\tilde{\Delta}^L_{ijk}(q,l,p)$ are given by

$$\tilde{\Delta}_{ijk}^{T}(q,l,p) = \frac{i(l-q)_k \delta_{ij} + i(p-l)_i \delta_{jk} + i(q-p)_j \delta_{ik}}{2(|q|+|l|+|p|)}.$$
(3.57)

$$\tilde{\Delta}_{ijk}^{L}(q,l,p) = \frac{iq_{i}\delta_{jk}(|l|-|p|)(|p|+|l|-|q|)}{6q^{2}(|p|+|l|)} + \frac{il_{j}\delta_{ki}(|p|-|q|)(|q|+|p|-|l|)}{6l^{2}(|q|+|p|)} + \frac{ip_{k}\delta_{ij}(|q|-|l|)(|l|+|q|-|p|)}{6p^{2}(|l|+|q|)},$$
(3.58)

(eq.(3.57) and eq.(3.58) can be obtained from eq.(2.25) and eq.(3.49) after some computation using the fact that $\bar{A}_{i,q}^{aT(0)}$ is transverse). In fact, $\tilde{\Delta}_{ijk}^{L}(q,l,p)$ does not contribute to the bulk action, since the fields multiplying it in the action are effectively transverse⁴. The third term in eq.(3.54) is given by

$$\hat{S}^{3rd\ term} = \frac{1}{6} \int d^3q d^3l d^3p A^{a(0)}_{i,q} A^{b(0)}_{j,l} A^{c(0)}_{k,p} \tilde{\Delta}^T_{ijk}(q,l,p) \delta^3(q+l+p), \tag{3.60}$$

it also has the same anti-symmetrization.

The canonical momentum of the source $A^a_{i,q}$ is given by

$$\hat{\Pi}^{a}_{i,q} = \frac{\delta \hat{S}_{bulk}}{\delta A^{a(0)}_{i,q}} = -|q|A^{a(0)}_{i,-q} - g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p A^{b(0)}_{j,-q-p} A^{c(0)}_{k,p} \tilde{\Delta}_{ijk}(q,-q-p,p)$$
(3.61)

Dirichlet Boundary Condition: Without adding any boundary action, the on-shell action, I_{os}^{D} is given by

$$I_{os}^{D}(A_{i,q}^{a(0)}) = -\frac{1}{2} \int_{-\infty}^{\infty} d^{3}q |q| A_{i,q}^{a(0)} A_{i,-q}^{a(0)} - \frac{1}{3} g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}q d^{3}p A_{i,q}^{a(0)} A_{j,-q-p}^{b(0)} A_{k,p}^{c(0)} \tilde{\Delta}_{ijk}^{T}(q,-q-p,p).$$

$$(3.62)$$

The Legendre transform of I_{os}^{D} becomes the boundary effective action in terms of dual operator $\Pi_{i,q}^{a}$, which is given by

$$\Gamma^{D}(\Pi^{a}_{i,q}) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^{3}q}{|q|} \hat{\Pi}^{a}_{i,q} \hat{\Pi}^{a}_{i,-q} + \frac{1}{3} g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}q d^{3}p \frac{\tilde{\Delta}_{ijk}(q,-q-p,p)}{|q||q+p||p|} \hat{\Pi}^{a}_{i,q} \hat{\Pi}^{b}_{j,-q-p} \hat{\Pi}^{c}_{k,p}.$$
 (3.63)

This action has exotic momentum dependent cubic interaction, which is classically marginal. Up to this order, we can evaluate 2-point and 3-point functions of the boundary CFT and the dual CFT.

Neumann Boundary Condition: The effective action in Neumann boundary condition can be obtained by Legendre transform of (3.63), which becomes

$$\Gamma^{N}(A_{i,q}^{a(0)}) = -\frac{1}{2} \int_{-\infty}^{\infty} d^{3}q |q| A_{i,q}^{a(0)} A_{i,-q}^{a(0)} - \frac{1}{3} g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}q d^{3}p A_{i,q}^{a(0)} A_{j,-q-p}^{b(0)} A_{k,p}^{c(0)} \tilde{\Delta}_{ijk}(q,-q-p,p),$$
(3.64)

with $J^N = \hat{\Pi}^a_{i,q}$. The generating functionals for each boundary condition are given by

$$I_{os}^{D}(A_{i,q}^{a(0)}) = W^{D}[A_{i,q}^{a(0)}] = \Gamma^{N}[A_{i,q}^{a(0)}] \text{ and } I_{os}^{N}(\Pi_{i,q}^{a}) = W^{N}(\Pi_{i,q}^{a}) = \Gamma^{D}[\Pi_{i,q}^{a}].$$
(3.65)

$$A_{i,q}^{a(0)}A_{j,-q-p}^{b(0)}A_{k,p}^{c(0)} = \varepsilon^3 \bar{A}_{i,q}^{aT(0)} \bar{A}_{j,-q-p}^{bT(0)} \bar{A}_{k,p}^{cT(0)} + O(\varepsilon^4).$$
(3.59)

As a result, at this order, $\tilde{\Delta}_{ijk}^{L}(q, l, p)$ disappears from the boundary on-shell action.

⁴The bulk solution of Yang-Mills fields up to second order in ε , requires terms only up to cubic in ε in \hat{S}_{bulk} . Using the expansion (3.50), of the boundary value of the Yang-Mills field $A_{i,q}^{a(0)}$ in the cubic interaction in eq.(3.54), it is easy to see that up to $O(\varepsilon^3)$ this term is effectively transverse

Massive and Self-Dual Boundary Condition: To impose self-dual and massive deformation as a boundary condition, we add the following boundary action to \hat{S}_{bulk} :

$$\hat{S}_{bdy} = \int_{-\infty}^{\infty} d^{3}q \left[-\beta \left(A_{i,q}^{a(0)} \Pi_{i,q}^{a} + \frac{1}{2m} \Pi_{i,q}^{a} \Pi_{i,-q}^{a} \right) + \frac{3}{2m} \alpha \beta g \epsilon^{abc} \Pi_{i,q}^{a} \int_{-\infty}^{\infty} d^{3}p d^{3}l A_{j,l}^{b(0)} A_{k,p}^{c(0)} \delta^{3}(-q+l+p) \tilde{\Delta}_{ijk}(-q,l,p) + \eta \epsilon_{ijk} \left(A_{i,q}^{a(0)}(iq_{j}) A_{k,-q}^{a(0)} - \frac{1}{3}g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p d^{3}l A_{i,q}^{a(0)} A_{j,l}^{b(0)} A_{k,p}^{c(0)} \delta^{3}(q+l+p) \right) + \frac{\gamma}{3}g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p d^{3}l A_{i,q}^{a(0)} A_{j,l}^{b(0)} A_{k,p}^{c(0)} \delta^{3}(q+l+p) \tilde{\Delta}_{ijk}(q,l,p) \right],$$
(3.66)

where α , β , γ and η are numerical constants which would be determined by imposing right boundary condition. Variation of $I_{os}^{MS}[A_{i,q}^{a(0)}] = \hat{S}_{bulk} + \hat{S}_{bdy}$ with respect to $A_{i,q}^{a(0)}$ and $\hat{\Pi}_{i,q}^{a}$ provides the following boundary conditions:

$$\hat{\Pi}^{a}_{i,q} = -mA^{a(0)}_{i,-q} + \frac{3}{2}\alpha g \epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p d^{3}l A^{b(0)}_{j,l} A^{c(0)}_{k,p} \delta^{3}(q+l+p)\tilde{\Delta}_{ijk}(q,l,p), \quad (3.67)$$

$$\begin{aligned} (\beta - 1)\Pi_{i,q}^{a} &= g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p d^{3}l A_{j,l}^{b(0)} A_{k,p}^{c(0)} \delta^{3}(q+l+p) (\gamma - \frac{3\alpha\beta(|l|+|p|)}{2m}) \tilde{\Delta}_{ijk}(q,l,p) 3.68) \\ &+ \eta\epsilon_{ijk} F_{jk,-q}^{a}, \end{aligned}$$

where

$$F^{a}_{ij,q} = -iq_{i}A^{a}_{j,q} + iq_{j}A^{a}_{i,q} - g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p d^{3}l A^{b(0)}_{i,l} A^{c(0)}_{j,p} \delta^{3}(-q+l+p),$$
(3.69)

is Yang-Mills field strength in momentum space. For the consistency between canonical momentum (3.61) and boundary condition (3.67), one requires

$$\alpha = -\frac{2}{3} \text{ and } A_{iq}^{a(0)} \to A_{iq}^{a(0)}|_{|q|=m}.$$
 (3.70)

We can use the second boundary condition (3.68) to impose non-abelian version of massive self-dual boundary condition [16], which is given by

$$A_{i,q}^{a(0)} = -\frac{1}{2m} \epsilon_{ijk} F_{jk,q}^a.$$
(3.71)

To do this, we plug eq.(3.67) into eq.(3.68). Then the massive self-dual boundary condition can be obtained if we impose the condition,

$$\gamma = 1 - 3\beta$$
 and $\eta = \frac{\beta - 1}{2}$. (3.72)

The on-shell action, dual CFT action and the source term can then be derived using this massive self-dual condition as,

$$I_{os}^{MS}[A_{i,q}^{a(0)}] = \frac{1}{2}(\beta - 1) \int d^3p \left[m A_{i,p}^{a(0)} A_{i,-p}^{a(0)} + \epsilon_{ijk} \left(A_{i,p}^{a(0)}(ip_j) A_{k,-p}^{a(0)} \right) \right]$$
(3.73)

$$- \frac{1}{3}g\epsilon^{abc}\int d^{3}ld^{3}pA^{a(0)}_{i,q}A^{b(0)}_{j,l}A^{c(0)}_{k,p}\delta^{3}(q+l+p)\bigg)\bigg],$$

$$\Gamma^{MS}[A^{a(0)}_{i,q}] = -\frac{1}{2}(\beta-1)\int d^{3}p\left[mA^{a(0)}_{i,p}A^{a(0)}_{i,-p} + \epsilon_{ijk}\left(A^{a(0)}_{i,p}(ip_{j})A^{a(0)}_{k,-p}\right) - \frac{1}{6}g\epsilon^{abc}\int d^{3}ld^{3}pA^{a(0)}_{i,q}A^{b(0)}_{j,l}A^{c(0)}_{k,p}\delta^{3}(q+l+p)\bigg)\bigg].$$
and $J_{MS} = (\beta-1)\left[mA^{a(0)}_{i,-p} + \epsilon_{ijk}\left(ip_{j}A^{a(0)}_{k,-p} - \frac{1}{4}g\epsilon^{abc}\int d^{3}ld^{3}pA^{b(0)}_{j,l}A^{c(0)}_{k,p}\delta^{3}(q+l+p)\bigg)\bigg],$

$$(3.75)$$

respectively. The on-shell effective action (3.73) turns out to be proportional to the non-abelian Chern-Simon action.

4 "Approximate" Electric-Magnetic Duality in SU(2)Yang-Mills in AdS_4

It is well-known fact that explicit electric-magnetic duality cannot be demonstrated for nonabelian gauge field theory, pure U(1) gauge theory equations of motion, on the other hand, are manifestly invariant. Exchanging electric and magnetic fields is possible even for Yang-Mills, but such a transformation is not a canonical transformation⁵.

There is, however, an attempt to construct a canonical transformation in SU(2) Yang-Mills, which is gives rise to an *approximate* electric-magnetic duality transformation, if one restrict to Yang-Mills action truncated up to cubic order interactions in weak field expansion[28].

To see this more clearly, let us explain the meaning of "approximate" electric-magnetic duality. The authors in [28] construct an infinitesimal canonical transformation which is a natural extension of U(1) electric-magnetic duality to SU(2) Yang-Mills, which is manifest symmetry in Yang-Mills action when the action only retains cubic order interactions in small amplitudes of gauge fields in it(i.e. they do not keep quartic order interactions). Therefore, if Yang-Mills coupling vanishes, then this symmetry becomes the usual duality in U(1). However, this is not precisely electric-magnetic duality in SU(2) Yang-Mills since the variation of electric field is not proportional to magnetic field even up to such a truncation. Therefore by "approximate" duality we mean that there exists a canonical transformation which is the most natural generalization of electric-magnetic duality in U(1). It is worth mentioning at this point that this has been demonstrated in a particular gauge for the Yang-Mills fields, namely the transverse gauge. In this gauge, components of gauge fields surviving in the action are all transverse. In any other gauge, the transformation may be difficult to implement.

In this section, we will discuss "approximate" electric-magnetic duality transformation for our system. The difference between flat space and AdS space here only comes from their boundaries. In general, electric-magnetic duality is not a manifest symmetry of the Lagrangian

⁵There is, in fact, a no-go theorem for this duality. At least in a particular gauge this has been demonstrated in [27].

but it is a symmetry of equations of motion. The total derivative terms in the Lagrangian which inhibit this manifestation disappear in the flat space, if we suppose that all the fields die off sufficiently fast at infinite boundary. However, Weyl transformed action(2.4) from AdS_4 has conformal boundary at $x^4 \equiv r = 0$ and gauge fields do not die off fast enough at this boundary. Therefore, we need to keep total derivative terms with respect to 'r'.

Now let us apply the canonical transformation of [28] to our case. Yang-Mills action (2.4) can be written in terms of the Legendre transform of the Hamiltonian as

$$S[A_i^a, E_i^a, A_r^a] = \int d^4x \left(-E_i^a \partial_r A_i^a - \mathcal{H}[\Pi_i^a, A_i^a] \right), \qquad (4.1)$$

where the canonical momentum $\Pi_i^a = -E_i^a$, the electric field, and the Hamiltonian density \mathcal{H} is given by

$$\mathcal{H} = \frac{1}{2} \left(E_i^a E_i^a - B_i^a B_i^a \right) + A_r^a \left(\partial_i E_i^a + \epsilon^{abc} E_i^b A_i^c \right), \tag{4.2}$$

where we set Yang-Mills coupling g = 1 for convenience, $B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a$, magnetic field, and negative sign in front of the magnetic field square in the Hamiltonian density appears since we are working in the Euclidean space. Notice the Legendre transform is taken with respect to the radial coordinate. The gauge field component A_r^a has no dynamics and in fact, it is a Lagrange multiplier, which gives rise to the Gauss law constraint. By imposing the Gauss law constraint, $(D_i E_i)^a = \partial_i E_i^a + \epsilon^{abc} E_i^b A_i^c = 0$, we can remove the terms which are proportional to A_r^a from the action. Another important point is the gauge choice. In [28], authors point out that it is very crucial to choose transverse gauge. Under such a choice, longitudinal parts of electric fields becomes quartic order in the weak field expansion in the action, so at the cubic approximation we will not need to worry about those terms. With all these conditions, the action can be expressed as

$$S[E_i^a, A_i^a] = \int d^4x \left[-E_i^{a,T} \partial_r A_i^a - \frac{1}{2} \left(E_i^{a,T} E_i^{a,T} - \bar{B}_i^a \bar{B}_i^a \right) - \frac{1}{2} \bar{B}_i^a \epsilon^{abc} \epsilon_{ijk} A_j^b A_k^c + O((A_i^a)^\alpha (E_i^a)^\beta) \right],$$
(4.3)

where α and β are positive integers which satisfy an inequality $\alpha + \beta \geq 4$ and $\bar{B}_i^a \equiv \epsilon_{ijk} \partial_j A_k^a$. It turns out that this action is invariant upto cubic order in small amplitude expansion of the fields under the following infinitesimal transformation:

$$E_i^{a,T} \rightarrow E_i^{a,T} + \eta \left(\bar{B}_i^a - \frac{3}{2} \epsilon^{abc} \epsilon_{ijk} (A_j^b A_k^c)^T \right) + \text{ higher order}, \qquad (4.4)$$

$$E_i^{a,L} \rightarrow E_i^{a,L} - \frac{1}{2} \eta \epsilon^{abc} \epsilon_{ijk} \left(A_j^b A_k^c - E_j^{b,T} \frac{1}{\nabla^2} E_k^{c,T} \right)^L + \text{ higher order}, \qquad (4.5)$$

$$A_i^a \to A_i^a - \eta \frac{1}{\nabla^2} \epsilon_{ijk} \partial_j E_k^{a,T} + \text{ higher order},$$
 (4.6)

and
$$A_r^a \to A_r^a$$
, (4.7)

where 'higher order' denotes cubic or higher than cubic order in weak fields expansion, η is the infinitesimal duality rotation angle and the superscripts T and L mean that only transverse and longitudinal parts of the terms would be kept respectively.

Under such transformation, the action changes as

$$S[E_i^a, A_i^a] \to S[E_i^a, A_i^a] + \frac{1}{2}\eta \int_{r=0} d^3x \left(A^a \cdot \nabla \times A^a + \epsilon^{abc} A^a \cdot A^b \times A^c + E^a \cdot \frac{1}{\nabla^2} (\nabla \times E^a) \right), \tag{4.8}$$

The action is invariant up to the boundary terms. These boundary terms will be treated as an infinitesimal boundary deformations. While the last term in the boundary action is non-local, first two terms are similar to the Chern-Simons term. Since the duality transformation is approximate, we are not able to get the relative factors correctly.

One may wonder if transformation (4.5) of the longitudinal part of the electric field will not be necessary since they appear at the quartic order and our transformations are applicable only up to cubic part of the action. However, the term proportional to A_r^a is eliminated from the action (4.1) by imposing the Gauss law constraint. To ensure that the Gauss law constraint is not affected up to this order requires the transformation (4.5).

5 Yang-Mills Instanton

In the previous section, we have developed various kinds of deformations to obtain the corresponding boundary actions. While there are many reasonable deformations, most of them are obtained by doing small amplitude expansion about perturbative classical solution. In this section, we will consider a nonperturbative solution in the bulk, namely, the instanton solution, and construct the boundary action corresponding to this Yang-Mills instanton solution in the bulk. The instanton solution in the flat space is known for a long time and we use the same solution[23, 24, 25]. The reason is that in four dimensions Yang-Mills theory is classically conformally invariant and the four dimensional anti-de Sitter space, AdS_4 , is conformally flat. As a result the instanton solution to Yang-Mills theory in the Euclidean AdS_4 has same form as that in the \mathbb{R}^4 . There is a crucial difference between these two cases because the Euclidean AdS_4 is conformally equivalent to \mathbb{R}^4_+ because of the semi-infinite range of the radial coordinate. This fact plays an important role in determining the boundary action. We start our discussion with the 't Hooft instanton[26] with winding number 1 which is a solution to the self-duality condition (3.36), and is given by

$$A_M^a(x, x_0, \rho) = -\frac{2}{g} \frac{\eta_{MN}^a (x - x_0)^N}{(x - x_0)^2 + \rho^2},$$
(5.1)

where, ρ is a real parameter which is size of the instanton and x_0^M indicates its position. For simplicity, we choose $x_0^4 = 0$, then our instanton solution is located on AdS boundary. The gauge condition is chosen as $\partial_M A^{aM} = 0$ and $\eta_{ij}^a = \epsilon^{aij}$, $\eta_{ir}^a = -\eta_{ri}^a = \delta_i^a$ for i = 1, 2, 3. Even if equations of motion are the same under the Weyl rescaling defined at the beginning in the Sec.2, the gauge condition does not. Lorentz gauge condition $\partial_M A^{aM} = 0$ in flat space is different from that in AdS space, which is $\partial_N(\sqrt{-G}G^{NM}(r)A_M^a) = 0$. However, the radial gauge is the same in both cases. It is therefore convenient to work in the radial gauge (For details of gauge transformation and the radial gauge solution of Yang-Mills instanton, See Appendix.D). The field strength of 't Hooft instanton solution is given by

$$F_{MN}^{a} = \frac{4}{g} \eta_{MN}^{a} \frac{\rho^{2}}{((x-x_{0})^{2} + \rho^{2})^{2}},$$
(5.2)

and the action has a finite value as $S[A_{instanton}] = \frac{8\pi^2}{g^2}$.

In flat \mathbb{R}^4_+ , the instanton solution(5.1) approaches pure gauge solution and the field strength(5.2) becomes zero at $x_4 = r \to \infty$. This region, however, gets mapped to the Poincare horizon in AdS_4 space under the Weyl scaling (See the beginning in Sec.2). Therefore, Yang-Mills instanton solutions do not change the boundary conditions at the horizon. Interestingly, the instanton solution does not become pure gauge solution at r = 0 and the field strength has the finite value. As shown in Appendix D, the Fefferman-Graham expansion of Yang-Mills instanton in the radial gauge near AdS boundary is given by

$$A_i^a = A_i^{a(0)} + rA_i^{a(1)} + O(r^2), (5.3)$$

with,

$$A_i^{a(0)} = -\frac{2}{g} \frac{\eta_{ij}^a (y - y_0)_j}{(y - y_0)^2 + \rho^2},$$
(5.4)

$$A_i^{a(1)} = -\frac{4}{g} \frac{\delta_i^a \rho^2}{((y - y_0)^2 + \rho^2)^2},$$
(5.5)

where we have defined a boundary coordinate $y^i \equiv x^i$, and $y^2 = \sum_{i=1}^3 y^i y^i$. $A_i^{a(1)}$ is related to $A_i^{a(0)}$ by the small r limit of the self-duality condition eq.(3.37), which in radial gauge becomes

$$A_i^{a(1)} = \frac{1}{2} \epsilon_{ijk} F_{jk}^{a(0)}.$$
(5.6)

It is easy to see that the boundary values of Yang-Mills instanton solution (5.4) and (5.5) satisfy the boundary condition (5.6). We want to write down this boundary condition in terms of the boundary field $A_i^{a(0)}$ only. Although there are various ways of expressing this boundary condition, we find representation of the Yang-Mills gauge field corresponding to the 't Hooft instanton solution in terms of a scalar function $\lambda(y)$ is most convenient for writing the boundary term.

The usual ansatz for Yang-Mills instanton solution is given by

$$A_{i}^{a(0)} = \frac{1}{g} \eta_{ij}^{a} \partial_{j} ln(\lambda(y)), \text{ where } \lambda(y) = \frac{\rho^{2}}{(y - y_{0})^{2} + \rho^{2}}.$$
(5.7)

This equation can be inverted to write $\lambda(y)$ as a non-local function of $A_i^{a(0)}$,

$$\lambda(y) = e^{\frac{g}{2}\epsilon^{ai}{}_{j}\int_{y_{0}}^{y}A_{i}^{a(0)}(z)dz^{j}}.$$
(5.8)

The $A_i^{a(1)}$ can be written in terms of $\lambda(y)$ as

$$A_i^{a(1)} = -\frac{4}{g} \delta_i^a \frac{\lambda^2(y)}{\rho^2}.$$
 (5.9)

With these, we can rewrite the self-dual boundary condition (5.6) in terms of $A_i^{a(0)}$ only as

$$\frac{1}{2}\epsilon_{ijk}F^{a(0)}_{jk}(z) = -\frac{4}{g\rho^2}\delta^a_i e^{g\int_0^z \epsilon^a_{ij}A^{a(0)}_i(\tilde{z})d\tilde{z}^j},$$
(5.10)

where $z \equiv y - y_0$.

We can now ask if it is possible to write down the boundary on-shell action such that the boundary condition (5.10) is the equation of motion of this boundary action. It is easy to seen that the left hand side of eq.(5.10) comes from the non-abelian Chern Simons action. The right hand side contains line integration in the exponent. This line integral resembles the Wilson line, but it, in fact, corresponds a non-local interaction. Such a term in the deformed boundary conditions cannot be obtained within the perturbative approach. Moreover, in the case of multi-instanton solution[26] in the bulk, the corresponding boundary condition would continue to have this type of non-local, although its precise form is different from eq.(5.10).

To discuss this boundary condition in general, we promote eq.(5.10) to a general boundary condition and treat ρ as a parameter in the corresponding boundary theory. The boundary on-shell action providing the boundary condition then takes the following form:

$$I_{os} = a \int_{-\infty}^{\infty} d^3 z \left[\epsilon_{ijk} \left(A_i^{a(0)}(z) \partial_j A_k^{a(0)}(z) - \frac{1}{3} g \epsilon^{abc} A_i^{a(0)}(z) A_j^{b(0)}(z) A_k^{c(0)}(z) \right) + \frac{1}{\rho^2} \mathcal{L}_{NL} \right],$$
(5.11)

where *a* is a real constant and \mathcal{L}_{NL} is a Lagrangian providing the non-local term in eq.(5.10) and we have pulled out ρ dependence explicitly. We now note that the coupling $\frac{1}{\rho^2}$ explicitly breaks the scaling symmetry as $z \to Lz$ and $A_i^{a(0)} \to \frac{1}{L}A_i^{a(0)}$ and its scaling property shows that it is a relevant coupling.

Since, we promote ρ as a parameter in the boundary on-shell action, we can take a limit as $\rho \to \infty$. In this limit, \mathcal{L}_{NL} will relatively suppressed, and the boundary theory becomes approximately pure Chern Simons theory.

6 Conclusion

We studied various boundary conditions for SU(2) Yang-Mills theory in AdS_4 background. One of the motivation was to introduce interactions in the boundary CFT. Momentum dependent cubic interactions in Yang-Mills theory lead to non-trivial interaction terms in the boundary action both in cases of Dirichlet and Neumann boundary conditions. We computed bulk Yang-Mills solutions to the first subleading order to incorporate the leading effects of Yang-Mills interaction. We found that in case of the Dirichlet boundary condition the boundary propagator is proportional to $|\vec{q}|$, where \vec{q} is three dimensional momentum. The cubic interaction has the form

$$\Delta_{ijk}^{D,abc}(q,l,p) \sim ig\epsilon^{abc}\delta^3(q+p+l)\frac{(l-q)_k\delta_{ij} + (p-l)_i\delta_{jk} + (q-p)_j\delta_{ik}}{2(|q|+|p|+|l|)}.$$
(6.12)

The Neumann boundary condition on the other hand has the propagator proportional to 1/|q|and the cubic interaction is

$$\Delta_{ijk}^{N,abc}(q,l,p) \sim \frac{\Delta_{ijk}^{D,abc}(q,l,p)}{|q||p||l|}.$$
(6.13)

Another motivation was to study more interesting boundary conditions like massive and self dual boundary conditions in the context of non-abelian gauge theory. While the massive boundary condition gives rise to massive gauge theory on the boundary, the self dual boundary condition takes the form of Bogomolnyi equation in the small r expansion around the boundary. The combined massive and self dual condition gives massive Chern Simons gauge theory action on the boundary. Equations of motion derived from this action were studied as self-duality conditions in odd dimensions[16].

We studied the effect of approximate electric-magnetic duality on SU(2) Yang-Mills theory defined on AdS_4 and resulting boundary contribution. Although the symmetry is not exact it seems to point towards a Chern-Simon like term on the boundary in addition to a non-local piece. It would be interesting to explore effects of duality on boundary conditions in the AdSspace, particularly in the context of supersymmetric gauge theories.

We also studied instanton solution in AdS_4 with unit charge. While it was a straightforward generalization of the solution in \mathbb{R}^4 due to conformal invariance of classical action and selfduality condition, implication of the solution are quite interesting. In contrast to what happens in AdS_5/CFT_4 correspondence, where D-instantons in AdS_5 do not modify the boundary condition, in AdS_4 case, the Yang-Mills instanton becomes pure gauge on the Poincare horizon and modifies the boundary condition on AdS boundary. We showed that the boundary action is the Chern Simons action with a non-local deformation. It would be interesting to understand this non-local deformation better.

In this paper we concentrated only on the gauge field sector, it would be interesting to combine it with analysis of fermion boundary conditions [5]. In particular, it would be interesting to classify supersymmetric boundary conditions⁶. This analysis however is beyond the scope of this work but we will address some of these issues in future.

Acknowledgments

We would like to thank Stanley Deser, Ashoke Sen, Rajesh Gopakumar, Suvrat Raju, Sudhakar Panda for useful discussion. Especially, Jae-Hyuk Oh thanks his W.J. We would also like to thank anonymous referee for useful suggestions and providing pointers in those directions. This work is partially supported by 11-R&D-HRI-5.02-0304 at Harish-Chandra Research Institute, India.

⁶For some work along these lines see [13].

Appendix

A Gauge Transformation

The most general form of non-abelian gauge transformation is given by

$$A_{M}^{a\prime}(x^{P})\frac{\sigma^{a}}{2} = V(x^{P})\left(A_{M}^{a}(x^{P})\frac{\sigma^{a}}{2} - \frac{i}{g}\partial_{M}\right)V^{-1}(x^{P}),\tag{A.1}$$

where

$$V(x) = exp\left(-ig\phi^a \frac{\sigma^a}{2}\right) \tag{A.2}$$

and σ^a are Pauli matrices. It turns out that the right gauge transformation up to quadratic order in gauge fields or gauge parameter is given by

$$A_M^{a\prime} \to A_M^a + \partial_M \phi^a - g \epsilon^{abc} A_M^b \phi^c + \frac{1}{2} g \epsilon^{abc} \phi^b \partial_M \phi^c + \text{ higher order}, \tag{A.3}$$

where ϕ^a is a gauge parameter which would be expanded as $\phi^a = \varepsilon \overline{\phi}^a + \varepsilon^2 \widetilde{\phi}^a + O(\varepsilon^3)$ and "higher" means that the higher orders in weak fields, ϕ^a or A^a_M . We evaluate this relation order by order in ε as

First order in
$$\varepsilon : \bar{A}_M^{a\prime} \to \bar{A}_M^a + \partial_M \bar{\phi}^a$$
 (A.4)

Second order in
$$\varepsilon : \tilde{A}^{a\prime}_M \to \tilde{A}^a_M + \partial_M \tilde{\phi}^a - g \epsilon^{abc} \bar{A}^b_M \bar{\phi}^c + \frac{1}{2} g \epsilon^{abc} \bar{\phi}^b \partial_M \bar{\phi}^c.$$
 (A.5)

Under these transformation, the field strengths are transformed as

First order in
$$\varepsilon : \bar{F}_{MN}^{a\prime} \to \bar{F}_{MN}^{a}$$
, (A.6)

Second order in
$$\varepsilon : \tilde{F}_{MN}^{a\prime} \to \tilde{F}_{MN}^a - g\varepsilon^{abc}\bar{\phi}^c\bar{F}_{MN}^b$$
 (A.7)

B Evaluation of the Second order Bulk Solution

We start with eq.(2.19). We plug the first order solution(2.11) into this and we get

$$0 = \left(\nabla^2 \tilde{A}^a_r - \partial_r \partial_j \tilde{A}^a_j\right) - g \epsilon^{abc} \left(\nabla^2 \bar{\phi}^b \partial_r \bar{\phi}^c + (\partial_j \bar{\phi}^b + \bar{A}^{bT}_j) \partial_j \partial_r \bar{\phi}^c + (\partial_j \bar{\phi}^c + \bar{A}^{cT}_j) \partial_r \bar{A}^{bT}_j\right).$$
(B.1)

We want to solve this equation in momentum space, so we perform Fourier transform for any fields appearing in the equation using eq.(2.12). Then, the equation becomes

$$q^{2}\tilde{A}^{a}_{r,q} - iq_{j}\partial_{r}\tilde{A}^{a}_{j,q} = -g\epsilon^{abc}\int_{-\infty}^{\infty} d^{3}p \left(-p_{j}q_{j}\bar{\phi}^{b}_{p}\partial_{r}\bar{\phi}^{c}_{q-p} + \bar{A}^{cT}_{jp}\partial_{r}\bar{A}^{bT}_{j,q-p} - ip_{j}\partial_{r}(\bar{\phi}^{c}_{p}\bar{A}^{bT}_{j,q-p})\right).$$
(B.2)

We would like to define RHS of this equation as source terms, which come from the first order solution. Under radial gauge(2.16), the first terms in both side on the above equation vanish since the gauge parameter $\bar{\phi}$ does not depend on r and $\tilde{A}_r^a = 0$. Using

$$\partial_r \bar{A}^a_{M,q}(r) = -|q| \bar{A}^a_{M,q}(r), \tag{B.3}$$

we get eq.(2.23).

To manipulate N = i equations, we substitute the first order solution(2.11) into eq.(2.20). Then, we obtain

$$0 = (\partial_r^2 + \nabla^2) \tilde{A}_i^a - \partial_i (\partial_r \tilde{A}_r^a + \partial_j \tilde{A}_j^a) - g \epsilon^{abc} \left\{ \partial_r (\partial_r \bar{\phi}^b (\partial_i \bar{\phi}^c + \bar{A}_i^{cT})) + \partial_j \left((\partial_j \bar{\phi}^b + \bar{A}_j^{bT}) (\partial_i \bar{\phi}^c + \bar{A}_i^{cT}) \right) - (\partial_j \bar{A}_i^{bT} - \partial_i \bar{A}_j^{bT}) (\partial_j \bar{\phi}^c + \bar{A}_j^{cT}) - \partial_r \bar{\phi}^c \partial_r \bar{A}_i^{bT} \right\}.$$
(B.4)

By performing Fourier transform, the momentum space expression of the equation becomes

$$0 = (\partial_{r}^{2} - q^{2})\tilde{A}_{iq}^{a} + iq_{i}(\partial_{r}\tilde{A}_{rq}^{a} - iq_{j}\tilde{A}_{jq}^{a}) - g\epsilon^{abc} \int_{-\infty}^{\infty} \left\{ \partial_{r}(\partial_{r}\bar{\phi}_{p}^{b}(-i(q-p)_{i}\bar{\phi}_{q-p}^{c} + \bar{A}_{i,q-p}^{cT}) - iq_{j}(-ip_{j}\bar{\phi}_{p}^{b} + \bar{A}_{j,p}^{bT})(-i(q-p)_{i}\bar{\phi}_{q-p}^{c} + \bar{A}_{i,q-p}^{cT}) - \partial_{r}\bar{\phi}_{p}^{c}\partial_{r}\bar{A}_{i,q-p}^{bT} - (-ip_{j}\bar{\phi}_{p}^{c} + \bar{A}_{j,p}^{cT})(-i(q-p)_{j}\bar{A}_{i,q-p}^{bT} + i(q-p)_{i}\bar{A}_{j,q-p}^{bT}) \right\} d^{3}p.$$

To obtain solutions, we plug eq.(B.2) into eq.(B.5) and we get

$$\begin{aligned} (\partial_{r}^{2} - q^{2})P_{ij}(q)\tilde{A}_{jq}^{a} &= g\epsilon^{abc} \int_{-\infty}^{\infty} d^{3}p \left\{ iP_{ij}(q)p_{j}\partial_{r}(\bar{\phi}_{q-p}^{c}\partial_{r}\bar{\phi}_{p}^{b}) + P_{ij}(q)\partial_{r}^{2}(\bar{\phi}_{p}^{b}\bar{A}_{i,q-p}^{cT}) \right. \\ &+ \frac{i}{2}q_{j}(p_{j}q_{i} - p_{i}q_{j})\bar{\phi}_{p}^{b}\bar{\phi}_{q-p}^{c} + \bar{\phi}_{p}^{c}\left((\partial_{r}^{2} + (2q-p)_{k}p_{k})\delta_{ij} - q_{i}q_{j}\right)\bar{A}_{j,q-p}^{bT} \\ &+ \bar{A}_{k,p}^{cT}\left(\frac{iq_{i}}{q^{2}}\delta_{jk}\partial_{r}^{2} - i(q-p)_{i}\delta_{jk} + iq_{k}\delta_{ij} - iq_{j}\delta_{ik}\right)\bar{A}_{j,q-p}^{bT} \right\}, \end{aligned}$$
(B.6)

where $P_{ij}(q) = \delta_{ij} - \frac{q_i q_j}{q^2}$ is a projection operator to the transverse part of gauge field. Contracting q_i to both sides of the equation, one can see that both sides are identically zero. The radial gauge condition eliminates the first term on the *RHS*. The terms proportional to $\bar{\phi}^b \bar{A}_i^c$ are combined and they are

$$\bar{\phi}^b \bar{A}_i^c \sim (|q-p|^2 - q^2) P_{ij}(q) \bar{\phi}_p^b \bar{A}_{i,q-p}^c, \tag{B.7}$$

using $\partial_r^2 A_{i,q}^a(r) = |q|^2 A_{i,q}^a(r)$. The term proportional to $\bar{\phi}^b \bar{\phi}^c$ can be written as

$$\frac{i}{2}q_j(p_jq_i - p_iq_j)\bar{\phi}^b_p\bar{\phi}^c_{q-p} = -\frac{i}{2}q^2P_{ij}(q)p_j\bar{\phi}^b_p\bar{\phi}^c_{q-p}.$$
(B.8)

These equations can be used to obtain eq.(2.24).

C Bulk Solutions in the Position Space

In Sec.2, we have obtained bulk solutions in the momentum space. In this section, we would provide position space expressions, which are given by

$$\bar{A}_{i}^{a}(r,x) = \partial_{i}\bar{\phi}^{a}(r,x) + \bar{A}_{i}^{aT}(r,x), \quad \bar{A}_{r}^{a} = \partial_{r}\bar{\phi}^{a}(r,x), \quad \partial_{i}\bar{A}_{i}^{aT}(r,x) = 0, \quad (C.1)$$

and $\bar{A}_{i,p}^{aT}(r,x) = \cosh(\sqrt{-\nabla^{2}r})\bar{A}_{i}^{aT(0)}(x) + \frac{1}{\sqrt{-\nabla^{2}}}\sinh(\sqrt{-\nabla^{2}r})\bar{A}_{i}^{aT(1)}(x),$

with

$$\bar{A}_{i}^{aT}(r,x) = e^{-\sqrt{-\nabla^{2}}r} \bar{A}_{i}^{aT(0)}(x), \qquad (C.2)$$

by the regularity condition. To maintain the radial gauge, we need

$$\bar{\phi}^a(r, x^i) \to \bar{\phi}^a(x^i).$$
 (C.3)

The position space expressions for solutions up to $O(\varepsilon^2)$, eq.(2.27) and eq.(2.26), are given by

$$\tilde{A}_{i}^{aT}(x) = g\epsilon^{abc}\bar{A}_{k}^{cT(0)}(x)\alpha_{ijk}(\partial_{l},\overleftarrow{\partial}_{l})e^{-(\sqrt{-\overleftarrow{\nabla}^{2}}+\sqrt{-\nabla^{2}})r}\bar{A}_{j}^{cT(0)}(x), \qquad (C.4)$$

$$\tilde{A}_{i}^{aL}(x) = -g\epsilon^{abc}\frac{\partial_{i}}{\nabla^{2}} \left(\bar{A}_{j}^{cT(0)}(x)\frac{e^{-(\sqrt{-\nabla^{2}}+\sqrt{-\nabla^{2}})r}}{\sqrt{-\nabla^{2}}}\sqrt{-\nabla^{2}}\bar{A}_{j}^{bT(0)}(x) \right), \quad (C.5)$$

where

$$\alpha_{ijk}(\partial_l, \overleftarrow{\partial}_l) = \frac{\left(\left(\frac{\overleftarrow{\partial_i}}{(\overleftarrow{\partial_i}+\partial_i)^2}\nabla^2 + \partial_i\right)\delta_{jk} - (\overleftarrow{\partial}_k + \partial_k)\delta_{ij} + (\overleftarrow{\partial}_j + \partial_j)\delta_{ik}\right)}{(\sqrt{-\overleftarrow{\nabla^2}} + \sqrt{-\nabla^2})^2 + (\overleftarrow{\partial_i} + \partial_i)^2}, \quad (C.6)$$

and the differential operators with arrows indicate that such operators act to the left and the operators without arrows would act to the right.

D Yang-Mills Instantons in Radial Gauge

The usual Yang-Mills instanton solution is given by

$$A_M^a(x, x_0, \rho) = -\frac{2}{g} \frac{\eta_{MN}^a (x - x_0)^N}{(x - x_0)^2 + \rho^2},$$
 (D.1)

For the further use, we need to transform this expression into radial gauge. In this section, we explicitly construct the gauge transformation from the above expression to radial gauge solution. First, we separate the instanton solution into r-directional and i directional pieces as

$$A_r^a(r, y, \rho) = \frac{2}{g} \frac{\delta_i^a(y - y_0)_i}{r^2 + (y - y_0)^2 + \rho^2},$$
 (D.2)

$$A_i^a(r, y, \rho) = -\frac{2}{g} \left(\frac{\delta_i^a r}{r^2 + (y - y_0)^2 + \rho^2} + \frac{\eta_{ij}^a (y - y_0)_j}{r^2 + (y - y_0)^2 + \rho^2} \right),$$
(D.3)

where $x^4 \equiv r$ and $x^i \equiv y^i$ for i = 1, 2, 3. For some gauge transformation, we want to eliminate A_r^a . It turns out that such gauge transformation is given by

$$V(x^P) = e^{-Z(x^P)},$$
 (D.4)

where

$$Z(x) = -\frac{i\sigma^a \delta_i^a (y - y_0)_i}{\sqrt{(y - y_0)^2 + \rho^2}} tan^{-1} \left(\frac{r}{\sqrt{(y - y_0)^2 + \rho^2}}\right).$$
 (D.5)

The question is that what is the form of A_i^a in the radial gauge. It has a form of

$$A_{i}^{a}(r, y, \rho)\frac{\sigma^{a}}{2} = e^{-Z(x^{P})}\frac{\sigma^{a}}{g}Q_{i}^{a}e^{Z(x^{P})},$$
(D.6)

where

$$Q_{i}^{a} = -\left(\frac{1}{\sqrt{(y-y_{0})^{2}+\rho^{2}}}tan^{-1}\left(\frac{r}{\sqrt{(y-y_{0})^{2}+\rho^{2}}}\right) + \frac{r}{r^{2}+(y-y_{0})^{2}+\rho^{2}}\right)\delta_{j}^{a}\Sigma_{ij}(D.7) - \frac{\eta_{ij}^{a}(y-y_{0})_{j}}{r^{2}+(y-y_{0})^{2}+\rho^{2}},$$

and

$$\Sigma_{ij} = \delta_{ij} - \frac{(y - y_0)_i (y - y_0)_j}{(y - y_0)^2 + \rho^2}.$$
 (D.8)

Even if we do not obtain a compact form of the solution, we might get the near boundary expansion of this instanton solution using the boundary expansion of $V(x^P)$ as

$$V(x^{P}) = 1 + \frac{i\sigma^{a}\delta^{a}_{j}(y-y_{0})_{j}r}{(y-y_{0})^{2} + \rho^{2}} + O(r^{3}).$$
(D.9)

With this, one can expand A_i^a near boundary as

$$A_i^a = A_i^{a(0)} + rA_i^{a(1)} + O(r^2), (D.10)$$

where

$$A_i^{a(0)} = -\frac{2}{g} \frac{\eta_{ij}^a (y - y_0)_j}{(y - y_0)^2 + \rho^2},$$
 (D.11)

$$A_i^{a(1)} = -\frac{4}{g} \frac{\delta_i^a \rho^2}{((y-y_0)^2 + \rho^2)^2}.$$
 (D.12)

 $A_i^{a(0)}$ should be an instanton solution of boundary effective action of the dual field theory with self-dual boundary condition.

References

- Peter Breitenlohner, Daniel Z Freedman, Phys. Lett. B115 (1982), 197, Peter Breitenlohner, Daniel Z Freedman, Ann. Phys. 144 (1982) 249.
- [2] Igor R. Klebanov, Edward Witten, Nucl. Phys. B556:89-114 (1999), arXiv:hep-th/9905104.
- [3] Edward Witten, arXiv:hep-th/0112258.
- [4] Amit Sever, Assaf Shomer, JHEP 0207 (2002) 027, arXiv:hep-th/0203168.
- [5] M. Henneaux, In *Tbilisi 1998, Mathematical methods in modern theoretical physics* 161-170 [hep-th/9902137].
- [6] R. C. Rashkov, Mod. Phys. Lett. A 14, 1783 (1999) [hep-th/9904098].
- [7] W. Mueck and K. S. Viswanathan, Phys. Rev. D 60, 081901 (1999) [hep-th/9906155].
- [8] D. Marolf, JHEP **0505**, 042 (2005) [hep-th/0412032].
- [9] Sebastian de Haro, Ioannis Papadimitriou, Anastasios C. Petkou, Phys.Rev.Lett.98:231601 (2007), arXiv:hep-th/0611315.
- [10] Sebastian de Haro, Anastasios C. Petkou, JHEP 0612:076 (2006), arXiv:hep-th/0606276.
- [11] Edward Witten, arXiv:hep-th/0307041.
- [12] Donald Marolf, Simon Ross, JHEP0611:085 (2006), arXiv:hep-th/0606113.
- [13] Sebastian de Haro, Peng Gao, Phys.Rev.D76:106008 (2007), arXiv:hep-th/0701144.
- [14] Shiraz Minwalla, Adv. Theor. Math. Phys. 2:781-846 (1998), arXiv:hep-th/9712074.
- [15] Robert G. Leigh, Anastasios C. Petkou, JHEP 0312 (2003) 020, arXiv:hep-th/0309177.
- [16] P.K. Townsend, K. Pilch, P. Van Nieuwenhuizen, Phys.Lett.B 136, 38 (1984).
- [17] Ho-Ung Yee, Phys.Lett. B598 (2004) 139-14, arXiv:hep-th/0402115.
- [18] Vijay Balasubramanian, Per Kraus, Commun.Math.Phys. 208 (1999) 413-428, arXiv:hep-th/9902121.
- [19] Roberto Emparan, Clifford V. Johnson, Robert C. Myers, Phys.Rev. D60 (1999) 104001, arXiv:hep-th/9903238.
- [20] Dileep P. Jatkar, Aninda Sinha, Phys.Rev.Lett.106:171601 (2011), arXiv:1101.4746.
- [21] Ioannis Papadimitriou, JHEP 0705:075 (2007), arXiv:hep-th/0703152.
- [22] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48, 975978 (1982), S. Deser, R. Jackiw, S. Templeton, Annals of Physics 281, 409 (2000).

- [23] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. 59B (1975) 85-87.
- [24] R. Rajaraman, "Solitons and Instantons", North-Holland Publishing.
- [25] Stefan Vandoren, Peter van Nieuwenhuizen, arXiv:0802.1862.
- [26] G. 't Hooft, Phys. Rev. D 14, 3432 (1976) [Erratum-ibid. D 18, 2199 (1978)].
- [27] Stanley Deser and Claudio Teitelboim, Phys. Rev. D 13, 15921597 (1976).
- [28] S. Deser, D. Seminara, Phys.Rev.D71:081502,2005, arXiv:hep-th/0503030.