Two effective computational schemes for a prototype of an excitable system

Cite as: AIP Advances 10, 105120 (2020); doi: 10.1063/5.0024417
Submitted: 7 August 2020 • Accepted: 19 September 2020 •
Published Online: 12 October 2020

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ABSTRACT

In this article, two recent computational schemes [the modified Khater method and the generalized $\exp(-\varphi(\Im))$–expansion method] are applied to the nonlinear predator–prey system for constructing novel explicit solutions that describe a prototype of an excitable system. Many distinct types of solutions are obtained such as hyperbolic, parabolic, and rational. Moreover, the Hamiltonian system's characteristics are employed to check the stability of the obtained solutions to show their ability to be applied in various applications. 2D, 3D, and contour plots are sketched to illustrate more physical and dynamical properties of the obtained solutions. Comparing our obtained solutions and that obtained in previous published research papers shows the novelty of our paper. The performance of the two used analytical schemes explains their effectiveness, powerfulness, practicality, and usefulness. In addition, their ability in employing various forms of nonlinear evolution equations is also shown.

I. INTRODUCTION

Recently, the bio-mathematics field has been attracting the attention of many researchers who study many distinct biological models from a mathematical point of view.[1–3] Examples of such biological models are transmission of impulses, the nervous system, the bacteria cell and its distribution, viruses, DNA, and so on.[4,5] These fundamental models have been mathematically formulated based on the experimental and statistical data that have been considered as functions and arbitrary constants for construction of these phenomena in isolation by using modern experimental biology.[6,7] Solving these mathematical models gives a clear representation of the formulated functions and parameters to control these bio-mathematical models.[8]

The above-mentioned bio-mathematical models have been the focus of many mathematicians and physicists to derive various and more accurate computational, semi–analytical, and numerical schemes.[9,10] These schemes aim to construct various explicit traveling wave and solitary wave solutions.[11–13] Examples of such analytical schemes are the Khater method, modified Khater method, extended fan–expansion method, complex hyperbolic method, exp–expansion method, sinh–cosh expansion method, extended tanh–expansion method, and extended simplest equation method,[14–17] while the semi-analytical schemes are the Adomian analysis, variational iterational method, and generalized Adomian decomposition method.[18–20]

This paper studies a biological model to explain a prototype of an excitable system. The computational solutions of the predator–prey (PP) system are investigated through the modified Khater and the generalized $\exp(-\varphi(\Im))$–expansion method. The mathematical model of this system is constructed by[21–25]

$$\begin{cases} \mathcal{H}_t = \mathcal{H}_{xx} - \beta\,\mathcal{H} + (1+\beta)\,\mathcal{H}^2 - \mathcal{H}^3 - \mathcal{H}\,\mathcal{E}, \\ \mathcal{E}_t = \mathcal{E}_{xx} + k\,\mathcal{H}\,\mathcal{E} - m\,\mathcal{E} - \delta\mathcal{E}^3, \end{cases} \tag{1}$$

where k, δ, m, β are positive arbitrary parameters. The dynamics of the diffusive PP model have assumed the following relations between the parameters, namely, $\left[m = \beta \text{ and } k + \frac{1}{\sqrt{\delta}} - m = 1\right]$. These assumptions transform system 1 to the following system:

$$\begin{cases} \mathcal{H}_t = \mathcal{H}_{xx} - \beta\mathcal{H} + \left(k + \frac{1}{\sqrt{\delta}}\right)\mathcal{H}^2 - \mathcal{H}^3 - \mathcal{H}\mathcal{E}, \\ \mathcal{E}_t = \mathcal{E}_{xx} + k\mathcal{H}\mathcal{E} - \beta\mathcal{E} - \delta\mathcal{E}^3. \end{cases} \tag{2}$$

Employing the following transformation $[\mathcal{H}(x,t) = \mathcal{H}(\Im), \Im = x - ct]$ leads to

$$\begin{cases} \mathcal{H}'' + c\mathcal{H}' - \beta\mathcal{H} + \left(k + \frac{1}{\sqrt{\delta}}\right)\mathcal{H}^2 - \mathcal{H}^3 - \mathcal{H}\mathcal{E} = 0, \\ \mathcal{E}'' + c\mathcal{E}' + k\mathcal{H}\mathcal{E} - \beta\mathcal{E} - \delta\mathcal{E}^3 = 0, \end{cases} \tag{3}$$

where c is a nonzero constant. Using the relation between $[\mathcal{E}$ and $\mathcal{H}]$ in the form $\left[\mathcal{E} = \frac{\mathcal{H}}{\sqrt{\delta}}\right]$ yields

$$\mathcal{H}'' + c\mathcal{H}' - \beta\mathcal{H} + k\mathcal{H}^2 - \mathcal{H}^3 = 0. \tag{4}$$

Evaluating the balance value of Eq. (4), lancing $\mathcal{H}''$ with $\mathcal{H}^3$ in Eq. (4) yields $N + 2 = 3N \Rightarrow N = 1$.

The organization of the remaining sections in this article is given as follows: Sec. II employs two computational schemes[26–28] to construct many traveling solutions of the nonlinear PP system. In addition, many distinct types of sketches are plotted. Section III investigates the stability property of the obtained solutions. Section IV illustrates the originality and novelty of our paper by comparing our obtained solutions with previously calculated solutions that have been obtained in previous published papers. Section V gives the conclusion of our article.

II. COMPUTATIONAL SOLUTIONS OF THE NONLINEAR PP MODEL

Here, we investigate the analytical solutions of the nonlinear PP model through the modified Khater (MK) and the generalized $\exp(-\varphi(\Im))$–expansion (GEE) method to evaluate the solitary wave solutions of the PP system.

A. The MK method

Based on the balance value and the MK method's framework, the general solutions of nonlinear PP model 4 are derived by

$$\mathcal{H}(\Im) = \sum_{i=1}^{n} a_i k^{if(\Im)} + \sum_{i=1}^{n} b_i k^{-if(\Im)} + a_0 = a_1 k^{f(\Im)} + a_0 + b_1 k^{-f(\Im)}, \tag{5}$$

where a_0, a_1, b_1, k are arbitrary constants. Additionally, $f(\Im)$ is the solution function of the next ODE,

$$f'(\Im) = \frac{\delta k^{f(\Im)} + \varrho k^{-f(\Im)} + \chi}{\ln(k)}, \tag{6}$$

where δ, ϱ, χ are arbitrary constants. Thus, using the MK scheme based on Eqs. (5) and (6) gives the following values of the above-mentioned parameters.

1. Family I

$$\left[a_0 \to \frac{1}{2}\left(k - \sqrt{-4\beta + 8\delta\varrho + k^2}\right), a_1 \to -\sqrt{2}\delta, b_1 \to 0, c \to -\frac{k}{\sqrt{2}},\right.$$
$$\left.\chi \to \frac{\sqrt{-4\beta + 8\delta\varrho + k^2}}{\sqrt{2}}, \text{where } (\delta \neq 0 \text{ and } k \neq 0)\right].$$

Thus, the explicit wave solutions of the PP system are given as follows:

In case of $[\chi^2 - 4\delta\varrho < 0$ and $\delta \neq 0]$,

$$\mathcal{H}_1(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} - \sqrt{8\delta\varrho - 2\chi^2}\tan\left(\frac{1}{2}\sqrt{4\delta\varrho - \chi^2}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k + \sqrt{2}\chi\right), \tag{7}$$

$$\mathcal{H}_2(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} - \sqrt{8\delta\varrho - 2\chi^2}\cot\left(\frac{1}{2}\sqrt{4\delta\varrho - \chi^2}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k + \sqrt{2}\chi\right). \tag{8}$$

In case of $[\chi^2 - 4\delta\varrho > 0$ and $\delta \neq 0]$,

$$\mathcal{H}_3(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} + \sqrt{2}\sqrt{\chi^2 - 4\delta\varrho}\tanh\left(\frac{1}{2}\sqrt{\chi^2 - 4\delta\varrho}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k + \sqrt{2}\chi\right), \tag{9}$$

$$\mathcal{H}_4(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} + \sqrt{2}\sqrt{\chi^2 - 4\delta\varrho}\coth\left(\frac{1}{2}\sqrt{\chi^2 - 4\delta\varrho}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k + \sqrt{2}\chi\right). \tag{10}$$

In case of $[\delta\varrho > 0$ and $\varrho \neq 0$ and $\delta \neq 0$ and $\chi = 0]$,

$$\mathcal{H}_5(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} - 2\sqrt{2}\sqrt{\delta\varrho}\tan\left(\sqrt{\delta\varrho}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k\right), \tag{11}$$

$$\mathcal{H}_6(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} + 2\sqrt{2}\sqrt{\delta\varrho}\cot\left(\sqrt{\delta\varrho}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k\right). \tag{12}$$

In case of $[\delta\varrho < 0$ and $\varrho \neq 0$ and $\delta \neq 0$ and $\chi = 0]$,

$$\mathcal{H}_7(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} + 2\sqrt{2}\sqrt{-\delta\varrho}\tanh\left(\sqrt{-\delta\varrho}\left(x - \frac{kt}{\sqrt{2}}\right)\right) + k\right), \tag{13}$$

$$\mathcal{H}_8(x,t) = \frac{1}{2}\Bigg(-\sqrt{-4\beta+8\delta\varrho+k^2}+2\sqrt{2}\sqrt{-\delta\varrho}\coth \times \left(\sqrt{-\delta\varrho}\left(x-\frac{kt}{\sqrt{2}}\right)\right)+k\Bigg). \quad (14)$$

In case of $[\chi = 0 \text{ and } \varrho = -\delta]$,

$$\mathcal{H}_9(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+k^2-8\varrho^2}-2\sqrt{2}\varrho\coth\left(\frac{kt\varrho}{\sqrt{2}}-x\varrho\right)+k\right). \quad (15)$$

In case of $[\chi = \delta = \kappa \text{ and } \varrho = 0]$,

$$\mathcal{H}_{10}(x,t) = \frac{1}{2}\left(-\sqrt{k^2-4\beta}+\frac{2\sqrt{2}\kappa e^{\kappa x}}{e^{\kappa x}-e^{\frac{\kappa kt}{\sqrt{2}}}}+k\right). \quad (16)$$

In case of $[\varrho = 0 \text{ and } \chi \neq 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{11}(x,t) = \frac{1}{2}\left(-\sqrt{k^2-4\beta}+\frac{2\sqrt{2}\delta\chi e^{\chi x}}{\delta e^{\chi x}-2e^{\frac{kt\chi}{\sqrt{2}}}}+k\right). \quad (17)$$

In case of $[\chi = \varrho = 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{12}(x,t) = \frac{1}{2}\left(-\sqrt{k^2-4\beta}-\frac{4}{kt-\sqrt{2}x}+k\right). \quad (18)$$

In case of $[\chi = 0 \text{ and } \varrho = \delta]$,

$$\mathcal{H}_{13}(x,t) = \frac{1}{2}\Bigg(-2\sqrt{2}\varrho\tan\left(C-\frac{kt\varrho}{\sqrt{2}}+x\varrho\right) - \sqrt{-4\beta+k^2+8\varrho^2}+k\Bigg). \quad (19)$$

In case of $[\chi^2 - 4\delta\varrho = 0]$,

$$\mathcal{H}_{14}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+8\delta\varrho+k^2}+\frac{4\sqrt{2}\delta\varrho\left(\frac{2}{x-\frac{kt}{\sqrt{2}}}+\chi\right)}{\chi^2}+k\right). \quad (20)$$

2. Family II

$$\left[a_0 \to \frac{1}{2}\left(k-\sqrt{-4\beta+8\delta\varrho+k^2}\right), a_1 \to 0, b_1 \to -\sqrt{2}\varrho, c \to \frac{k}{\sqrt{2}}, \chi \to \frac{\sqrt{-4\beta+8\delta\varrho+k^2}}{\sqrt{2}}, \text{where } (\varrho \neq 0 \text{ and } k \neq 0)\right].$$

Thus, the explicit wave solutions of the PP system are given as follows:

In case of $[\chi^2 - 4\delta\varrho < 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{15}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+8\delta\varrho+k^2} + \frac{4\sqrt{2}\delta\varrho}{\chi-\sqrt{4\delta\varrho-\chi^2}\tan\left(\frac{1}{2}\sqrt{4\delta\varrho-\chi^2}\left(\frac{kt}{\sqrt{2}}+x\right)\right)}+k\right), \quad (21)$$

$$\mathcal{H}_{16}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+8\delta\varrho+k^2} + \frac{4\sqrt{2}\delta\varrho}{\chi-\sqrt{4\delta\varrho-\chi^2}\cot\left(\frac{1}{2}\sqrt{4\delta\varrho-\chi^2}\left(\frac{kt}{\sqrt{2}}+x\right)\right)}+k\right). \quad (22)$$

In case of $[\chi^2 - 4\delta\varrho > 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{17}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+8\delta\varrho+k^2} + \frac{4\sqrt{2}\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\tanh\left(\frac{1}{2}\sqrt{\chi^2-4\delta\varrho}\left(\frac{kt}{\sqrt{2}}+x\right)\right)+\chi}+k\right), \quad (23)$$

$$\mathcal{H}_{18}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta+8\delta\varrho+k^2} + \frac{4\sqrt{2}\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\coth\left(\frac{1}{2}\sqrt{\chi^2-4\delta\varrho}\left(\frac{kt}{\sqrt{2}}+x\right)\right)+\chi}+k\right). \quad (24)$$

In case of $[\delta\varrho > 0 \text{ and } \varrho \neq 0 \text{ and } \delta \neq 0 \text{ and } \chi = 0]$,

$$\mathcal{H}_{19}(x,t) = \frac{1}{2}\Bigg(-\sqrt{-4\beta+8\delta\varrho+k^2}-2\sqrt{2}\sqrt{\delta\varrho}\cot \times \left(\sqrt{\delta\varrho}\left(\frac{kt}{\sqrt{2}}+x\right)\right)+k\Bigg), \quad (25)$$

$$\mathcal{H}_{20}(x,t) = \frac{1}{2}\Bigg(-\sqrt{-4\beta+8\delta\varrho+k^2}+2\sqrt{2}\sqrt{\delta\varrho}\tan \times \left(\sqrt{\delta\varrho}\left(\frac{kt}{\sqrt{2}}+x\right)\right)+k\Bigg). \quad (26)$$

In case of $[\delta\varrho < 0 \text{ and } \varrho \neq 0 \text{ and } \delta \neq 0 \text{ and } \chi = 0]$,

$$\mathcal{H}_{21}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} - 2\sqrt{2}\sqrt{-\delta\varrho}\coth \times \left(\sqrt{-\delta\varrho}\left(\frac{kt}{\sqrt{2}} + x\right)\right) + k\right), \tag{27}$$

$$\mathcal{H}_{22}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} - 2\sqrt{2}\sqrt{-\delta\varrho}\tanh \times \left(\sqrt{-\delta\varrho}\left(\frac{kt}{\sqrt{2}} + x\right)\right) + k\right). \tag{28}$$

In case of $[\chi = 0 \text{ and } \varrho = -\delta]$,

$$\mathcal{H}_{23}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + k^2 - 8\varrho^2} - 2\sqrt{2}\varrho \times \tanh\left(\frac{kt\varrho}{\sqrt{2}} + x\varrho\right) + k\right). \tag{29}$$

In case of $[\chi = \frac{\varrho}{2} = \kappa \text{ and } \delta = 0]$,

$$\mathcal{H}_{24}(x,t) = \frac{1}{2}\left(-\sqrt{k^2 - 4\beta} - \frac{4\sqrt{2}\kappa}{e^{\kappa\left(\frac{kt}{\sqrt{2}} + x\right)} - 2} + k\right). \tag{30}$$

In case of $[\chi = \delta = 0 \text{ and } \varrho \neq 0]$,

$$\mathcal{H}_{25}(x,t) = \frac{1}{2}\left(-\sqrt{k^2 - 4\beta} - \frac{2\sqrt{2}}{\frac{kt}{\sqrt{2}} + x} + k\right). \tag{31}$$

In case of $[\chi = 0 \text{ and } \varrho = \delta]$,

$$\mathcal{H}_{26}(x,t) = \frac{1}{2}\left(-2\sqrt{2}\varrho\cot\left(C + \frac{kt\varrho}{\sqrt{2}} + x\varrho\right) - \sqrt{-4\beta + k^2 + 8\varrho^2} + k\right). \tag{32}$$

In case of $[\delta = 0 \text{ and } \chi \neq 0 \text{ and } \varrho \neq 0]$,

$$\mathcal{H}_{27}(x,t) = \frac{1}{2}\left(-\sqrt{k^2 - 4\beta} + \frac{2\sqrt{2}\chi\varrho}{\varrho - \chi e^{\chi\left(\frac{kt}{\sqrt{2}} + x\right)}} + k\right). \tag{33}$$

In case of $[\chi^2 - 4\delta\varrho = 0]$,

$$\mathcal{H}_{28}(x,t) = \frac{1}{2}\left(-\sqrt{-4\beta + 8\delta\varrho + k^2} + \frac{2\chi^2\left(kt + \sqrt{2}x\right)}{\sqrt{2}kt\chi + 2\chi x + 4} + k\right). \tag{34}$$

3. Family III

$$\Bigg[a_0 \to \frac{1}{4}\left(-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} + k\right), a_1 \to -\sqrt{2}\delta, b_1 \to 0, c \to \frac{k - 3\sqrt{k^2 - 4\beta}}{2\sqrt{2}}, \chi \to \frac{1}{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2}, \text{ where } \left(\delta \neq 0 \text{ and } k - 3\sqrt{k^2 - 4\beta} \neq 0\right)\Bigg].$$

Thus, the explicit wave solutions of the PP system are given as follows:

In case of $[\chi^2 - 4\delta\varrho < 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{29}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} - 2\sqrt{8\delta\varrho - 2\chi^2}\tan\left(\frac{1}{8}\sqrt{4\delta\varrho - \chi^2} \times \left(\sqrt{2}t\left(k - 3\sqrt{k^2 - 4\beta}\right) + 4x\right)\right) + k + 2\sqrt{2}\chi\Bigg], \tag{35}$$

$$\mathcal{H}_{30}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} - 2\sqrt{8\delta\varrho - 2\chi^2}\cot\left(\frac{1}{8}\sqrt{4\delta\varrho - \chi^2} \times \left(\sqrt{2}t\left(k - 3\sqrt{k^2 - 4\beta}\right) + 4x\right)\right) + k + 2\sqrt{2}\chi\Bigg]. \tag{36}$$

In case of $[\chi^2 - 4\delta\varrho > 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{31}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} + 2\sqrt{2}\sqrt{\chi^2 - 4\delta\varrho}\tanh\left(\frac{1}{8}\sqrt{\chi^2 - 4\delta\varrho} \times \left(\sqrt{2}t\left(k - 3\sqrt{k^2 - 4\beta}\right) + 4x\right)\right) + k + 2\sqrt{2}\chi\Bigg], \tag{37}$$

$$\mathcal{H}_{32}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} + 2\sqrt{2}\sqrt{\chi^2 - 4\delta\varrho}\coth\left(\frac{1}{8}\sqrt{\chi^2 - 4\delta\varrho} \times \left(\sqrt{2}t\left(k - 3\sqrt{k^2 - 4\beta}\right) + 4x\right)\right) + k + 2\sqrt{2}\chi\Bigg]. \tag{38}$$

In case of $[\delta\varrho > 0 \text{ and } \varrho \neq 0 \text{ and } \delta \neq 0 \text{ and } \chi = 0]$,

$$\mathcal{H}_{33}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2 - 4\beta} + k^2} + \sqrt{k^2 - 4\beta} - 4\sqrt{2}\sqrt{\delta\varrho} \times \tan\left(\sqrt{\delta\varrho}\left(\frac{t\left(k - 3\sqrt{k^2 - 4\beta}\right)}{2\sqrt{2}} + x\right)\right) + k\Bigg], \tag{39}$$

$$\mathcal{H}_{34}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + 4\sqrt{2}\sqrt{\delta\varrho} \times \cot\left(\sqrt{\delta\varrho}\left(\frac{t\left(k-3\sqrt{k^2-4\beta}\right)}{2\sqrt{2}} + x\right)\right) + k\Bigg]. \tag{40}$$

In case of $[\delta\varrho < 0 \text{ and } \varrho \neq 0 \text{ and } \delta \neq 0 \text{ and } \chi = 0]$,

$$\mathcal{H}_{35}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + 4\sqrt{2}\sqrt{-\delta\varrho} \times \tanh\left(\sqrt{-\delta\varrho}\left(\frac{t\left(k-3\sqrt{k^2-4\beta}\right)}{2\sqrt{2}} + x\right)\right) + k\Bigg], \tag{41}$$

$$\mathcal{H}_{36}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + 4\sqrt{2}\sqrt{-\delta\varrho} \times \coth\left(\sqrt{-\delta\varrho}\left(\frac{t\left(k-3\sqrt{k^2-4\beta}\right)}{2\sqrt{2}} + x\right)\right) + k\Bigg]. \tag{42}$$

In case of $[\chi = 0 \text{ and } \varrho = -\delta]$,

$$\mathcal{H}_{37}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2k\left(\sqrt{k^2-4\beta}+k\right) - 4(\beta + 8\varrho^2)} + \sqrt{k^2-4\beta} + 4\sqrt{2}\varrho \times \coth\left(\frac{1}{4}\varrho\left(\sqrt{2}t\left(k-3\sqrt{k^2-4\beta}\right) + 4x\right)\right) + k\Bigg]. \tag{43}$$

In case of $[\chi = \delta = \kappa \text{ and } \varrho = 0]$,

$$\mathcal{H}_{38}(x,t) = \frac{1}{4}\Bigg[\sqrt{k^2-4\beta} - \sqrt{2k\left(\sqrt{k^2-4\beta}+k\right) - 4\beta} + \frac{4\sqrt{2}\kappa e^{\frac{\kappa kt}{2\sqrt{2}}+\kappa x}}{e^{\frac{\kappa kt}{2\sqrt{2}}+\kappa x} - e^{\frac{3\kappa t\sqrt{k^2-4\beta}}{2\sqrt{2}}}} + k\Bigg]. \tag{44}$$

In case of $[\varrho = 0 \text{ and } \chi \neq 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{39}(x,t) = \frac{1}{4}\Bigg[\sqrt{k^2-4\beta} - \sqrt{2k\left(\sqrt{k^2-4\beta}+k\right) - 4\beta} + \frac{4\sqrt{2}\delta\chi e^{\frac{kt\chi}{2\sqrt{2}}+\chi x}}{\delta e^{\frac{kt\chi}{2\sqrt{2}}+\chi x} - 2e^{\frac{3t\chi\sqrt{k^2-4\beta}}{2\sqrt{2}}}} + k\Bigg]. \tag{45}$$

In case of $[\chi = 0 \text{ and } \varrho = \delta]$,

$$\mathcal{H}_{40}(x,t) = \frac{1}{4}\Bigg[-4\sqrt{2}\varrho\tan\left(C + \varrho\left(\frac{t\left(k-3\sqrt{k^2-4\beta}\right)}{2\sqrt{2}} + x\right)\right) - \sqrt{2}\sqrt{-2\beta + k\sqrt{k^2-4\beta} + k^2 + 16\varrho^2} + \sqrt{k^2-4\beta} + k\Bigg]. \tag{46}$$

In case of $[\chi^2 - 4\delta\varrho = 0]$,

$$\mathcal{H}_{41}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + \frac{8\sqrt{2}\delta\varrho\left(\frac{8}{\sqrt{2}t\left(k-3\sqrt{k^2-4\beta}\right)+4x} + \chi\right)}{\chi^2} + k\Bigg]. \tag{47}$$

4. Family IV

$$\Bigg[a_0 \to \frac{1}{4}\left(-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + k\right), a_1 \to 0, b_1 \to -\sqrt{2}\varrho, c \to \frac{3\sqrt{k^2-4\beta}-k}{2\sqrt{2}}, \chi \to \frac{1}{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2}, \text{ where } \left(\varrho \neq 0 \text{ and } 3\sqrt{k^2-4\beta} \neq k\right)\Bigg].$$

Thus, the explicit wave solutions of the PP system are given as follows:

In case of $[\chi^2 - 4\delta\varrho < 0 \text{ and } \delta \neq 0]$,

$$\mathcal{H}_{42}(x,t) = \frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta + 16\delta\varrho + k\sqrt{k^2-4\beta} + k^2} + \sqrt{k^2-4\beta} + \frac{8\sqrt{2}\delta\varrho}{\chi - \sqrt{4\delta\varrho-\chi^2}\tan\left(\frac{1}{8}\sqrt{4\delta\varrho-\chi^2}\left(\sqrt{2}t\left(3\sqrt{k^2-4\beta}-k\right)+4x\right)\right)} + k\Bigg], \tag{48}$$

$$\mathcal{H}_{43}(x,t)=\frac{1}{4}\left[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}+\frac{8\sqrt{2}\delta\varrho}{\chi-\sqrt{4\delta\varrho-\chi^2}\cot\left(\frac{1}{8}\sqrt{4\delta\varrho-\chi^2}\left(\sqrt{2}t\left(3\sqrt{k^2-4\beta}-k\right)+4x\right)\right)}+k\right]. \tag{49}$$

In case of $\left[\chi^2-4\delta\varrho>0 \text{ and } \delta\neq 0\right]$,

$$\mathcal{H}_{44}(x,t)=\frac{1}{4}\left[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}+\frac{8\sqrt{2}\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\tanh\left(\frac{1}{8}\sqrt{\chi^2-4\delta\varrho}\left(\sqrt{2}t\left(3\sqrt{k^2-4\beta}-k\right)+4x\right)\right)+\chi}+k\right], \tag{50}$$

$$\mathcal{H}_{45}(x,t)=\frac{1}{4}\left[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}+\frac{8\sqrt{2}\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\coth\left(\frac{1}{8}\sqrt{\chi^2-4\delta\varrho}\left(\sqrt{2}t\left(3\sqrt{k^2-4\beta}-k\right)+4x\right)\right)+\chi}+k\right]. \tag{51}$$

In case of $[\delta\varrho>0 \text{ and } \varrho\neq 0 \text{ and } \delta\neq 0 \text{ and } \chi=0]$,

$$\mathcal{H}_{46}(x,t)=\frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}-4\sqrt{2}\sqrt{\delta\varrho}\times\cot\left(\sqrt{\delta\varrho}\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)\right)+k\Bigg], \tag{52}$$

$$\mathcal{H}_{47}(x,t)=\frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}+4\sqrt{2}\sqrt{\delta\varrho}\times\tan\left(\sqrt{\delta\varrho}\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)\right)+k\Bigg]. \tag{53}$$

In case of $[\delta\varrho<0 \text{ and } \varrho\neq 0 \text{ and } \delta\neq 0 \text{ and } \chi=0]$,

$$\mathcal{H}_{48}(x,t)=\frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}-4\sqrt{2}\sqrt{-\delta\varrho}\times\coth\left(\sqrt{-\delta\varrho}\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)\right)+k\Bigg], \tag{54}$$

$$\mathcal{H}_{49}(x,t)=\frac{1}{4}\Bigg[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2}+\sqrt{k^2-4\beta}-4\sqrt{2}\sqrt{-\delta\varrho}\times\tanh\left(\sqrt{-\delta\varrho}\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)\right)+k\Bigg]. \tag{55}$$

In case of $[\chi=0 \text{ and } \varrho=-\delta]$,

$$\mathcal{H}_{50}(x,t)=\frac{1}{4}\Bigg[-\sqrt{2k\left(\sqrt{k^2-4\beta}+k\right)-4(\beta+8\varrho^2)}+\sqrt{k^2-4\beta}+4\sqrt{2}\varrho\times\tanh\left(\frac{1}{4}\varrho\left(\sqrt{2}t\left(k-3\sqrt{k^2-4\beta}\right)-4x\right)\right)+k\Bigg]. \tag{56}$$

In case of $\left[\chi=\frac{\varrho}{2}=\kappa \text{ and } \delta=0\right]$,

$$\mathcal{H}_{51}(x,t)=\frac{1}{4}\left(\sqrt{k^2-4\beta}-\sqrt{2k\left(\sqrt{k^2-4\beta}+k\right)-4\beta}-\frac{8\sqrt{2}\kappa}{e^{\kappa\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)}-2}+k\right). \tag{57}$$

In case of $[\chi = \delta = 0 \text{ and } \varrho \neq 0]$,

$$\mathcal{H}_{52}(x,t) = \frac{1}{4}\left(\sqrt{k^2-4\beta} - \sqrt{2k\left(\sqrt{k^2-4\beta}+k\right)-4\beta} - \frac{16\sqrt{2}}{\sqrt{2}t\left(3\sqrt{k^2-4\beta}-k\right)+4x} + k\right). \tag{58}$$

In case of $[\chi = 0 \text{ and } \varrho = \delta]$,

$$\mathcal{H}_{53}(x,t) = \frac{1}{4}\left[-4\sqrt{2}\varrho\cot\left(C+\varrho\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)\right) - \sqrt{2}\sqrt{-2\beta+k\sqrt{k^2-4\beta}+k^2+16\varrho^2} + \sqrt{k^2-4\beta}+k\right]. \tag{59}$$

In case of $[\delta = 0 \text{ and } \chi \neq 0 \text{ and } \varrho \neq 0]$,

$$\mathcal{H}_{54}(x,t) = \frac{1}{4}\left(\sqrt{k^2-4\beta} - \sqrt{2k\left(\sqrt{k^2-4\beta}+k\right)-4\beta} + \frac{4\sqrt{2}\chi\varrho}{\varrho - \chi e^{\chi\left(\frac{t\left(3\sqrt{k^2-4\beta}-k\right)}{2\sqrt{2}}+x\right)}} + k\right). \tag{60}$$

In case of $[\chi^2 - 4\delta\varrho = 0]$,

$$\mathcal{H}_{55}(x,t) = \frac{1}{4}\left[-\sqrt{2}\sqrt{-2\beta+16\delta\varrho+k\sqrt{k^2-4\beta}+k^2} + \sqrt{k^2-4\beta} + \frac{4\chi^2\left(3t\sqrt{k^2-4\beta}-kt+2\sqrt{2}x\right)}{\sqrt{2}t\chi\left(3\sqrt{k^2-4\beta}-k\right)+4\chi x+8} + k\right]. \tag{61}$$

B. The GEE method

Based on the balance value and the GEE method's framework, the general solutions of nonlinear PP model 4 are derived by

$$\mathcal{H}(\Im) = \sum_{i=0}^{n} a_i e^{-i\phi(\Im)} = a_1 e^{-\phi(\Im)} + a_0, \tag{62}$$

where a_0, a_1 are arbitrary constants. Additionally, $\phi(\Im)$ is the solution function of the next ODE,

$$\phi'(\Im) = \lambda + \mu e^{\phi(\Im)} + \frac{\sigma}{e^{\phi(\Im)}}, \tag{63}$$

where λ, μ, σ are arbitrary constants. Thus, using the GEE scheme based on Eqs. (62) along (63) gives the following values of the abovementioned parameters.

1. Family I

$$\left[a_1 \to \sqrt{2}\sigma,\ c \to -\frac{-2a_0^2(\beta+2\mu\sigma)+a_0^4+4\mu^2\sigma^2}{\sqrt{2}\left(a_0^3-2a_0\mu\sigma\right)},\ k \to a_0\left(\frac{\beta}{a_0^2-2\mu\sigma}+1\right)-\frac{2\mu\sigma}{a_0},\ \lambda \to \frac{a_0^2+2\mu\sigma}{\sqrt{2}a_0},\ \text{where } (a_0 \neq 0 \text{ and } \sigma \neq 0)\right].$$

Therefore, the explicit wave solutions of the PP system when $\sigma = 1$ are as follows:

In case of $[\lambda^2 - 4\mu > 0 \text{ and } \mu \neq 0]$,

$$\mathcal{H}_{56}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu)+a_0^4+4\mu^2\right)}{\sqrt{2}\left(a_0^3-2a_0\mu\right)}+\vartheta\right)\right)}, \tag{64}$$

$$\mathcal{H}_{57}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu)+a_0^4+4\mu^2\right)}{\sqrt{2}\left(a_0^3-2a_0\mu\right)}+\vartheta\right)\right)}. \tag{65}$$

In case of $[\lambda^2 - 4\mu > 0 \text{ and } \mu = 0]$,

$$\mathcal{H}_{58}(x,t) = a_0 + \frac{\sqrt{2}\lambda}{e^{\lambda\left(x - \frac{t\left(-2a_0^2(\beta+2\mu)+a_0^4+4\mu^2\right)}{\sqrt{2}\left(a_0^3-2a_0\mu\right)}+\vartheta\right)} - 1}. \tag{66}$$

In case of $[\lambda^2 - 4\mu = 0 \text{ and } \mu \neq 0 \text{ and } \lambda \neq 0]$,

$$\mathcal{H}_{59}(x,t) = a_0 - \frac{\sqrt{2}\lambda^2\left(x - \frac{t\left(-2a_0^2(\beta+2\mu) + a_0^4 + 4\mu^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\right)} + \vartheta\right)}{2\lambda\left(x - \frac{t\left(-2a_0^2(\beta+2\mu) + a_0^4 + 4\mu^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\right)} + \vartheta\right) + 4}. \tag{67}$$

In case of $[\lambda^2 - 4\mu = 0 \text{ and } \mu = 0 \text{ and } \lambda = 0]$,

$$\mathcal{H}_{60}(x,t) = a_0 + \frac{\sqrt{2}}{x - \frac{t\left(-2a_0^2(\beta+2\mu) + a_0^4 + 4\mu^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\right)} + \vartheta}. \tag{68}$$

In case of $[\lambda^2 - 4\mu < 0 \text{ and } \mu \neq 0]$,

$$\mathcal{H}_{61}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{4\mu - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu) + a_0^4 + 4\mu^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\right)} + \vartheta\right)\right)}, \tag{69}$$

$$\mathcal{H}_{62}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{4\mu - \lambda^2}\cot\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu) + a_0^4 + 4\mu^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\right)} + \vartheta\right)\right)}. \tag{70}$$

When $\lambda = 0$:
In case of $[\sigma > 0 \text{ and } \mu > 0]$,

$$\mathcal{H}_{63}(x,t) = a_0 + \sqrt{2}\mu\sqrt{\frac{\sigma}{\mu}}\cot\left(\sqrt{\mu\sigma}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu\sigma) + a_0^4 + 4\mu^2\sigma^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\sigma\right)} + \vartheta\right)\right), \tag{71}$$

$$\mathcal{H}_{64}(x,t) = a_0 + \sqrt{2}\mu\sqrt{\frac{\sigma}{\mu}}\tan\left(\sqrt{\mu\sigma}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu\sigma) + a_0^4 + 4\mu^2\sigma^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\sigma\right)} + \vartheta\right)\right). \tag{72}$$

In case of $[\sigma\mu > 0 \text{ and } \mu < 0]$,

$$\mathcal{H}_{65}(x,t) = a_0 + \frac{\sqrt{2}\mu\tanh\left(\sqrt{-\mu\sigma}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu\sigma) + a_0^4 + 4\mu^2\sigma^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\sigma\right)} + \vartheta\right)\right)}{\sqrt{-\frac{\mu}{\sigma}}}, \tag{73}$$

$$\mathcal{H}_{66}(x,t) = a_0 + \frac{\sqrt{2}\mu\coth\left(\sqrt{-\mu\sigma}\left(x - \frac{t\left(-2a_0^2(\beta+2\mu\sigma) + a_0^4 + 4\mu^2\sigma^2\right)}{\sqrt{2}\left(a_0^3 - 2a_0\mu\sigma\right)} + \vartheta\right)\right)}{\sqrt{-\frac{\mu}{\sigma}}}. \tag{74}$$

2. Family II

$$\left[a_1 \to \sqrt{2}\sigma, c \to -\frac{a_0^2 + \beta - 2\mu\sigma}{\sqrt{2}a_0}, k \to \frac{\beta - 2\mu\sigma}{a_0} + a_0,\ \lambda \to \frac{a_0^2 - \beta + 2\mu\sigma}{\sqrt{2}a_0}, \text{ where } (a_0 \neq 0 \text{ and } \sigma \neq 0)\right].$$

Therefore, the explicit wave solutions of the PP system when $\sigma = 1$ are as follows:
In case of $\left[\lambda^2 - 4\mu > 0 \text{ and } \mu \neq 0\right]$,

$$\mathcal{H}_{67}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}, \tag{75}$$

$$\mathcal{H}_{68}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}. \tag{76}$$

In case of $\left[\lambda^2 - 4\mu > 0 \text{ and } \mu = 0\right]$,

$$\mathcal{H}_{69}(x,t) = \frac{\sqrt{2}\lambda}{e^{\lambda\left(-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta\right)}-1} + a_0. \tag{77}$$

In case of $\left[\lambda^2 - 4\mu = 0 \text{ and } \mu \neq 0 \text{ and } \lambda \neq 0\right]$

$$\mathcal{H}_{70}(x,t) = \frac{\lambda^2\left(2a_0(x+\vartheta) - \sqrt{2}t\left(a_0^2+\beta-2\mu\right)\right)}{2a_0^2\lambda t - 2\sqrt{2}a_0(\lambda(x+\vartheta)+2) + 2\lambda t(\beta-2\mu)} + a_0. \tag{78}$$

In case of $\left[\lambda^2 - 4\mu = 0 \text{ and } \mu = 0 \text{ and } \lambda = 0\right]$,

$$\mathcal{H}_{71}(x,t) = \frac{\sqrt{2}}{-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta} + a_0. \tag{79}$$

In case of $\left[\lambda^2 - 4\mu < 0 \text{ and } \mu \neq 0\right]$,

$$\mathcal{H}_{72}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}, \tag{80}$$

$$\mathcal{H}_{73}(x,t) = a_0 - \frac{2\sqrt{2}\mu}{\lambda - \sqrt{4\mu-\lambda^2}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(-\frac{t\left(a_0^2+\beta-2\mu\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}. \tag{81}$$

When $\lambda = 0$ In case of $[\sigma > 0 \text{ and } \mu > 0]$,

$$\mathcal{H}_{74}(x,t) = \sqrt{2}\mu\sqrt{\frac{\sigma}{\mu}}\cot \times\left(\sqrt{\mu\sigma}\left(-\frac{t\left(a_0^2+\beta-2\mu\sigma\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right) + a_0, \tag{82}$$

$$\mathcal{H}_{75}(x,t) = \sqrt{2}\mu\sqrt{\frac{\sigma}{\mu}}\tan \times\left(\sqrt{\mu\sigma}\left(-\frac{t\left(a_0^2+\beta-2\mu\sigma\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right) + a_0. \tag{83}$$

In case of $[\sigma\mu > 0 \text{ and } \mu < 0]$,

$$\mathcal{H}_{76}(x,t) = \frac{\sqrt{2}\mu\tanh\left(\sqrt{-\mu\sigma}\left(-\frac{t\left(a_0^2+\beta-2\mu\sigma\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}{\sqrt{-\frac{\mu}{\sigma}}} + a_0, \tag{84}$$

$$\mathcal{H}_{77}(x,t) = \frac{\sqrt{2}\mu\coth\left(\sqrt{-\mu\sigma}\left(-\frac{t\left(a_0^2+\beta-2\mu\sigma\right)}{\sqrt{2}a_0}+x+\vartheta\right)\right)}{\sqrt{-\frac{\mu}{\sigma}}} + a_0. \tag{85}$$

III. STABILITY PROPERTY

Here, we study the stability property of the constructed traveling wave solutions based on the Hamiltonian system[29,30] as follows:

According to Eq. (13),

$$\begin{aligned}\Xi = \frac{1}{4c}\Big[&\mathrm{Li}_2\left(-e^{20(c-1)}\right) - \mathrm{Li}_2\left(-e^{-20(c+1)}\right) - \mathrm{Li}_2\left(-e^{20(c+1)}\right)\\ &+ \mathrm{Li}_2\left(-e^{20-20c}\right) - 4\Big(-625c + 5c\log\left(e^{20(c+1)}+1\right)\\ &- 5(c-1)\log\left(e^{20(c-1)}+1\right) - 2\log\left(e^{20-10c}+e^{10c}\right)\\ &- 5(c+1)\log\left(e^{-20(c+1)}+1\right) + 5\log\left(e^{20(c+1)}+1\right)\\ &+ 2\log\left(e^{-10c}+e^{10(c+2)}\right) + 5(c-1)\log\left(e^{20-20c}+1\right)\Big)\Big],\end{aligned} \quad (86)$$

where Ξ is the momentum of the Hamiltonian system, and thus,

$$\frac{\partial \Xi}{\partial \omega}\Big|_{c=1} = 18.61370564 > 0,$$

where c is the wave frequency. Consequently, this solution is stable.

According to Eq. (73),

$$\begin{aligned}\Xi = \frac{1}{4c}\Big[&-\sqrt{2}\mathrm{Li}_2\left(-e^{20c-32}\right) - \sqrt{2}\mathrm{Li}_2\left(-e^{-20c-8}\right) + \sqrt{2}\mathrm{Li}_2\left(-e^{-4(5c+8)}\right)\\ &+ \sqrt{2}\mathrm{Li}_2\left(-e^{20c-8}\right) + 4\Big(450c - \log\left(e^{-20c}+e^{32}\right) + \log\left(e^{20c}+e^{32}\right)\\ &- 8\sqrt{2}\log\left(e^{-4(5c+8)}+1\right) - 2\sqrt{2}\log\left(e^{20c-8}+1\right) - \log\left(e^{20c+8}+1\right)\\ &+ \log\left(e^{8-20c}+1\right) + 8\sqrt{2}\log\left(e^{20c-32}+1\right)\\ &+ \sqrt{2}\Big((5c+2)\log\left(e^{-20c-8}+1\right) + 5c\Big(\log\left(e^{20c}+e^{8}\right)\\ &- \log\left(e^{20c}+e^{32}\right) - \log\left(e^{-4(5c+8)}+1\right) + \log(\cosh(2(5c+8)))\\ &- \log(\cosh(10c+4)) - \log(\cosh(4-10c))\\ &+ \log(\cosh(16-10c))\Big)\Big) + 2\sqrt{2}\big(4\log(\cosh(2(5c+8)))\\ &- \log(\cosh(10c+4)) + \log(\cosh(4-10c))\\ &- 4\log(\cosh(16-10c))\big)\Big)\Big],\end{aligned} \quad (87)$$

and thus,

$$\frac{\partial \Xi}{\partial \omega}\Big|_{c=\frac{77}{9\sqrt{2}}} = -3.54399 < 0.$$

Consequently, this solution is unstable. Therefore, using the same steps in the other obtained solutions yields a good result of the stability property of each of them.

IV. RESULTS AND DISCUSSION

Here, we show the originality and novelty of our solutions by comparing the two techniques we used, the obtained solutions, and the sketches shown.

1. Comparison between our two used methods is as follows:
    - Both schemes are similar when $e = K, \varrho = \sigma, \chi = \lambda$, and $\delta = \mu$.
    - The MK method's solutions are more than those obtained by the GEE method.

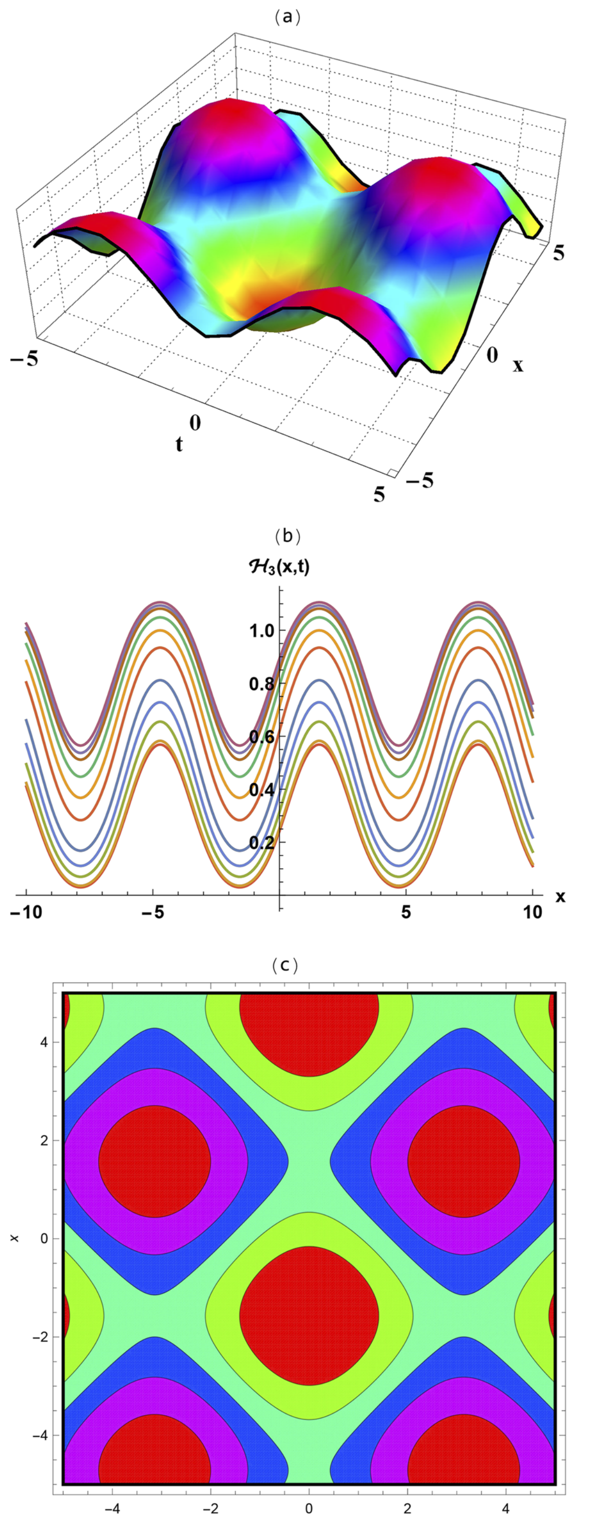

FIG. 1. Cone solitary solution sketches for Eq. (9) in (a) 3D, (b) 2D, and (c) contour plots.

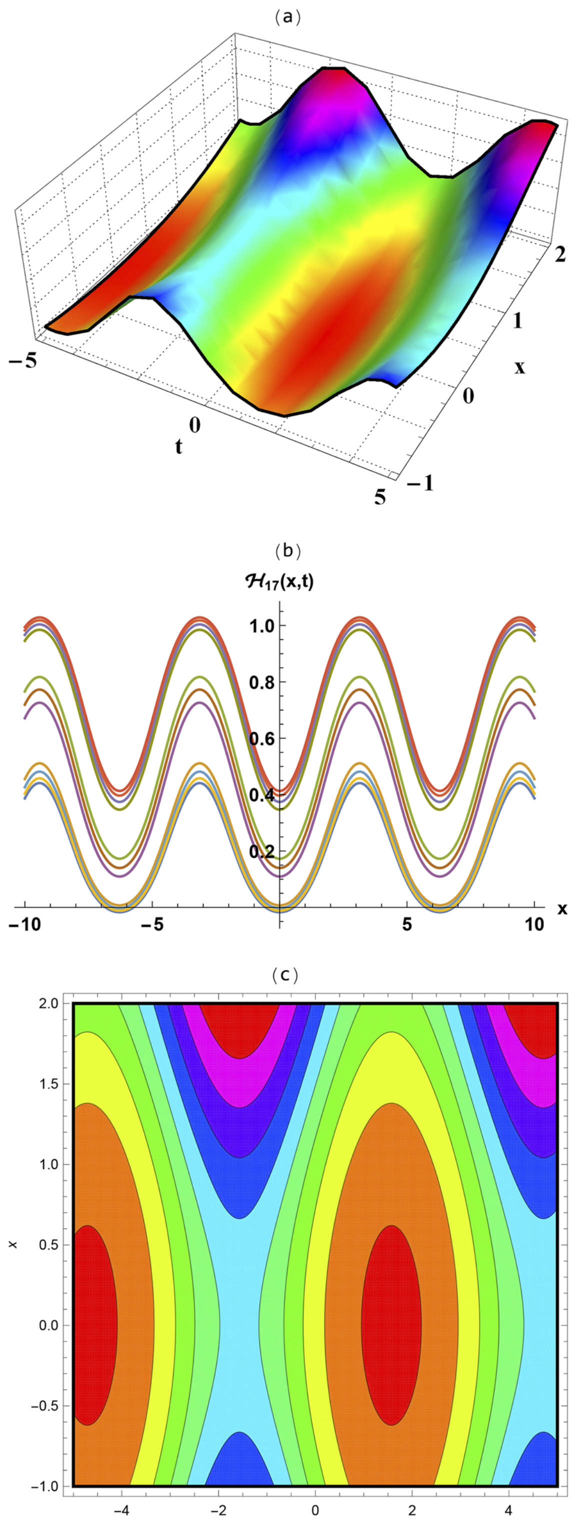

FIG. 2. Breather solitary solution sketches for Eq. (23) in (a) 3D, (b) 2D, and (c) contour plots.

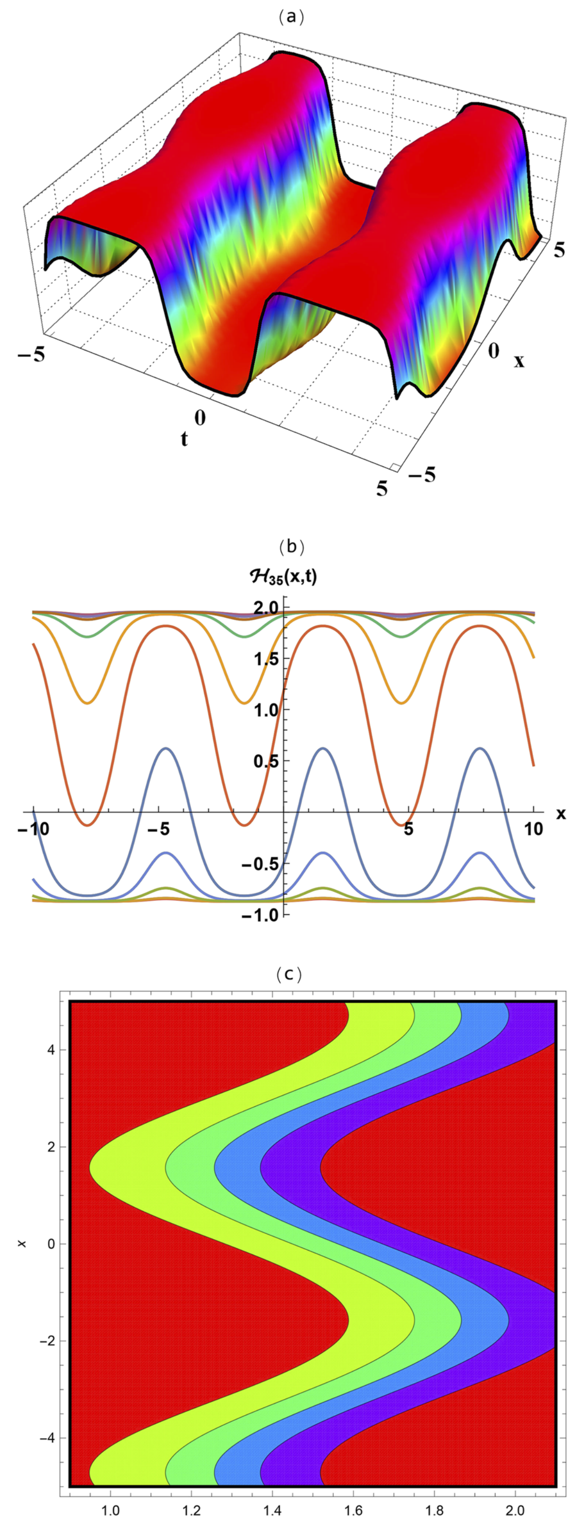

FIG. 3. Periodic kind antikink solitary sketches for Eq. (41) in (a) 3D, (b) 2D, and (c) contour plots.

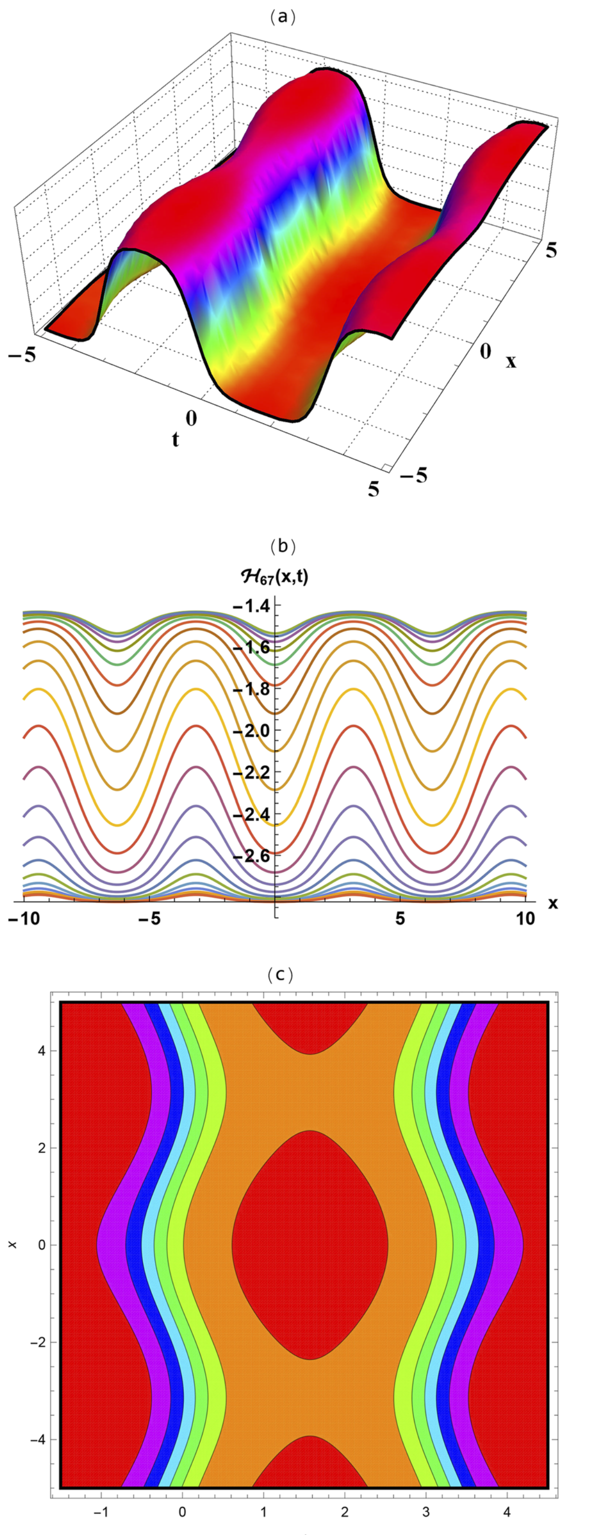

FIG. 4. Periodic kind antikink solitary sketches for Eq. (75) in (a) 3D, (b) 2D, and (c) contour plots.

2. Comparison between the results is as follows:
   - Equation (13) is equal to Eq. (53) in Ref. 31 in case of $\left[\sqrt{-4\,\beta + 8\,\delta\,\varrho + k^2} = k - 2\right]$.
   - In Ref. 32, Abdelrahman *et al.* used the $\exp(-\varphi(\Im))$-expansion method, while we used the general form of this method, so all results obtained in Ref. 32 are covered in our paper.
   - All the other solutions obtained by us are new and different from that obtained in Refs. 31 and 32.
3. Our sketches:

   Here, we explain the figures shown and their physical meaning as follows:

   Figures 1–4 show, respectively, the cone and periodic kink of Eqs. (9), (23), (41), and (75) with the following parameter values: $\beta = -1$, $\delta = 1$, $k = \sqrt{2}$, $\chi = 5$, $\varrho = 6$ and $\beta = -1$, $\delta = 1$, $k = \sqrt{2}$, $\chi = 5$, $\varrho = 6$ and $\beta = -1$, $\delta = -1$, $k = 4$, $\varrho = 1$ and $a_0 = \sqrt{2}$, ($\beta = 2$, $\lambda = 5$, $\mu = 6$, $\vartheta = 1$.

V. CONCLUSION

Our paper has investigated the analytical solutions of the nonlinear PP model through two computational techniques (the MK and GEE techniques). Many novel solutions have been obtained in hyperbolic, trigonometric, and rational formulas that have been sketched in conical and periodic shapes. Moreover, the stability property of the obtained solutions has been successfully checked. The comparison between our obtained solutions and that obtained in previous published papers has been discussed. The powerfulness, effectiveness, and superiority of the two computational schemes we used have been explained through their performance when being employed as analytical schemes.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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