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Local stability of mappings on multi-normed spaces

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Abstract

First we investigate the Hyers–Ulam stability of the Cauchy functional equation for mappings from bounded (unbounded) intervals into Banach spaces. Then we study the Hyers–Ulam stability of the functional equation $f(xy) = xg(y) + h(x)y$ for mappings from bounded (unbounded) intervals into multi-normed spaces.

MSC: 39B72

Keywords: Multi-normed space; Hyers–Ulam stability; Cauchy functional equation

1 Introduction

The concept of stability for a functional equation $(*)$ arises when the functional equation $(*)$ is replaced by an inequality that acts as the equation perturbation. In 1940, Ulam [16] posed the first question concerning the stability of homomorphisms between groups. Hyers [4] answered the question of Ulam in the context of Banach spaces. Hyers's stability theorem was generalized by some authors, and they considered cases where the Cauchy difference was unbounded (see [1, 3, 8, 10, 12, 13]). The stability problem for the Cauchy functional equation on a bounded domain was first proved by Skof [14]. The stability problem for the functional equation

$$f(xy) = xf(y) + f(x)y \tag{1.1}$$

on the interval $(0,1]$ was posed by Maksa [7]. Tabor [15] and Páles [11] proved the Hyers–Ulam stability of functional equation (1.1) for real-valued functions on the intervals $(0,1]$ and $[1,+\infty)$, respectively. In this paper, we use some ideas from the works [6, 9, 14] to investigate the Hyers–Ulam stability of the Cauchy functional equation and (1.1) for mappings from bounded (unbounded) intervals into multi-normed spaces.

Let $(E,\|\cdot\|)$ be a complex linear space. For given $k \in \mathbb{N}$, we denote by E^k the linear space consisting of k-tuples $(x_1,\dots,x_k)$, where $x_1,\dots,x_k \in E$. The linear operations on E^k are defined coordinatewise. We write $(0,\dots,0,x_i,0,\dots,0)$ for an element in E^k, when x_i appears in the ith coordinate. We denote the zero element of either E or E^k by 0.

Definition 1.1 ([2]) A *multi-norm* on $\{E^n : n \in \mathbb{N}\}$ is a sequence $\{\|\cdot\|_n\}_n$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$ and the following axioms are satisfied for each $n \in \mathbb{N}$:

(A_1) $\|x\|_1 = \|x\|$ for each $x \in E$, and $\|\cdot\|_n$ is a norm on E^n;
(A_2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \le (\max_{1\le i\le n} |\alpha_i|)\|(x_1,\dots,x_n)\|_n$ for each $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ and each $x_1,\dots,x_n \in E$;
(A_3) $\|(x_{\sigma(1)},\dots,x_{\sigma(n)})\|_n = \|(x_1,\dots,x_n)\|_n$ for each permutation σ on $\{1,\dots,n\}$;
(A_4) $\|(x_1,\dots,x_n,0)\|_{n+1} = \|(x_1,\dots,x_n)\|_n$ for each $x_1,\dots,x_n \in E$;
(A_5) $\|(x_1,\dots,x_{n-1},x_n,x_n)\|_{n+1} = \|(x_1,\dots,x_{n-1},x_n)\|_n$ for each $x_1,\dots,x_n \in E$;
In this case $\{(E^n,\|\cdot\|_n) : n \in \mathbb{N}\}$ is called a *multi-normed space.*

Example 1.2 ([2]) Let $(E,\|\cdot\|)$ be a normed space. For given $n \in \mathbb{N}$, define $\|\cdot\|_n$ on E^n by $\|x_1,\dots,x_n\|_n = \max_{1\le i\le n}\|x_i\|$. This gives a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

For details and many other examples, we refer the readers to [2]. We now have the following consequences of the axioms.

Proposition 1.3 ([2]) *Let $\{(E^n,\|\cdot\|_n) : n \in \mathbb{N}\}$ be a multi-normed space. Then*
(i) $\|(x,\dots,x)\|_n = \|x\|$ *for each* $x \in E$;
(ii) $\|(\alpha_1 x_1,\dots,\alpha_n x_n)\|_n = \|(x_1,\dots,x_n)\|_n$ *for each* $x_1,\dots,x_n \in E$ *and* $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ *with* $|\alpha_1| = \dots = |\alpha_n| = 1$;
(iii) $\max_{1\le i\le n}\|x_i\| \le \|(x_1,\dots,x_n)\|_n \le \sum_{i=1}^n \|x_i\| \le n\max_{1\le i\le n}\|x_i\|$ *for each* $x_1,\dots,x_n \in E$.

Proof (*i*) follows from (A_1) and (A_5). To prove (*ii*), it follows from (A_2) that

$$\|(\alpha_1 x_1,\dots,\alpha_n x_n)\|_n \le \|(x_1,\dots,x_n)\|_n = \|(\bar{\alpha}_1\alpha_1 x_1,\dots,\bar{\alpha}_n\alpha_n x_n)\|_n \le \|(\alpha_1 x_1,\dots,\alpha_n x_n)\|_n.$$

This proves (*ii*). To prove (*iii*), since $\|\cdot\|_n$ is a norm on E^n, we have by (A_4)

$$\begin{aligned}\|(x_1,\dots,x_n)\|_n &\le \|(x_1,0,\dots,0)\|_n + \dots + \|(0,0,\dots,0,x_n)\|_n \\ &= \sum_{i=1}^n \|x_i\| \le n\max_{1\le i\le n}\|x_i\|.\end{aligned}$$

On the other hand, for each $1 \le i \le n$, we have by (*ii*)

$$\begin{aligned}\|x_i\| &= \frac{1}{2}\|(x_1,\dots,x_i,\dots,x_n) + (-x_1,\dots,x_i,\dots,-x_n)\|_n \\ &\le \frac{1}{2}\big[\|(x_1,x_2,\dots,x_n)\|_n + \|(x_1,x_2,\dots,x_n)\|_n\big] = \|(x_1,x_2,\dots,x_n)\|_n.\end{aligned}$$

Hence we get $\max_{1\le i\le n}\|x_i\| \le \|(x_1,\dots,x_n)\|_n$. □

Item (*iii*) of Proposition 1.3 implies that if $(E,\|\cdot\|)$ is a Banach space, then $(E^n,\|\cdot\|_n)$ is a Banach space for each $n \in \mathbb{N}$. We use the term *multi-Banach space* for $\{(E^n,\|\cdot\|_n) : n \in \mathbb{N}\}$ when $(E,\|\cdot\|)$ is Banach.

Definition 1.4 Let $\{(E^n,\|\cdot\|_n) : n \in \mathbb{N}\}$ be a multi-normed space. A sequence $\{x_n\}_n$ in E is said to be a *multi-Cauchy sequence* in E if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\|(x_n - x_m,\dots,x_{n+k-1} - x_{m+k-1})\|_k < \varepsilon, \quad n,m \ge n_0.$$

A sequence $\{x_n\}_n$ in E is called *multi-convergent* to a in E if, for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \left\| (x_n - a, \ldots, x_{n+k-1} - a) \right\|_k < \varepsilon, \quad n \geq m.$$

In this case we write

$$\mathop{\mathbf{Lim}}_{n\to\infty} x_n = a.$$

Applying the triangle inequality for the norm $\|\cdot\|_k$ and property (*iii*) of Proposition 1.3, we deduce the following result.

Lemma 1.5 ([9]) *Suppose that* $k \in \mathbb{N}$ *and* $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in E^k$. *For each* $j \in \{1, \ldots, k\}$, *let* $\{x_{n,j}\}_n$ *be a sequence in E such that* $\lim_{n\to\infty} x_{n,j} = x_j$. *Then*

$$\lim_{n\to\infty} (x_{n,1} - y_1, \ldots, x_{n,k} - y_k) = (x_1 - y_1, \ldots, x_k - y_k).$$

It is clear that each multi-convergent sequence is a multi-Cauchy sequence and convergent. In multi-Banach spaces a multi-Cauchy sequence is multi-convergent.

In this paper, using some ideas from [6, 9, 14], we investigate the Hyers–Ulam stability of functional equation (1.1) for mappings from subsets of $\mathbb{R}$ into multi-normed spaces.

2 Stability of functional equation (1.1)

Theorem 2.1 *Let* $f : [0, c) \to E$ *be a function satisfying*

$$\sup_{n\in\mathbb{N}} \left\| \bigl(f(x_1 + y_1) - f(x_1) - f(y_1), \ldots, f(x_n + y_n) - f(x_n) - f(y_n)\bigr) \right\|_n \leq \delta \tag{2.1}$$

for some $\delta > 0$ *and all* $x_1, \ldots, x_n, y_1, \ldots, y_n \in (0, c]$ *with* $x_i + y_i \in (0, c]$ *for all* $i \in \{1, \ldots, n\}$. *Then there exists an additive function* $A : \mathbb{R} \to E$ *such that*

$$\sup_{n\in\mathbb{N}} \left\| \bigl(A(x_1) - f(x_1), \ldots, A(x_n) - f(x_n)\bigr) \right\|_n \leq 3\delta, \quad x \in [0, c). \tag{2.2}$$

Proof We extend the function f to $[0, +\infty)$. For this we represent arbitrary $x \geq 0$ by $x = n(c/2) + \alpha$, where n is an integer and $0 \leq \alpha < c/2$. Then we define a function $\varphi : [0, +\infty) \to E$ by $\varphi(x) = nf(c/2) + f(\alpha)$. It is clear that $\varphi(x) = f(x)$ for all $x \in [0, c/2)$. If $x \in [c/2, c)$, then $\varphi(x) = f(c/2) + f(x - c/2)$. We claim that

$$\sup_{n\in\mathbb{N}} \left\| \bigl(\varphi(x_1) - f(x_1), \ldots, \varphi(x_n) - f(x_n)\bigr) \right\|_n \leq \delta, \quad x_1, \ldots, x_n \in [0, c). \tag{2.3}$$

Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in [0, c)$. We set $\Omega = \{i : 1 \leq i \leq n, x_i \in [c/2, c)\}$ and $|\Omega| = m$. If Ω is empty, then $\varphi(x_i) = f(x_i)$ for all $1 \leq i \leq n$, and consequently the claim is true. For the case $m \geq 1$, we have $\varphi(x_i) - f(x_i) = f(c/2) + f(x_i - c/2) - f(x_i)$ for all $i \in \Omega$. Let $j_1, \ldots, j_m \in \Omega$. Then (A_4) and (2.1) imply

$$\left\| \bigl(\varphi(x_1) - f(x_1), \ldots, \varphi(x_n) - f(x_n))\bigr) \right\|_n = \left\| \bigl(\varphi(x_{j_1}) - f(x_{j_1}), \ldots, \varphi(x_{j_m}) - f(x_{j_m})\bigr) \right\|_m \leq \delta,$$

which proves (2.3). We now prove that

$$\sup_{n\in\mathbb{N}}\left\| \bigl(\varphi(x_1+y_1)-\varphi(x_1)-\varphi(y_1),\ldots,\varphi(x_n+y_n)-\varphi(x_n)-\varphi(y_n)\bigr)\right\|_n \le 2\delta \tag{2.4}$$

for all $x_1,\ldots,x_n,y_1,\ldots,y_n \in [0,+\infty)$. For given $n\in\mathbb{N}$ and $x_i, y_i \ge 0$, let $x_i = n_i(c/2)+\alpha_i$ and $y_i = m_i(c/2)+\beta_i$, where m_i and n_i are integers and $0\le \alpha_i,\beta_i < c/2$. We set $\Delta = \{i : 1\le i\le n, \alpha_i+\beta_i \in [c/2,c)\}$. Then it is easy to show that

$$\varphi(x_i+y_i)-\varphi(x_i)-\varphi(y_i) = f(\alpha_i+\beta_i)-f(\alpha_i)-f(\beta_i), \quad i\notin\Delta, \tag{2.5}$$

$$\varphi(x_i+y_i)-\varphi(x_i)-\varphi(y_i) = \varphi(\alpha_i+\beta_i)-f(\alpha_i)-f(\beta_i), \quad i\in\Delta. \tag{2.6}$$

To prove (2.4), we need to consider three cases as follows.

Case 1. Suppose that Δ is empty. Then (2.4) follows from (2.1) and (2.5).

Case 2. Suppose $|\Delta| = n$. Then, by (2.1), (2.3), and (2.6), we have

$$\begin{aligned}
&\left\| \bigl(\varphi(x_1+y_1)-\varphi(x_1)-\varphi(y_1),\ldots,\varphi(x_n+y_n)-\varphi(x_n)-\varphi(y_n)\bigr)\right\|_n \\
&\quad\le \left\| \bigl(\varphi(\alpha_1+\beta_1)-f(\alpha_1+\beta_1),\ldots,\varphi(\alpha_n+\beta_n)-f(\alpha_n+\beta_n)\bigr)\right\|_n \\
&\qquad+ \left\| \bigl(f(\alpha_1+\beta_1)-f(\alpha_1)-f(\beta_1),\ldots,f(\alpha_n+\beta_n)-f(\alpha_n)-f(\beta_n)\bigr)\right\|_n \\
&\quad\le 2\delta.
\end{aligned}$$

Case 3. Suppose that Δ is not empty and $|\Delta| = m < n$. Then

$$\begin{aligned}
&\left\| \bigl(\varphi(x_1+y_1)-\varphi(x_1)-\varphi(y_1),\ldots,\varphi(x_n+y_n)-\varphi(x_n)-\varphi(y_n)\bigr)\right\|_n \\
&\quad\le \left\| \bigl(\varphi(\alpha_{j_1}+\beta_{j_1})-f(\alpha_{j_1}+\beta_{j_1}),\ldots,\varphi(\alpha_{j_m}+\beta_{j_m})-f(\alpha_{j_m}+\beta_{j_m})\bigr)\right\|_m \\
&\qquad+ \left\| \bigl(f(\alpha_1+\beta_1)-f(\alpha_1)-f(\beta_1),\ldots,f(\alpha_n+\beta_n)-f(\alpha_n)-f(\beta_n)\bigr)\right\|_n \\
&\quad\le 2\delta.
\end{aligned}$$

Hence we have proved (2.4). Letting $y_i = x_i$ for $1\le i\le n$ in (2.4), we get

$$\sup_{n\in\mathbb{N}}\left\| \bigl(\varphi(2x_1)-2\varphi(x_1),\ldots,\varphi(2x_n)-2\varphi(x_n)\bigr)\right\|_n \le 2\delta$$

for all $x_1,\ldots,x_n,y_1,\ldots,y_n \in [0,+\infty)$. Replacing $x_1,\ldots,x_n$ by $2^k x_1,\ldots,2^k x_n$ in the above inequality and dividing both sides of the resulting inequality by 2^{k+1}, we obtain

$$\sup_{n\in\mathbb{N}}\left\| \left(\frac{\varphi(2^{k+1}x_1)}{2^{k+1}}-\frac{\varphi(2^k x_1)}{2^k},\ldots,\frac{\varphi(2^{k+1}x_n)}{2^{k+1}}-\frac{\varphi(2^k x_n)}{2^k}\right)\right\|_n \le \frac{\delta}{2^k}$$

for all $x_1,\ldots,x_n \in [0,+\infty)$. Then

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\| \left(\frac{\varphi(2^{k+1}x_1)}{2^{k+1}}-\frac{\varphi(2^m x_1)}{2^m},\ldots,\frac{\varphi(2^{k+1}x_n)}{2^{k+1}}-\frac{\varphi(2^m x_n)}{2^m}\right)\right\|_n \\
&\quad= \sup_{n\in\mathbb{N}}\left\| \sum_{i=m}^{k}\left(\frac{\varphi(2^{i+1}x_1)}{2^{i+1}}-\frac{\varphi(2^i x_1)}{2^i},\ldots,\frac{\varphi(2^{i+1}x_n)}{2^{i+1}}-\frac{\varphi(2^i x_n)}{2^i}\right)\right\|_n
\end{aligned}$$

$$\le \sup_{n\in\mathbb{N}} \sum_{i=m}^{k} \left\| \left(\frac{\varphi(2^{i+1}x_1)}{2^{i+1}} - \frac{\varphi(2^{i}x_1)}{2^{i}}, \ldots, \frac{\varphi(2^{i+1}x_n)}{2^{i+1}} - \frac{\varphi(2^{i}x_n)}{2^{i}} \right) \right\|_n$$
$$\le \sum_{i=m}^{k} \frac{\delta}{2^i} \tag{2.7}$$

for all $x_1, \ldots, x_n \in [0, +\infty)$. For fixed $x \in [0, +\infty)$, replacing x_j by $2^{j-1}x$ for all $1 \le j \le n$ in the above inequality, we obtain

$$\sup_{n\in\mathbb{N}} \left\| \left(\frac{\varphi(2^{k+1}x)}{2^{k+1}} - \frac{\varphi(2^{m}x)}{2^{m}}, \ldots, \frac{\varphi(2^{k+n}x)}{2^{k+1}} - \frac{\varphi(2^{m+n-1}x)}{2^{m}} \right) \right\|_n \le \sum_{i=m}^{k} \frac{\delta}{2^i}. \tag{2.8}$$

Then (A_2) and (2.8) yield

$$\sup_{n\in\mathbb{N}} \left\| \left(\frac{\varphi(2^{k+1}x)}{2^{k+1}} - \frac{\varphi(2^{m}x)}{2^{m}}, \ldots, \frac{\varphi(2^{k+n}x)}{2^{k+n}} - \frac{\varphi(2^{m+n-1}x_n)}{2^{m+n-1}} \right) \right\|_n$$
$$= \sup_{n\in\mathbb{N}} \left\| \left(\frac{\varphi(2^{k+1}x)}{2^{k+1}} - \frac{\varphi(2^{m}x)}{2^{m}}, \ldots, \frac{1}{2^{n-1}} \left[\frac{\varphi(2^{k+n}x)}{2^{k+1}} - \frac{\varphi(2^{m+n-1}x_n)}{2^{m}} \right] \right) \right\|_n$$
$$\le \sup_{n\in\mathbb{N}} \left\| \left(\frac{\varphi(2^{k+1}x)}{2^{k+1}} - \frac{\varphi(2^{m}x)}{2^{m}}, \ldots, \frac{\varphi(2^{k+n}x)}{2^{k+1}} - \frac{\varphi(2^{m+n-1}x_n)}{2^{m}} \right) \right\|_n$$
$$\le \sum_{i=m}^{k} \frac{\delta}{2^i}.$$

Hence $\{\frac{\varphi(2^m x)}{2^m}\}_m$ is a multi-Cauchy sequence in E for all $x \in [0, +\infty)$. So it is multi-convergent in the multi-Banach space $\{(E^n, \|\cdot\|_n) : n \in \mathbb{N}\}$. Consider the function $A_0 : [0, +\infty) \to E$ defined by

$$A_0(x) = \mathop{\mathbf{Lim}}_{m\to\infty} \frac{\varphi(2^m x)}{2^m}.$$

Letting $m = 0$ in (2.7), we get

$$\sup_{n\in\mathbb{N}} \left\| \left(\frac{\varphi(2^{k+1}x_1)}{2^{k+1}} - \varphi(x_1), \ldots, \frac{\varphi(2^{k+1}x_n)}{2^{k+1}} - \varphi(x_n) \right) \right\|_n \le \sum_{i=0}^{k} \frac{\delta}{2^i}$$

for all $x_1, \ldots, x_n \in [0, +\infty)$. Letting $k \to \infty$ and utilizing Lemma 1.5, we infer that

$$\sup_{n\in\mathbb{N}} \left\| \left(A_0(x_1) - \varphi(x_1), \ldots, A_0(x_n) - \varphi(x_n) \right) \right\|_n \le 2\delta, \quad x_1, \ldots, x_n \in [0, +\infty).$$

Using (2.3), we have

$$\sup_{n\in\mathbb{N}} \left\| \left(A_0(x_1) - f(x_1), \ldots, A_0(x_n) - f(x_n) \right) \right\|_n$$
$$\le \sup_{n\in\mathbb{N}} \left\| \left(A_0(x_1) - \varphi(x_1), \ldots, A_0(x_n) - \varphi(x_n) \right) \right\|_n$$
$$+ \sup_{n\in\mathbb{N}} \left\| \left(\varphi(x_1) - f(x_1), \ldots, \varphi(x_n) - f(x_n) \right) \right\|_n$$
$$\le 3\delta, \quad x_1, \ldots, x_n \in [0, c).$$

We now extend A_0 to a function $A : \mathbb{R} \to E$ given by

$$A(x) = \begin{cases} A_0(x), & x \geq 0; \\ -A_0(-x), & x < 0. \end{cases}$$

We show that A is additive. For given $x, y \in \mathbb{R}$, since $A(x+y) - A(x) - A(y)$ is symmetric in x and y, we may assume the following cases:

(i) If $x, y \geq 0$ or $x, y < 0$, then we get $A(x+y) = A(x) + A(y)$.

(ii) If $x \geq 0, y < 0$ and $x + y \geq 0$, then

$$\begin{aligned} A(x+y) - A(x) - A(y) &= A_0(x+y) - A_0(x) + A_0(-y) \\ &= \bigl[A_0(x+y) + A_0(-y)\bigr] - A_0(x) \\ &= A_0(x) - A_0(x) = 0. \end{aligned}$$

(iii) If $x \geq 0, y < 0$ and $x + y < 0$, then

$$\begin{aligned} A(x+y) - A(x) - A(y) &= -A_0(-x-y) - A_0(x) + A_0(-y) \\ &= A_0(-y) - \bigl[A_0(-x-y) + A_0(x)\bigr] \\ &= A_0(-y) - A_0(-y) = 0. \end{aligned}$$

Hence $A : \mathbb{R} \to E$ is an additive function satisfying (2.2), which completes the proof. □

Theorem 2.2 *Let $c \geq 0$ and $f : [c, +\infty) \to E$ be a function satisfying*

$$\sup_{n\in\mathbb{N}} \bigl\| \bigl(f(x_1+y_1) - f(x_1) - f(y_1), \ldots, f(x_n+y_n) - f(x_n) - f(y_n)\bigr) \bigr\|_n \leq \delta \tag{2.9}$$

for some $\delta > 0$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in [c, +\infty)$. Then there exists an additive function $A : \mathbb{R} \to E$ such that

$$\sup_{n\in\mathbb{N}} \bigl\| \bigl(A(x_1) - f(x_1), \ldots, A(x_n) - f(x_n)\bigr) \bigr\|_n \leq \delta, \quad x_1, \ldots, x_n \in [c, +\infty). \tag{2.10}$$

Proof Using the same argument as in the proof of Theorem 2.1, there exists an additive function $T : [c, +\infty) \to E$ such that

$$\sup_{n\in\mathbb{N}} \bigl\| \bigl(T(x_1) - f(x_1), \ldots, T(x_n) - f(x_n)\bigr) \bigr\|_n \leq \delta, \quad x_1, \ldots, x_n \in [c, +\infty).$$

We extend T from $[c, +\infty)$ to $\mathbb{R}$. First, we extend T from $[c, +\infty)$ to $[0, +\infty)$ by defining

$$\widetilde{T}(x) = \begin{cases} T(x), & x \in [c, +\infty); \\ T(x+c) - T(c), & x \in [0, c). \end{cases}$$

It is easy to see that $\widetilde{T}:[0,+\infty)\to E$ is additive. Now, we extend $\widetilde{T}$ to the additive function $A:\mathbb{R}\to E$ by defining

$$A(x)=\begin{cases}\widetilde{T}(x), & x\geq 0;\\ -\widetilde{T}(-x), & x<0.\end{cases}$$ □

For convenience, we use the following abbreviation for a given mapping $f:(0,1]\to E$:

$$Df(x,y):=f(xy)-xf(y)-f(x)y,\quad x,y\in(0,1].$$

Theorem 2.3 *Let $\{(E^n,\|\cdot\|_n):n\in\mathbb{N}\}$ be a multi-Banach space. Suppose that ε is a non-negative real number and $f:(0,1]\to E$ is a mapping satisfying*

$$\big\|\big(Df(x_1,y_1),\dots,Df(x_n,y_n)\big)\big\|_n\leq\varepsilon\prod_{i=1}^{n}x_iy_i, \tag{2.11}$$

for all $n\in\mathbb{N}$ and all $x_1,\dots,x_n,y_1,\dots,y_n\in(0,1]$. Then there exists a function $\Delta:(0,1]\to E$ satisfying functional equation (1.1) and the inequality

$$\sup_{n\in\mathbb{N}}\big\|\big(\Delta(t_1)-f(t_1),\dots,\Delta(t_n)-f(t_n)\big)\big\|_n\leq\varepsilon,\quad t_1,\dots,t_n\in(0,1].$$

Proof For each $n\in\mathbb{N}$, it follows from (A_2) and (2.11) that

$$\begin{aligned}&\left\|\left(\prod_{i=2}^{n}x_iy_iDf(x_1,y_1),\dots,\prod_{i=1}^{n-1}x_iy_iDf(x_n,y_n)\right)\right\|_n\\&\quad\leq\big\|\big(Df(x_1,y_1),\dots,Df(x_n,y_n)\big)\big\|_n\leq\varepsilon\prod_{i=1}^{n}x_iy_i\end{aligned}$$

for all $x_1,\dots,x_n,y_1,\dots,y_n\in(0,1]$. Then

$$\left\|\left(\frac{Df(x_1,y_1)}{x_1y_1},\dots,\frac{Df(x_n,y_n)}{x_ny_n}\right)\right\|_n\leq\varepsilon,\quad x_1,\dots,x_n,y_1,\dots,y_n\in(0,1]. \tag{2.12}$$

If we define the mapping $g:(0,1]\to E$ by

$$g(x)=\frac{f(x)}{x},$$

then (2.12) means

$$\big\|\big(g(x_1y_1)-g(x_1)-g(y_1),\dots,g(x_ny_n)-g(x_n)-g(y_n)\big)\big\|_n\leq\varepsilon \tag{2.13}$$

for all $x_1,\dots,x_n,y_1,\dots,y_n\in(0,1]$. Let us define the mapping $G:[0,+\infty)\to E$ by $G(t)=g(e^{-t})$. Replacing $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ by $(e^{-t_1},\dots,e^{-t_n})$ and $(e^{-s_1},\dots,e^{-s_n})$, respectively, in (2.13), we get

$$\sup_{n\in\mathbb{N}}\big\|\big(G(t_1+s_1)-G(t_1)-G(s_1),\dots,G(t_n+s_n)-G(t_n)-G(s_n)\big)\big\|_n\leq\varepsilon$$

for all $t_1,\dots,t_n,s_1,\dots,s_n \in [0,+\infty)$. As in the proof of Theorem 2.1, there exists an additive mapping $A:\mathbb{R}\to E$ such that

$$\sup_{n\in\mathbb{N}}\big\|\big(A(x_1)-G(x_1),\dots,A(x_n)-G(x_n)\big)\big\|_n \le \varepsilon, \quad x_1,\dots,x_n\in[0,+\infty).$$

Letting $x_i = -\ln t_i$ in the above inequality and using the definitions of G and g, we obtain

$$\sup_{n\in\mathbb{N}}\left\|\left(A(-\ln t_1)-\frac{f(t_1)}{t_1},\dots,A(-\ln t_n)-\frac{f(t_n)}{t_n}\right)\right\|_n \le \varepsilon, \quad t_1,\dots,t_n\in(0,1]. \tag{2.14}$$

Applying (A_2) and (2.14), we get

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\big\|\big(t_1A(-\ln t_1)-f(t_1),\dots,t_nA(-\ln t_n)-f(t_n)\big)\big\|_n\\
&\quad=\sup_{n\in\mathbb{N}}\left\|\left(t_1\left[A(-\ln t_1)-\frac{f(t_1)}{t_1}\right],\dots,t_n\left[A(-\ln t_n)-\frac{f(t_n)}{t_n}\right]\right)\right\|_n\\
&\quad\le\sup_{n\in\mathbb{N}}\left\|\left(A(-\ln t_1)-\frac{f(t_1)}{t_1},\dots,A(-\ln t_n)-\frac{f(t_n)}{t_n}\right)\right\|_n\\
&\quad\le\varepsilon, \quad t_1,\dots,t_n\in(0,1].
\end{aligned}\tag{2.15}$$

If we define the function $\Delta:(0,1]\to E$ by $\Delta(x)=xA(-\ln x)$, then $\Delta(xy)=\Delta(x)y+x\Delta(y)$ for all $x,y\in(0,1]$ and (2.15) implies

$$\sup_{n\in\mathbb{N}}\big\|\big(\Delta(t_1)-f(t_1),\dots,\Delta(t_n)-f(t_n)\big)\big\|_n\le\varepsilon, \quad t_1,\dots,t_n\in(0,1].$$ □

Theorem 2.4 *Let $\mathbb{X}$ be a normed space. Suppose that $f,g,h:(0,+\infty)\to\mathbb{X}$ are mappings satisfying*

$$\big\|f(xy)-xg(y)-h(x)y\big\|\le\psi(x,y), \quad x,y\in(0,+\infty), \tag{2.16}$$

where $\psi:(0,+\infty)\to[0,+\infty)$ is a mapping satisfying and

$$\lim_{x\to\infty}\frac{\psi(x,b)}{x}=\lim_{y\to\infty}\frac{\psi(a,y)}{y}=0, \quad a,b\in(0,+\infty).$$

Then

$$\begin{aligned}
&g(ab)=ag(b)+bg(a)-abg(1);\\
&h(ab)=ah(b)+bh(a)-abh(1), \quad a,b\in(0,+\infty).
\end{aligned}\tag{2.17}$$

Moreover, if

$$\lim_{x\to\infty}\psi(a/x,x)=0 \quad\text{or}\quad \lim_{x\to\infty}\psi(ax,1/x)=0, \quad a\in(0,+\infty),$$

then

$$f(ab)=af(b)+f(a)b-abf(1), \quad a,b\in(0,+\infty). \tag{2.18}$$

Proof We define the functions $F, G, H : (0, +\infty) \to \mathbb{X}$ by

$$F(x) = \frac{f(x)}{x}, \qquad G(x) = \frac{g(x)}{x}, \qquad H(x) = \frac{h(x)}{x}, \quad x \in (0, +\infty).$$

By (2.16), we have

$$\left\| F(xy) - G(x) - H(y) \right\| \leq \frac{\psi(x,y)}{xy}, \quad x, y \in (0, +\infty). \tag{2.19}$$

Then from (2.19) we get

$$G(x) = \lim_{y\to\infty} \left[F(xy) - H(y) \right], \quad x \in (0, +\infty);$$

$$H(y) = \lim_{x\to\infty} \left[F(xy) - G(x) \right], \quad y \in (0, +\infty).$$

Therefore, for all $a, b \in (0, +\infty)$, we have

$$\begin{aligned} G(ab) - G(a) - G(b) &= \lim_{x\to\infty} \left[F(abx) - H(x) \right] - \lim_{x\to\infty} \left[F(ax) - H(x) \right] - G(b) \\ &= \lim_{x\to\infty} \left[F(abx) - F(ax) \right] - \lim_{x\to\infty} \left[F(abx) - H(ax) \right] \\ &= \lim_{x\to\infty} \left[H(ax) - F(ax) \right] = -G(1), \\ H(ab) - H(a) - H(b) &= \lim_{x\to\infty} \left[F(abx) - G(x) \right] - \lim_{x\to\infty} \left[F(ax) - G(x) \right] - H(b) \\ &= \lim_{x\to\infty} \left[F(abx) - F(ax) \right] - \lim_{x\to\infty} \left[F(abx) - G(ax) \right] \\ &= \lim_{x\to\infty} \left[G(ax) - F(ax) \right] = -H(1). \end{aligned}$$

Moreover, if $\lim_{x\to\infty} \psi(a/x, x) = 0$, then by replacing x with x/y in (2.19) and letting $y \to \infty$ in the resulting inequality, we get

$$F(x) = \lim_{y\to\infty} \left[G(x/y) + H(y) \right], \quad x \in (0, +\infty).$$

Consequently,

$$\begin{aligned} F(ab) - F(a) - F(b) &= \lim_{x\to\infty} \left[G(ab/x) + H(x) \right] - \lim_{x\to\infty} \left[G(a/x) + H(x) \right] - F(b) \\ &= \lim_{x\to\infty} \left[G(ab/x) - G(a/x) \right] - \lim_{x\to\infty} \left[G(ab/x) + H(x/a) \right] \\ &= -\lim_{x\to\infty} \left[G(a/x) + H(x/a) \right] = -F(1). \end{aligned}$$

Hence we get $f(ab) = af(b) + f(a)b - abf(1)$ for all $a, b \in (0, +\infty)$. By a similar argument, we get the result if $\lim_{x\to\infty} \psi(ax, 1/x) = 0$ for all $a \in (0, +\infty)$. □

Corollary 2.5 *Let* $\mathbb{X}$ *be a normed space. Suppose that* $f, h : (0, +\infty) \to \mathbb{X}$ *are mappings satisfying*

$$\left\| f(xy) - xf(y) - h(x)y \right\| \leq \psi(x,y), \quad x, y \in (0, +\infty), \tag{2.20}$$

where $\psi : (0, +\infty) \to [0, +\infty)$ *is a mapping satisfying and*

$$\lim_{x\to\infty} \frac{\psi(x,b)}{x} = \lim_{y\to\infty} \frac{\psi(a,y)}{y} = 0, \quad a, b \in (0, +\infty).$$

Then

$$h(ab) = ah(b) + h(a)b, \qquad f(ab) = af(b) + bh(a) = af(b) + bf(a) - abf(1) \tag{2.21}$$

for all $a, b \in (0, +\infty)$.

Corollary 2.6 *Let* $\mathbb{X}$ *be a normed space and* $p, q, r, s \in (-\infty, 1)$. *Suppose that* ε, δ, θ *are nonnegative real numbers and* $f, g : (0, +\infty) \to \mathbb{X}$ *are mappings satisfying*

$$\|f(xy) - xf(y) - g(x)y\| \le \varepsilon + \delta x^p y^q + \theta x^r y^s, \quad x, y \in (0, +\infty).$$

Then f *and* g *satisfy* (2.21).

Corollary 2.7 *Let* $\mathbb{X}$ *be a normed space and* $p, q \in (-\infty, 1)$ *with* $pq < 0$. *Suppose that* ε *is a nonnegative real number and* $f, g, h : (0, +\infty) \to \mathbb{X}$ *are mappings satisfying*

$$\|f(xy) - xg(y) - h(x)y\| \le \varepsilon\bigl(x^p + y^q\bigr), \quad x, y \in (0, +\infty).$$

Then f, g, *and* h *satisfy* (2.17) *and* (2.18).

Remark 2.8 By similar reasoning as in the proof of Theorem 2.4, it can be shown that Theorem 2.4 is also valid if the domains of functions f, g, ψ are $(-\infty, 0)$ or $\mathbb{R} \setminus \{0\}$.

The following theorem is an improved version of the main result in [6].

Theorem 2.9 *Let* $\mathbb{X}$ *be a normed space. Suppose that* ε *is a nonnegative real number and* $f, g, h : (0, +\infty) \to \mathbb{X}$ *are mappings satisfying*

$$\|f(xy) - xg(y) - h(x)y\| \le \varepsilon, \quad x, y \in (0, +\infty). \tag{2.22}$$

Then

(i) f, g *satisfy* (2.17).

(ii) *there exists a function* $\varphi : (0, +\infty) \to \mathbb{X}$ *satisfying* (1.1) *and*

$$\|f(x) - \varphi(x) - xf(1)\| \le 4x\varepsilon, \quad x \in [1, +\infty). \tag{2.23}$$

Moreover, φ *is unique on the domain* $[1, +\infty)$.

Proof (i) follows from Theorem 2.4. It suffices to prove (ii). Letting $x = 1$, $y = 1$, and $x = y = 1$ in (2.22), respectively, we obtain

$$\|f(y) - g(y) - h(1)y\| \le \varepsilon, \qquad \|f(x) - xg(1) - h(x)\| \le \varepsilon, \qquad \|f(1) - g(1) - h(1)\| \le \varepsilon$$

for all $x, y \in (0, +\infty)$. Therefore

$$\begin{aligned}
&\|f(xy) - xf(y) - f(x)y + xyf(1)\| \\
&\quad \le \|f(xy) - xg(y) - h(x)y\| \\
&\qquad + \|xg(y) - xf(y) + xyh(1)\| \\
&\qquad + \|h(x)y - f(x)y + xyg(1)\| \\
&\qquad + \|xyf(1) - xyg(1) - xyh(1)\| \\
&\quad \le \varepsilon(1 + x + y + xy), \quad x, y \in (0, +\infty).
\end{aligned} \tag{2.24}$$

Define the functions $F : (0, +\infty) \to \mathbb{X}$ and $T : \mathbb{R} \to \mathbb{X}$ by

$$F(x) = \frac{f(x)}{x}, \quad x \in (0, +\infty), \qquad T(t) = F(e^t), \quad t \in \mathbb{R}.$$

Then by (2.24) we have

$$\|F(xy) - F(y) - F(x) + f(1)\| \le \varepsilon(1 + 1/x + 1/y + 1/xy), \quad x, y \in (0, +\infty). \tag{2.25}$$

Replacing x and y by e^t and e^s in (2.25), respectively, we get

$$\|T(s + t) - T(s) - T(t) + T(0)\| \le \varepsilon(1 + e^{-s} + e^{-t} + e^{-s-t}), \quad t, s \in \mathbb{R}.$$

Hence

$$\|T(s + t) - T(s) - T(t) + T(0)\| \le 4\varepsilon, \quad t, s \in [0, +\infty).$$

Then by [5, Lemma 2.27] there exists a unique additive function $A : \mathbb{R} \to \mathbb{X}$ such that

$$\|T(x) - A(x) - T(0)\| \le 4\varepsilon, \quad x \in [0, +\infty).$$

Therefore

$$\|f(x) - xA(\ln x) - xf(1)\| \le 4x\varepsilon, \quad x \in [1, +\infty).$$

We put $\varphi(x) := xA(\ln x)$, $x \in (0, +\infty)$. Then we get $\varphi(xy) = x\varphi(y) + \varphi(x)y$ and (2.23). To prove the uniqueness of φ, let ψ be another function satisfying (2.23) and $\psi(xy) = x\psi(y) + \psi(x)y$ for all $x \in [1, +\infty)$. It is easy to see that $\varphi(x^n) = nx^{n-1}\varphi(x)$ and $\psi(x^n) = nx^{n-1}\psi(x)$ for all $n \in \mathbb{N}$ and $x \in [1, +\infty)$. Then

$$\begin{aligned}
nx^{n-1}\|\varphi(x) - \psi(x)\| &= \|\varphi(x^n) - \psi(x^n)\| \\
&\le \|f(x^n) - \varphi(x^n) - x^n f(1)\| + \|f(x^n) - \psi(x^n) - x^n f(1)\| \\
&\le 8x^n\varepsilon, \quad x \in [1, +\infty), n \in \mathbb{N}.
\end{aligned}$$

Hence

$$\|\varphi(x) - \psi(x)\| \le \frac{8x\varepsilon}{n}, \quad x \in [1, +\infty), n \in \mathbb{N}.$$

Letting $n \to \infty$, we conclude that $\varphi(x) = \psi(x)$ for all $x \in [1, +\infty)$. □

For convenience, we use the following abbreviation for given mappings $f, g, h : [c, +\infty) \to E$, where $c \ge 0$:

$$D_{g,h}f(x, y) := f(xy) - xg(y) - h(x)y, \quad x, y \in [c, +\infty).$$

Theorem 2.10 *Let $\{(E^n, \|\cdot\|_n) : n \in \mathbb{N}\}$ be a multi-Banach space. Suppose that ε is a non-negative real number and $f, g, h : [1, +\infty) \to E$ are mappings satisfying*

$$\sup_{n\in\mathbb{N}} \big\| \big(D_{g,h}f(x_1, y_1), \dots, D_{g,h}f(x_n, y_n)\big) \big\|_n \le \varepsilon \tag{2.26}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in [1, +\infty)$. Then there exists a function $\Delta : (0, +\infty) \to E$ satisfying functional equation (1.1) and the following inequalities:

$$\sup_{n\in\mathbb{N}} \left\| \left(\frac{\Delta(x_1)}{x_1} - \frac{f(x_1)}{x_1} + f(1), \dots, \frac{\Delta(x_n)}{x_n} - \frac{f(x_n)}{x_n} + f(1) \right) \right\|_n \le 4\varepsilon$$

for all $x_1, \dots, x_n \in [1, +\infty)$ and

$$\big\| \big(\Delta(x_1) - f(x_1) + x_1 f(1), \dots, \Delta(x_n) - f(x_n) + x_n f(1)\big) \big\|_n \le 4c\varepsilon$$

for all $x_1, \dots, x_n \in [1, c]$.

Proof It follows from (2.26) that $\|D_{g,h}f(x, y)\| \le \varepsilon$ for all $x, y \in [1, +\infty)$. Then g and h satisfy (2.17) by Theorem 2.9. Applying axiom (A_2) in (2.26), we infer

$$\sup_{n\in\mathbb{N}} \left\| \left(\frac{D_{g,h}f(x_1, y_1)}{x_1 y_1}, \dots, \frac{D_{g,h}f(x_n, y_n)}{x_n y_n} \right) \right\|_n \le \varepsilon \tag{2.27}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in [1, +\infty)$. Letting $x_1 = \cdots = x_n = 1$, $y_1 = \cdots = y_n = 1$, and $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 1$ in (2.27), respectively, we obtain

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \left\| \left(\frac{f(y_1)}{y_1} - \frac{g(y_1)}{y_1} - h(1), \dots, \frac{f(y_n)}{y_n} - \frac{g(y_n)}{y_n} - h(1) \right) \right\|_n \le \varepsilon, \\
&\sup_{n\in\mathbb{N}} \left\| \left(\frac{f(x_1)}{x_1} - \frac{h(x_1)}{x_1} - g(1), \dots, \frac{f(x_n)}{x_n} - \frac{h(x_n)}{x_n} - g(1) \right) \right\|_n \le \varepsilon, \\
&\big\| f(1) - g(1) - h(1) \big\| \le \varepsilon, \quad x_1, \dots, x_n, y_1, \dots, y_n \in [1, +\infty)
\end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in [1, +\infty)$. Let $F, G, H : [1, +\infty) \to E$ be mappings defined by

$$F(x) = \frac{f(x)}{x}, \qquad G(x) = \frac{g(x)}{x}, \qquad H(x) = \frac{h(x)}{x}.$$

Therefore we conclude from (2.27) and the above inequalities that

$$\begin{aligned}&\big\|\big(F(x_1y_1)-F(x_1)-F(y_1)+F(1),\dots,F(x_1y_1)-F(x_1)-F(y_1)+F(1)\big)\big\|_n\\&\quad\le\big\|\big(F(x_1y_1)-G(y_1)-H(x_1),\dots,F(x_ny_n)-G(y_n)-H(x_n)\big)\big\|_n\\&\qquad+\big\|\big(G(y_1)-F(y_1)+H(1),\dots,G(y_n)-F(y_n)+H(1)\big)\big\|_n\\&\qquad+\big\|\big(H(x_1)-F(x_1)+G(1),\dots,H(x_n)-F(x_n)+G(1)\big)\big\|_n\\&\qquad+\big\|\big(F(1)-G(1)-H(1),\dots,F(1)-G(1)-H(1)\big)\big\|_n\\&\quad\le 4\varepsilon,\quad n\in\mathbb{N},x_1,\dots,x_n,y_1,\dots,y_n\in[1,+\infty).\end{aligned}$$

Letting $T(t)=F(e^t)$ and replacing x_i and y_i by e^{t_i} and e^{s_i} in the above inequality, respectively, we get

$$\sup_{n\in\mathbb{N}}\big\|\big(T(s_1+t_1)-T(s_1)-T(t_1)+T(0),\dots,T(s_n+t_n)-T(s_n)-T(t_n)+T(0)\big)\big\|_n\le 4\varepsilon$$

for all $s_1,\dots,s_n,t_1,\dots,t_n\in[0,+\infty)$. By Theorem 2.2 there exists an additive function $A:\mathbb{R}\to E$ such that

$$\sup_{n\in\mathbb{N}}\big\|\big(A(t_1)-T(t_1)+T(0),\dots,A(t_n)-T(t_n)+T(0)\big)\big\|_n\le 4\varepsilon$$

for all $t_1,\dots,t_n\in[0,+\infty)$. Hence

$$\sup_{n\in\mathbb{N}}\big\|\big(A(\ln x_1)-F(x_1)+F(1),\dots,A(\ln x_n)-F(x_n)+F(1)\big)\big\|_n\le 4\varepsilon$$

for all $x_1,\dots,x_n\in[1,+\infty)$. Define $\Delta:(0,+\infty)\to E$ by $\Delta(x)=x^{-1}A(\ln x)$. Then $\Delta(xy)=x\Delta(y)+\Delta(x)y$ for all $x,y\in(0,+\infty)$ and

$$\sup_{n\in\mathbb{N}}\left\|\left(\frac{\Delta(x_1)}{x_1}-\frac{f(x_1)}{x_1}+f(1),\dots,\frac{\Delta(x_n)}{x_n}-\frac{f(x_n)}{x_n}+f(1)\right)\right\|_n\le 4\varepsilon$$

for all $x_1,\dots,x_n\in[1,+\infty)$. Using axiom (A_2), we get

$$\big\|\big(\Delta(x_1)-f(x_1)+x_1f(1),\dots,\Delta(x_n)-f(x_n)+x_nf(1)\big)\big\|_n\le 4c\varepsilon$$

for all $x_1,\dots,x_n\in[1,c]$. □

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