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HOLOGRAPHIC SUPERCONDUCTOR FOR A LIFSHITZ FIXED POINT

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We consider the gravity dual of strongly coupled system at a Lifshitz-fixed point and finite temperature, which was constructed in a recent work arXiv:0909.0263. We construct an Abelian–Higgs model in that background and calculate condensation and conductivity using holographic techniques. We find that condensation happens and DC conductivity blows up when temperature turns below a critical value. We also study the zero temperature limit of strongly coupled system at the Lifshitz-fixed point.

Keywords: AdS/CFT; holography; superconductor.

1. Introduction

AdS/CFT correspondence[1] is one of the most interesting results in the sense that it opened a window of connecting the string theory to the QCD and condensed matter systems. The connection between gauge theory and strings has a long history since the appearance of string models of hadrons in 1960's: for example, it has been observed that the elementary excitations of a lattice gauge theory in the strong coupling limit can be represented by strings formed by color-electric fluxes. It is also suggested that in a certain limit all the degrees of freedom in the gauge theory should be represented by the flux lines (strings) instead of fields (see Refs. 3 and 2 and references therein). Therefore it is natural to expect an exact duality between gauge fields and strings although its precise formulation was obtained only recently. The semiclassic version of this duality can be stated as gauge/gravity duality and such duality has become a powerful tool to understand the strongly coupled QCD and the properties of quark–gluon plasma in heavy ion collisions at RHIC[4–6,a] as well as the low energy hadron physics.

More recently, it has been attempted to use this correspondence to describe certain condensed matter systems such as the quantum Hall effect,[8] Nernst effect,[9–11] superconductor[12,14,15] and fractional quantum Hall effect (FQHE).[16] These phenomena were suggested to have dual gravitational descriptions. As pointed out in Refs. 17–19, there is a large class of interesting strongly correlated electronic and atomic systems that can be created and studied in experiments and there are nonrelativistic systems which have Schrödinger symmetry.[17–19] However, the dynamics of such systems near a critical point is described by a relativistic conformal field theory or sometimes more subtle scaling theory having Lifshitz symmetry.[20]

To describe the finite temperature version of such scaling system, four-dimensional black hole solutions with asymptotically Lifshitz space–times were investigated.[21–25] The Lifshitz black holes in arbitrary dimensions were also found in a different class of action.[26] Recently an analytical solution in yet another action was proposed for $z = 2$ in four dimensions.[27] Additionally, Lifshitz black holes in three-dimensional massive gravity and four-dimensional R^2 gravity were also discussed.[28,29] Embedding those black holes with the action in Ref. 20 into string theory was addressed in Ref. 30.

In Ref. 12, a model of a strongly coupled system which shows superconductivity was constructed based on holography, which is an Abelian–Higgs model in a warped space–time. While the electrons in real materials are nonrelativistic, the model in Ref. 12 is relativistic. Therefore it is natural to ask whether one can develop a similar model with nonrelativistic kinematics,[37] especially at Lifshitz-like fixed point. One purpose of this paper is to answer this question. We find that there is a critical temperature, like the relativistic case, below which a charged scalar field condensate and the (DC) conductivity blows up. We also calculated the frequency dependent conductivity.

This paper is organized as follows. In Sec. 2, we check thermodynamics of the Lifshitz black hole and chemical potential background. In the following section, we study the superconductive phases in the Lifshitz background. We obtain similar results as in the usual AdS black hole background. In Sec. 4, we consider the zero temperature limit of our system and we conclude there is no hard gap for the AC conductivity.

[a]For recent review, see Ref. 7.

2. Gravity Dual of the Lifshitz Fixed Point

We begin with the equilibrium properties of the strongly coupled thermal field at Lifshitz-like fixed point, by analyzing the Lifshitz black hole solutions.

2.1. *Lifshitz black hole solutions*

The Lifshitz scaling is defined by

$$t \to \lambda^z t\,, \quad x \to \lambda x\,, \tag{1}$$

where z is called dynamical exponent. The metric with this symmetry was first found in Ref. 20:

$$ds^2 = L^2\left(-\frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2}\right), \tag{2}$$

where $0 < r < \infty$ and L sets the scale for the radius of curvature. For $z = 1$, this geometry is anti-de Sitter space–time. For $z > 1$, it is a candidate for the dual gravity of a field theory with Lifshitz scaling.

The tidal forces diverge on the "horizon" at $r \to \infty$ unless $z = 1$ and this implies that the metric (2) has no global extension.[31] To describe the physics of the dual field theory at finite temperature, black hole solutions with asymptotical Lifshitz metric (2) were studied in Refs. 21–25.

Recently, a 4D black hole solution which asymptotes to the Lifshitz space–time (2) was constructed.[27] The action is

$$\begin{aligned} S = {} & \frac{1}{2}\int d^4x\sqrt{-g}(R - 2\Lambda) \\ & - \int d^4x\sqrt{-g}\left(\frac{e^{-2\phi}}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + (e^{-2\phi} - 1)\right), \end{aligned} \tag{3}$$

where $\Lambda = -\frac{z^2+z+4}{2}$, $m^2 = 2z$ and $F = dA$. The gravitational constant and curvature radius are set by $8\pi G_4 = 1$ and $L = 1$ respectively. With this convention the equations of motion are

$$\begin{aligned} & F^2 = -4\,, \quad \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}e^{-2\phi}F^{\mu\nu}\right) = m^2 A^\nu\,, \\ & R_{\mu\nu} = e^{-2\phi}F_{\mu\lambda}F_\nu^{\lambda} + m^2 A_\mu A_\nu + \Lambda g_{\mu\nu} + \left(2e^{-2\phi} - 1\right)g_{\mu\nu}\,, \end{aligned} \tag{4}$$

and the black hole solution of this system is[b]

$$\begin{aligned} & ds^2 = -f(r)\,\frac{dt^2}{r^{2z}} + \frac{dx^2 + dy^2}{r^2} + \frac{dr^2}{r^2 f(r)}\,, \\ & f(r) = 1 - \frac{r^2}{r_H^2}\,, \quad e^{-2\phi} = 1 + \frac{r^2}{r_H^2}\,, \quad A = \frac{f(r)}{\sqrt{2}r^2}\,dt\,. \end{aligned} \tag{5}$$

[b]There is a factor $1/\sqrt{2}$ missing in the expression (2.5) of the massive vector field in Ref. 27.

In the rest of this paper, we will use this solution to discuss the transport and superconductivity.

2.2. *Thermodynamics*

We first review the thermodynamics of this black hole proposed in Ref. 27. Our calculational procedure follows.[31] According to the AdS/CFT dictionary, the partition function of the bulk theory is identified to that of the dual field theory. The path integral over metrics is dominated by the saddle point g_*, and the partition function is

$$Z = e^{-S_E[g_*]}\,, \tag{6}$$

where $S_E[g_*]$ is the Euclidean action evaluated on the saddle. This action must contain extrinsic boundary terms and intrinsic boundary terms in order to render the finiteness of the on-shell action. This was already given in Ref. 27:

$$\begin{aligned} S_E = &-\frac{1}{2}\int d^4x\sqrt{g}(R-2\Lambda) \\ &+\int d^4x\sqrt{g}\left(\frac{e^{-2\phi}}{4}F^2+\frac{m^2}{2}A^2+\left(e^{-2\phi}-1\right)\right) \\ &+\int_{r\to 0} d^3x\sqrt{\gamma}K-\frac{1}{2}\int_{r\to 0} d^3x\sqrt{\gamma}\left(-\frac{27}{8}+\frac{7}{2}\phi+\frac{7}{2}\phi^2\right) \\ &-\frac{1}{2}\int_{r\to 0} d^3x\sqrt{\gamma}\left(\left(\frac{17}{2}+7\phi\right)A^2+\frac{13}{2}A^4\right), \end{aligned} \tag{7}$$

where γ is the induced metric on the boundary $r \to 0$ and K is the trace of the extrinsic curvature. The Dirichlet boundary condition is imposed on the massive vector field.[c]

A saddle is obtained by Wick rotating Eq. (5):

$$ds_*^2 = f(r)\frac{d\tau^2}{r^{2z}} + \frac{dx^2+dy^2}{r^2} + \frac{dr^2}{r^2 f(r)}\,, \quad A = -i\frac{f(r)}{\sqrt{2}r^2}\,d\tau\,. \tag{8}$$

The temperature of the system is

$$T = \frac{1}{\beta} = \frac{1}{2\pi r_H^z}\,, \tag{9}$$

determined by the absence of the conical singularity at $r = r_H$.

We can now evaluate the action (7):

$$S_E[g_*] = -\beta\frac{L_x L_y}{2r_H^4} = -2\pi^2 L_x L_y T\,, \tag{10}$$

[c]That is $c_N = 0$ in the expression (3.3) in Ref. 27. We rewrote that expression into Euclidean space and substitute the specific values of c_0–c_5. There is a minus sign difference of c_1–c_5 here from those given in App. A of Ref. 27.

and the free energy

$$\mathcal{F} = -T \log Z = TS_E[g_*] = -\frac{L_x L_y}{2r_H^4} = -2\pi^2 L_x L_y T^2 \,, \tag{11}$$

as given in Ref. 27.

As a check, the entropy

$$\mathcal{S} = -\frac{\partial \mathcal{F}}{\partial T} = 4\pi^2 L_x L_y T \,, \tag{12}$$

coincides with the Bekenstein–Hawking entropy $\mathcal{S} = 2\pi A$ with the area of the event horizon $A = L_x L_y / r_H^2$ and the unit convention $8\pi G = 1$.

The boundary stress tensor resulting from (7) is[d]

$$\begin{aligned} T_{\mu\nu} &\equiv -\frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}} \\ &= K_{\mu\nu} - \left(\frac{17}{2} + 7\phi + 13A^2\right) A_\mu A_\nu - K\gamma_{\mu\nu} \\ &\quad + \frac{1}{2}\left(-\frac{27}{8} + \frac{7}{2}\phi + \frac{7}{2}\phi^2\right)\gamma_{\mu\nu} \\ &\quad + \frac{1}{2}\left(\left(\frac{17}{2} + 7\phi\right)A^2 + \frac{13}{2}A^4\right)\gamma_{\mu\nu} \,, \end{aligned} \tag{13}$$

then the internal energy and pressure of boundary theory following[27] are respectively

$$\begin{aligned} \mathcal{E} &= -L_x L_y \sqrt{-\gamma} T_t^t = \frac{L_x L_y}{2r_H^4} \,, \\ \mathcal{P} &= \frac{1}{2} L_x L_y \sqrt{-\gamma} T_i^i = L_x L_y \sqrt{-\gamma} T_x^x = \frac{L_x L_y}{2r_H^4} \,. \end{aligned} \tag{14}$$

Thus we have

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{1}{2} T\mathcal{S} \,. \tag{15}$$

The first law of thermodynamics $\mathcal{E} + \mathcal{P} = T\mathcal{S}$ is satisfied as given in Ref. 27.

2.3. *Finite chemical potential*

Now we consider a probe gauge field fluctuation A_μ in the Lifshitz black hole background. That means we need to add another Maxwell term to the original action (3)[e]

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \,. \tag{16}$$

[d] There are also some minus sign differences between the expression we give and the one in (3.4) of Ref. 27.

[e] In this subsection and the rest of this paper, A and F should be distinguished with those in the original action (3) where the vector field is not a gauge field. From now on, by A_μ and F we mean the Maxwell fluctuation and its strength.

For simplicity, we ignore the backreaction. It means that A_μ is a small perturbation and the metric is still the same as (5). This vector field is expected to support a charge current operator J_μ in the dual field. The equation of motion of A_μ is

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}F^{\mu\nu}\big) = 0\,. \tag{17}$$

If we only consider the zero component of A_μ, $A = \phi(r)dt$, then we have

$$\phi'' + \frac{z-1}{r}\phi' = 0\,. \tag{18}$$

Near the boundary,

$$\phi = \phi_{(0)} + \phi_{(1)}r^{2-z}\,, \tag{19}$$

where $\phi_{(0)}$ and $\phi_{(1)}$ are chemical potential and charge density respectively in the dual field theory if $z < 2$. In the special case $z = 2$,

$$A_0 = \mu_0 - \rho\log\frac{r}{r_*}\,. \tag{20}$$

The coefficient ρ of the log term is precisely the charge density, which in grand canonical ensemble, is defined as the derivative of the boundary action term with respect to the chemical potential μ_0 (see also Ref. 38 for related discussion).

3. Superconductivity

In this section, we will build an Abelian–Higgs model[33,12] in the Lifshitz black hole background and study the superconductive phase. We introduce a new U(1) gauge field A_μ which is different from that in the action (6) and also introduce a complex scalar ψ. We assume that the background response is negligible for simplicity.

3.1. *Superconductive phases*

Considering the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\nabla\psi - iA\psi|^2 - V(|\psi|)\,, \tag{21}$$

we have equations of motion for A and ψ are respectively

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}F^{\mu\nu}\big) = iq[\psi^*(\partial^\nu - iqA^\nu)\psi - \psi(\partial^\nu + iqA^\nu)\psi^*]\,, \tag{22}$$

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}(\partial^\mu\psi - iqA^\mu\psi)\big) - iqA^\mu(\partial_\mu\psi - iqA_\mu\psi) - \frac{\psi}{2|\psi|}V'(|\psi|) = 0\,. \tag{23}$$

We will work in the probe limit, in which A_μ and ψ are taken to be small so that their backreaction on the space–time metric can be ignored. The metric is still a four-dimensional Lifshitz black hole with $z = 2$ in (5).

Taking the ansatz $A = \phi(r)dt$, $\psi = \psi(r)$, these equations of motion reduce to

$$\phi'' + \frac{z-1}{r}\phi' - \frac{2\psi^2}{r^2 f(r)}\phi = 0\,, \tag{24}$$

$$\psi'' + \left[\frac{f'(r)}{f(r)} - \frac{z+1}{r}\right]\psi' + \frac{r^{2z-2}\phi^2}{f^2(r)}\psi - \frac{V'(\psi)}{2r^2 f(r)} = 0\,, \tag{25}$$

where ψ can be taken to be real which is allowed by the r-component of (22). For simplicity we will specialize to a simple potential $V(\psi) = m^2|\psi|^2$ with $m^2 < 0$ but above the Breitenlohner–Freedman bound. Then near the boundary $r \to 0$ the bulk fields behave as

$$\phi = \mu + \rho r^{2-z} + \cdots\,, \tag{26}$$

$$\psi = \psi_{(0)} r^{\nu_-} + \psi_{(1)} r^{\nu_+} + \cdots \tag{27}$$

with $\nu_\pm = \frac{z+2}{2} \pm \sqrt{m^2 + \left(\frac{z+2}{2}\right)^2}$. At the horizon $\phi(r_H) = 0$ and (25) implies

$$\psi'(r_H) = -\frac{m^2}{2r_H}\psi(r_H)\,. \tag{28}$$

For $z = 2$, as we mentioned before, there is a Log singularity for the second term on the right-hand side of (26). We will study the regularized on-shell action in the following. Including $\phi(r)$ and $\psi(r)$, the bulk action can be rewritten as

$$S_{\text{bulk}} = V_3 \int dr\sqrt{-g}\left[-\frac{1}{2}\,g^{rr}g^{tt}(\partial_r\phi)^2 - g^{rr}(\partial_r\psi)^2 - g^{tt}\phi^2\psi^2 - m^2\psi^2\right]. \tag{29}$$

After doing the above integral by part and using the equations of motion, we have

$$\begin{aligned} S_{\text{on-shell}} = V_3 \Bigg[& \sqrt{-g}\phi\left(-\frac{1}{2}\,g^{rr}g^{tt}\partial_r\phi\right)\Bigg|_{r_B}^{r_H} \\ & + \sqrt{-g}\psi(-g^{rr}\partial_r\psi)\Big|_{r_B}^{r_H} + \int_{r_B}^{r_H} \sqrt{-g}g^{tt}\phi^2\psi^2\Bigg]. \end{aligned} \tag{30}$$

Actually, the properties of boundary behaviors of all three terms in (30) heavily depend on the parameter $\nu_\pm$ related to m. In our discussion, the asymptotical behaviors of the second and the third term in (30) are suppressed by $\psi(r)$. Using the asymptotical behaviors of $\phi(r)$ and $\psi(r)$, with Eq. (26) replaced by

$$\phi(r) = \mu_0 - \rho\log\frac{r}{r_*} + \cdots\,, \tag{31}$$

the action can be given by

$$S = S_{\text{on-shell}} = V_3\left[-\frac{1}{2}\,\rho\left(\mu_0 - \rho\log\frac{\epsilon}{r_*}\right) + \cdots\right], \tag{32}$$

where r_* was introduced in (20). Finally the regulated Euclidean total action of boundary field theory is given by

$$S = \frac{V_2}{T}\left(-\frac{1}{2}\rho\mu_{\rm sc} + \int_{r_B}^{r_H} \sqrt{-g}g^{tt}\phi_c^2\psi_c^2\right), \tag{33}$$

where $\mu_{\rm sc} = \mu_0$. We need to integrate the classical solution ϕ_c and ψ_c. The free energy is obtained by Legendre transformation:

$$F_{\rm sc} = TS + \mu_{\rm sc}\rho V_2 = V_2\left(\frac{1}{2}\rho\mu_{\rm sc} + \int_{r_B}^{r_H} \sqrt{-g}g^{tt}\phi_c^2\psi_c^2\right). \tag{34}$$

For the normal state, the free energy is given by setting $\psi_c = 0$ in (33)

$$F_n = V_2\left(\frac{1}{2}\,\rho\mu_n\right) = V_2\left(\frac{1}{2}\,\rho^2\log\frac{r_H}{r_*}\right), \tag{35}$$

where we use the horizon regularity condition $A_0(r_H) = 0$ and $\mu_n = \rho\log\frac{r_H}{r_*}$.

In the case with condensation, $\nu_\pm$ is simplified to $\nu_- = 1$, $\nu_+ = 3$ with $m^2 = -3$. Note that $\int_\epsilon^{r_H} \sqrt{-g}g^{tt}\phi_c^2\psi_c^2$ has a finite value.

The condensate of the scalar operator $\mathcal{O}$ is encoded in the dual field ψ by

$$\langle\mathcal{O}\rangle = \psi_{(1)} \tag{36}$$

with the boundary condition $\psi_{(0)} = 0$. We can solve Eqs. (24) and (25) numerically and finally get a condensation curve shown in Fig. 1. Near the critical temperature, this curve is similar to that in BCS theory and that in $z = 1$ holographic superconductor.[12] $\langle\mathcal{O}\rangle$ goes to a finite value as the temperature turns below a critical value. By dimensional analysis, $T_c \sim \mu$.

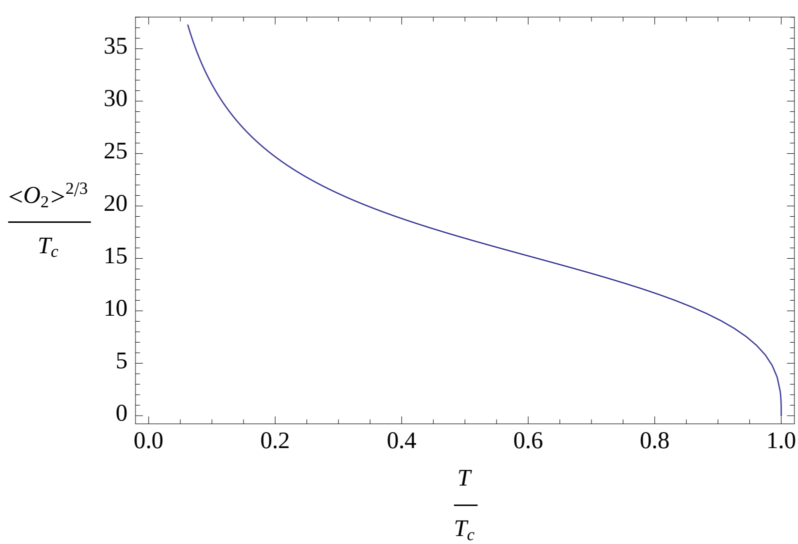

Fig. 1. Condensation curve at $z = 2$, $m^2 = -3$, $\langle O_2\rangle = \psi_{(1)}$.

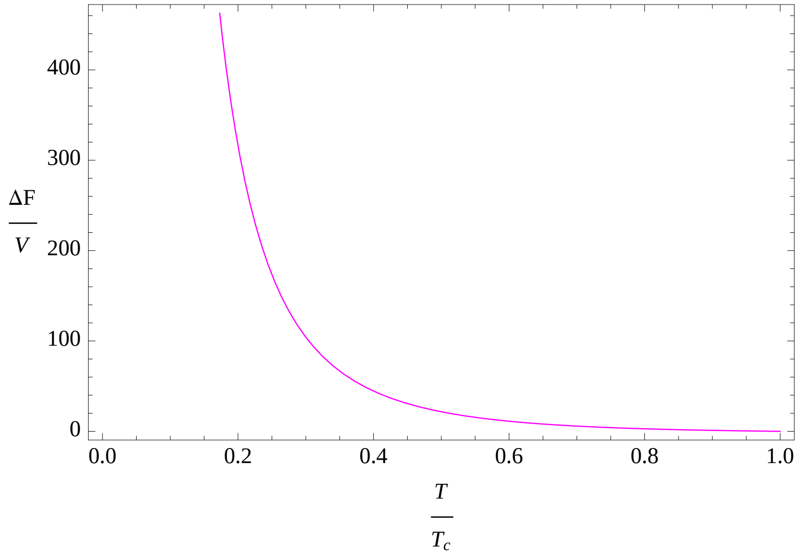

Fig. 2. Difference of free energy curve with and without condensation, $\Delta F = F_n - F_{\rm sc}$.

We plot the difference of free energy difference

$$\Delta F = F_n - F_{\rm sc} = V_2\left(\frac{1}{2}\rho(\mu_n - \mu_{\rm sc}) + \int_\epsilon^{r_H} \sqrt{-g}g^{tt}\phi_c^2\psi_c^2\right), \tag{37}$$

where μ_n, $\mu_{\rm sc}$ mean chemical potential at the normal and superconducting phase respectively. Figure 2 demonstrates that the free energy for superconducting state is lower below critical temperature.

3.2. *Conductivity*

In order to compute the electric conductivity, we follow the procedure in Ref. 31. In this paper we work in the probe limit so that the fluctuation of the metric or the massive gauge field is ignored. For the conductivity only fluctuation $A_x(r)$ is relevant and let us work in the zero spatial momentum limit. Together with the background ϕ and the fluctuation A_x,

$$A = \phi(r)dt + A_x(r)e^{-i\omega t}\, dx\,. \tag{38}$$

From the Maxwell equation,

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}F^{\mu x}\big) = 2\psi^2 A^x\,, \tag{39}$$

we find

$$A_x'' + \left[\frac{f'(r)}{f(r)} - \frac{z-1}{r}\right]A_x' + \left[\frac{\omega^2 r^{2z-2}}{f^2(r)} - \frac{2\psi^2}{r^2 f(r)}\right]A_x = 0\,. \tag{40}$$

At the horizon we choose the infalling boundary condition,

$$A_x \propto f(r)^{-i\omega r_H^z/2} . \tag{41}$$

Near the boundary, the field behaves as

$$A_x = A_{x(0)} + A_{x(1)} r^z + \cdots , \tag{42}$$

where $A_{x(0)}$ gives the background electric field in the dual field theory $E_x = i\omega A_{x(0)}$ and $A_{x(1)}$ is related to the expectation of electric current J_x.

For the gauge field (38), the Maxwell action reduces to

$$S = -\frac{2}{4} \int d^4x \sqrt{-g} \left[g^{rr} (g^{xx} A_x'^2 + g^{tt} \phi'^2) - g^{tt} g^{xx} \omega^2 A_x^2 \right] , \tag{43}$$

then the expectation value of the electric current can be obtained from this action,

$$\langle J^x \rangle = \frac{\delta S_{\text{on-shell}}}{\delta A_{x(0)}} = -\lim_{r\to 0} \frac{\delta S}{\delta \partial_r A_x} , \tag{44}$$

with the notation $\partial_r A_x = A_x'(r)$. Finally, we obtain

$$\langle J^x \rangle = \lim_{r\to 0} \sqrt{-g} g^{rr} (g^{xx} A_x') = z A_{x(1)} , \tag{45}$$

which gives the conductivity

$$\sigma(\omega) = \frac{\langle J^x \rangle}{E_x} = -\frac{i}{\omega} \frac{z A_{x(1)}}{A_{x(0)}} . \tag{46}$$

All left is to solve Eq. (40) in order to obtain the electric conductivity in (46). In particular, we plot the conductivity at $T < T_c$. Near $\omega = 0$, we observed a pole for the imaginary part. It means that DC conductivity becomes a delta function when condensation happens.

Apparently, there is a gap in Fig. 3. However, as pointed out in Ref. 39, it may not be the genuine gap. One way to see this is to work out the real part of the conductivity with low frequency at the zero temperature limit with full backreaction to the gravity background. However, since the Lifshitz geometry we

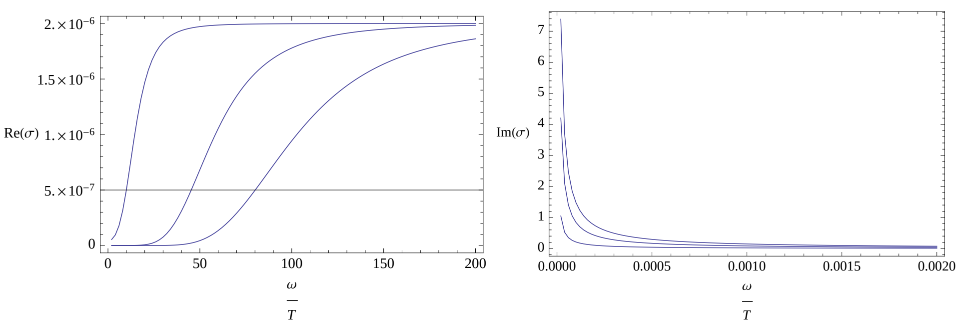

Fig. 3. Conductivity at $T < T_c$. The temperature $T = 0.512T_c$, $T = 0.155T_c$ and $T = 0.090T_c$ from up to down in left figure and opposite in the right one.

used is not a pure gravity solution like RN black hole solution, which also depends on a background scalar and vector. Together with the condensation field and Maxwell field we used to describe the strongly coupled electrons, basically we have four fields plus gravity coupled together. This complex system will bring us principle difficulties even for numerical calculation.

In order to avoid this five fields (or more than five, since metric contains several components even for some symmetry ansatz) coupled problem, we shall again use the probe approximation to try to approach the zero temperature limit of the system in the next section.

4. Zero Temperature Limit at $z = 2$

We shall consider the zero temperature limit explicitly. As we mentioned before, solutions of equations of motion (24) and (25) heavily depends on the parameter m^2. When we take the zero temperature limit, equations of motion reduce to

$$\phi'' + \frac{z-1}{r}\phi' - \frac{2\psi^2}{r^2}\phi = 0\,, \tag{47}$$

$$\psi'' - \frac{z+1}{r}\psi' + r^{2z-2}\phi^2\psi - \frac{m^2\psi}{r^2} = 0\,. \tag{48}$$

Now the horizon localizes at $r = \infty$. In order to determine the leading order behavior near $r = \infty$, we try the following ansatz:

$$\phi = r^{-\lambda_1}\,, \quad \psi = \psi_0 - \psi_1 r^{-\lambda_2}\,, \tag{49}$$

where we assume that λ_1 and λ_2 both are not negative. Note that we have used scaling symmetry to set the coefficient in ϕ to one. Using (47) and $z = 2$ we obtain

$$\psi_0^2 = \frac{\lambda_1^2}{2}\,. \tag{50}$$

Note that (50) is also true for $\psi_0 = 0$. Using the ansatz (49) and (48) we have the equation near $r = \infty$

$$-\psi_1\lambda_2(\lambda_2 - 2)r^{-\lambda_2-2} + \big(\psi_0 + \psi_1 r^{-\lambda_2}\big)r^{2-2\lambda_1} - m^2 r^{-2}\big(\psi_0 + \psi_1 r^{-\lambda_2}\big) = 0\,. \tag{51}$$

There are several consistent boundary conditions we would like report as follows.

4.1. $\psi_0 = \lambda_1 = 0$

For $\psi_0 = \lambda_1 = 0$, by analyzing (51) we found that near $r = \infty$ we require

$$\psi_1 = 0\,. \tag{52}$$

Then we have the trivial asymptotic solution for ψ. Actually it is easy to observe that

$$\phi = \text{const}\,, \quad \psi = 0 \tag{53}$$

is an exact solution for equations of motion (47) and (48). However, this solution with $\psi = 0$ is not interesting because it gives no condensation.

4.2. $\psi_0 = \lambda_1 \neq 0$

In this case, (51) reduces to

$$-\psi_1\lambda_2(\lambda_2-2)r^{-\lambda_2-2} + \psi_0 r^{2-2\lambda_1} - m^2 r^{-2}\psi_0 = 0\,. \tag{54}$$

4.2.1. $m^2 \neq 0$

For nonvanishing m^2, the only consistent boundary conditions is

$$m^2 = 1 \quad \text{and} \quad \lambda_1 = 2\,. \tag{55}$$

In this case, $\lambda_2 > 0$.

4.2.2. $m^2 = 0$

For vanishing m^2, the only consistent boundary condition is

$$-\lambda_2 - 2 = 2 - 2\lambda_1 \quad \text{and} \quad \psi_1 = \frac{\lambda_1}{\lambda_2(\lambda_2-2)}\,. \tag{56}$$

4.3. *Numerical result*

From now on we focus on the $m^2 = 0$ case and try to solve the system. In this case the IR asymptotic solutions become

$$\phi = r^{-\lambda_1}\,, \qquad \phi' = -\lambda_1 r^{-\lambda_1-1}\,, \tag{57}$$

$$\psi = \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_1}{(2\lambda_1-4)(2\lambda_1-6)}r^{4-2\lambda_1}\,, \quad \psi' = \frac{\lambda_1}{(2\lambda_1-6)}r^{3-2\lambda_1}\,. \tag{58}$$

Let us turn to the UV boundary ($r = 0$) asymptotic behaviors. For normalizable $\psi(r \to 0)$ we have

$$\phi = \mu + \rho r^{2-z} + \cdots\,, \tag{59}$$

$$\psi = \psi_{(0)} r^{\nu_-} + \psi_{(1)} r^{\nu_+} + \cdots \tag{60}$$

with $\nu_\pm = \frac{z+2}{2} \pm \sqrt{m^2 + \left(\frac{z+2}{2}\right)^2}$. For $z = 2$ and $m^2 = 0$, the second term for ϕ has a Log divergence and $\nu_- = 0$ as we mentioned before. Now we choose the normalization condition $\psi_{(0)} = 0$ as a UV boundary condition.

4.3.1. ψ *solution*

Now one can numerically integrate the IR solution to UV boundary and adjust λ_1 so that the solution for ψ is normalizable. We shall show the numerical result for ψ in Fig. 4. Using the shooting method, we found $\lambda_1 = 2.43308$.

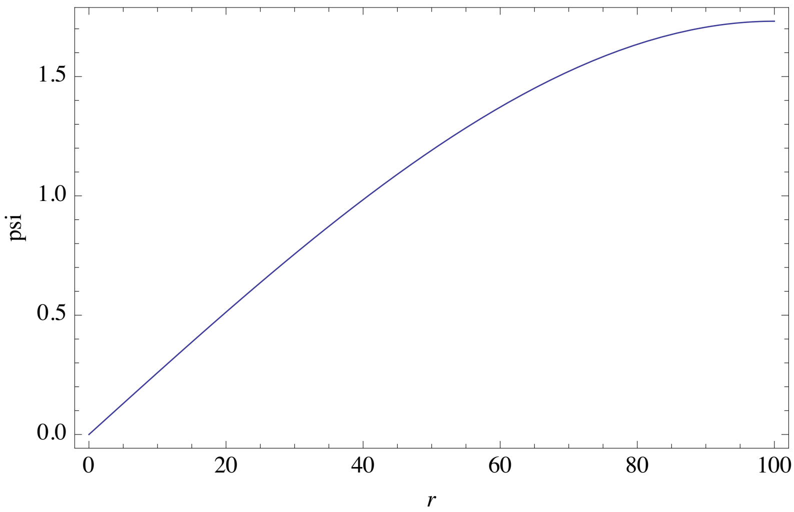

Fig. 4. Solution of ψ at $z = 2$, $m^2 = 0$, $\lambda_1 = 2.43308$.

4.3.2. *Conductivity at zero temperature limit*

Consider the conductivity at zero temperature limit. The equation of motion for A_x reduces to

$$A_x'' - \frac{z-1}{r} A_x' + \left[\omega^2 r^{2z-2} - \frac{2\psi^2}{r^2}\right] A_x = 0\,. \tag{61}$$

For $z = 2$, the above equation can be rewritten as

$$\frac{A_x''}{r^2} - \frac{A_x'}{r^3} + \left[\omega^2 - \frac{2\psi^2}{r^4}\right] A_x = 0\,. \tag{62}$$

Using the new variable $u = r^2$ we obtain the equation of motion

$$-A_x'' + \frac{\psi^2}{2u^2} A_x = \frac{\omega^2}{4} A_x\,. \tag{63}$$

This is nothing but Schrödinger equation with energy $\frac{\omega^2}{4}$. Compare with $z = 1$, where one does not need to change the variable in order to obtain the Schrödinger equation, for $z = 2$ we need a new variable $u = r^2$. It is easy to see that condensation field control the potential by $V(u) = \frac{\psi^2}{2u^2}$. And a little algebra will show that conductivity is related to reflection coefficient of the wave under the potential. In another word, conductivity is proportional to the transmission coefficient $\tilde{T}$. More explicitly, the approximate Jeffreys–Wentzel–Kramers–Brillouin (JWKB) solution for $\tilde{T}$ is

$$\tilde{T} \sim \exp\left\{-\frac{1}{\hbar}\int_a^b dr\sqrt{V(r) - E}\right\}, \tag{64}$$

where $a < r < b$ is the region where V is higher than E.

Following the analysis in Ref. 39, we shall ask whether there is a hard gap in conductivity. The way to answer this question is to see whether the conductivity in zero temperature limit vanishes or not. Let us focus on the condensation field both for $m^2 = 1$ and $m^2 = 0$. For both of them, the potential $\sqrt{V(r)}$ is integrable near the boundary and vanishes near the horizon. That means transmission coefficient $\tilde{T}$ from (64) will be nonzero even at zero temperature. Therefore a nonzero tunneling probability is always there. Due to the gap behavior for conductivity

$$\mathrm{Re}[\sigma] \sim e^{-\Delta/T}\,, \tag{65}$$

which means when $T \to 0$, it will strictly be zero, and we conclude that there is no hard gap.

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