Research Article

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On a functional equation that has the quadratic-multiplicative property

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Abstract: In this article, we obtain the general solution and prove the Hyers-Ulam stability of the following quadratic-multiplicative functional equation:

$$\phi(st - uv) + \phi(sv + tu) = [\phi(s) + \phi(u)][\phi(t) + \phi(v)]$$

by using the direct method and the fixed point method.

Keywords: fixed point method, Hyers-Ulam stability, quadratic functional equation, direct method

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1 Introduction

The problem of the stability of functional equations has been posed by Ulam [1]. Hyers [2] gave a positive answer for the stability of the Cauchy equation:

$$f(x + y) = f(x) + f(y).$$

From that time many other functional equations and inequalities have been studied (see [3–9]).

Let E_1 and E_2 be real vector spaces. A mapping $f : E_1 \to E_2$ is called a quadratic mapping if f is a solution of the quadratic functional equation:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.1}$$

It is well known that for each quadratic mapping $f : E_1 \to E_2$, there exists a unique symmetric biadditive mapping $B : E_1 \times E_1 \to E_2$ satisfying $f(x) = B(x, x)$ for all $x \in E_1$.

The Hyers-Ulam stability of the quadratic functional equation was first proved by Skof [10] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [11] demonstrated that Skof's theorem is also valid if E_1 is replaced by an abelian group G.

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Theorem 1.1. [10] *Let G be an abelian group and E be a real Banach space. If a mapping $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \delta$$

for some $\delta > 0$ and all $x, y \in G$, then there exists a unique quadratic mapping $q : G \to E$ such that

$$\|f(x) - q(x)\| \leqslant \frac{1}{2}\delta \qquad \text{for all } x \in G.$$

Thereafter, Czerwik [12] proved the Hyers-Ulam stability of the quadratic functional equation with nonconstant bound.

Theorem 1.2. [12] *Let E_1 and E_2 be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p)$$

for some $\varepsilon > 0$ and all $x, y \in E_1$, then there exists a unique quadratic mapping $q : E_1 \to E_2$ such that

$$\|f(x) - q(x)\| \leqslant \frac{2\varepsilon}{|4 - 2^p|}\|x\|^p \qquad \text{for all } x \in E_1.$$

In 1991, Hammer and Volkmann [13] investigated mappings $f : \mathbb{R} \to \mathbb{R}$ which satisfy (1.1) and

$$f(xy) = f(x)f(y) \quad (x, y \in \mathbb{R}). \tag{1.2}$$

It will turn out that $f(x) = 0$ and $f(x) = x^2$ are the only ones, which are continuous solutions of system of two functional equations (1.1) and (1.2).

In this article, we obtain the general solution and prove the Hyers-Ulam stability of the following quadratic-multiplicative functional equation:

$$\phi(st - uv) + \phi(sv + tu) = [\phi(s) + \phi(u)][\phi(t) + \phi(v)] \tag{1.3}$$

by using the direct method and the fixed point method.

Throughout this article, $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{C}$ stands for the set of all complex numbers.

2 General solution of the functional equation (1.3)

In this section, we obtain the general solution of the functional equation (1.3). We need the following results.

Theorem 2.1. [13] *A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the system of the following functional equations:*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad f(xy) = f(x)f(y), \quad x, y \in \mathbb{R},$$

if and only if there exists an additive-multiplicative function $w : \mathbb{C} \to \mathbb{C}$ such that $f(x) = |w(x)|^2$ for all $x \in \mathbb{R}$.

Theorem 2.2. [14] *A function $f : \mathbb{R} \to \mathbb{R}$ is additive-multiplicative, i.e., f satisfies the system of the following functional equations:*

$$f(x+y) = f(x) + f(y), \quad f(xy) = f(x)f(y), \quad x, y \in \mathbb{R},$$

if and only if $f \equiv 0$ or $f(x) = x$ for all $x \in \mathbb{R}$.

Theorem 2.3. *If a function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies (1.3) for all $s, t, u, v \in \mathbb{R}$, then $\phi \equiv 0$, $2\phi \equiv 1$ or $\phi(t) = t^2$ for all $t \in \mathbb{R}$.*

Proof. Let ϕ be a constant function. Then, $\phi \equiv 0$ or $2\phi \equiv 1$. We assume that ϕ is not constant. Letting $t = u = v = 0$ in (1.3), we obtain

$$2\phi(0) = 2[\phi(s) + \phi(0)]\phi(0), \quad s \in \mathbb{R}.$$

This implies that $\phi(0) = 0$, since ϕ is not constant. Letting $u = v = 0$ in (1.3), we get

$$\phi(st) = \phi(s)\phi(t) \quad s, t \in \mathbb{R}. \tag{2.1}$$

Putting $t = 1$ in (2.1), we get $\phi(1) = 1$, since ϕ is not constant. Letting $t = v = 1$ in (1.3) and using $\phi(1) = 1$, we obtain

$$\phi(s - u) + \phi(s + u) = 2\phi(s) + 2\phi(u) \quad s, u \in \mathbb{R}.$$

By Theorem 2.1, there exists an additive-multiplicative function $w : \mathbb{C} \to \mathbb{C}$ such that $\phi(t) = |w(t)|^2$ for all $t \in \mathbb{R}$. Since $\phi(1) = 1$, letting $u = v = 1$ in (1.3), we get

$$\Re[w(s)\overline{w(t)}] = \Re[w(s)w(t)], \quad s, t \in \mathbb{R}.$$

Then, $w(s) \in \mathbb{R}$ or $w(t) \in \mathbb{R}$. This implies $w(\mathbb{R}) \subseteq \mathbb{R}$. Therefore, $w : \mathbb{R} \to \mathbb{R}$ is an additive-multiplicative function, and by Theorem 2.2 we conclude $w(t) = t$ for all $t \in \mathbb{R}$. Hence, $\phi(t) = t^2$ for all $t \in \mathbb{R}$. □

Corollary 2.4. *If a nonzero mapping $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ satisfies (1.3) for all $s, t, u, v \in \mathbb{R}$, then $\phi(t) = t^2$ for all $t \in \mathbb{R}$.*

Proposition 2.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a derivation and satisfy (1.3) for all $s, t, u, v \in \mathbb{R}$. Then, $\phi \equiv 0$.*

Proof. Since ϕ is a derivation, we get $\phi(0) = \phi(1) = 0$. Letting $u = v = 0$ in (1.3), we get $\phi(st) = \phi(s)\phi(t)$ for all $s, t \in \mathbb{R}$. Hence, $\phi(s) = \phi(s)\phi(1) = 0$ for all $s \in \mathbb{R}$. □

3 Hyers-Ulam stability of the functional equation (1.3): direct method

In this section, we prove the Hyers-Ulam stability of the functional equation (1.3) by using the direct method.

For notational convenience, we use the following abbreviation, for a given mapping $\phi : \mathbb{R} \to \mathbb{R}$,

$$D_\phi(s, t, u, v) = \phi(st - uv) + \phi(sv + tu) - [\phi(s) + \phi(u)][\phi(t) + \phi(v)].$$

Theorem 3.1. *Let $\varepsilon \in \{-1, 1\}$ and $\vartheta : \mathbb{R}^4 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\vartheta(2^{n\varepsilon}s, 2^{n\varepsilon}t, 2^{n\varepsilon}u, 2^{n\varepsilon}v)}{4^{n\varepsilon}} < \infty$$

for all $s, t, u, v \in \mathbb{R}$. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\phi(0) = 0$ and

$$|D_\phi(s, t, u, v)| \leqslant \vartheta(s, t, u, v), \quad s, t, u, v \in \mathbb{R}. \tag{3.1}$$

Then, there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$ which satisfies (1.3) and

$$|\phi(s) - Q(s)| \leqslant \begin{cases} \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty} \frac{\vartheta(2^n s, 1, 2^n s, 1)}{4^n} + \sum_{n=0}^{\infty} \frac{\vartheta(2^n s, 1, 0, 0)}{4^n}, & \varepsilon = 1, \\ \dfrac{1}{4}\displaystyle\sum_{n=1}^{\infty} 4^n \vartheta\left(\frac{s}{2^n}, 1, \frac{s}{2^n}, 1\right) + \sum_{n=1}^{\infty} 4^n \vartheta\left(\frac{s}{2^n}, 1, 0, 0\right), & \varepsilon = -1 \end{cases} \tag{3.2}$$

for all $s \in \mathbb{R}$, where $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) \neq 1$.

Proof. We assume that $\varepsilon = 1$. Letting $u = v = 0$ in (3.1), we get

$$|\phi(st) - \phi(s)\phi(t)| \leqslant \vartheta(s, t, 0, 0), \quad s, t \in \mathbb{R}. \tag{3.3}$$

Putting $t = 1$ in (3.3), we get

$$|\phi(s) - \phi(s)\phi(1)| \leqslant \vartheta(s, 1, 0, 0), \quad s \in \mathbb{R}. \tag{3.4}$$

Letting $t = v = 1$ in (3.1), we get

$$|\phi(s + u) + \phi(s - u) - 2\phi(1)[\phi(s) + \phi(u)]| \leqslant \vartheta(s, 1, u, 1), \quad s, u \in \mathbb{R}. \tag{3.5}$$

Then, it follows from (3.4) and (3.5) that

$$|\phi(s + u) + \phi(s - u) - 2[\phi(s) + \phi(u)]| \leqslant \psi(s, u), \quad s, u \in \mathbb{R},$$

where $\psi(s, u) := \vartheta(s, 1, u, 1) + 2\vartheta(s, 1, 0, 0) + 2\vartheta(u, 1, 0, 0)$. By [15], there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$ such that

$$Q(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{4^n}, \quad \|\phi(s) - Q(s)\| \leqslant \frac{1}{4}\sum_{n=0}^{\infty} \frac{\psi(2^n s, 2^n s)}{4^n}, \quad s \in \mathbb{R}.$$

Then from (3.1) and (3.3), we obtain

$$Q(st - uv) + Q(sv + tu) = [Q(s) + Q(u)][Q(t) + Q(v)], \quad s, t, u, v \in \mathbb{R}.$$

Since $Q(0) = 0$, we infer $Q(1) \in \{0, 1\}$. Hence by Theorem 2.3, we get $Q(s) = s^2$ for all $s \in \mathbb{R}$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) = 0$. Then, we obtain (3.2).

Similarly, we can obtain the result for $\varepsilon = -1$. Hence, the proof of the theorem is now complete. □

Corollary 3.2. *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying*

$$|D_\phi(s, t, u, v)| \leqslant \varepsilon, \quad s, t, u, v \in \mathbb{R} \tag{3.6}$$

for some $\varepsilon \geqslant 0$. Then, we have the following assertions:

(i) *If $\phi(0) \neq 0$, then ϕ is bounded.*

(ii) *If $\phi(0) = 0$, then there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$, which satisfies (1.3) and*

$$|\phi(s) - Q(s)| \leqslant \frac{5}{3}\varepsilon, \quad s \in \mathbb{R},$$

where $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) \neq 1$.

Proof. Let $\phi(0) \neq 0$. Letting $t = u = v = 0$ in (3.6), we obtain

$$|2\phi(0) - 2[\phi(s) + \phi(0)]\phi(0)| \leqslant \varepsilon, \quad s \in \mathbb{R}.$$

Then, we get

$$|\phi(s)| \leqslant |\phi(0) - 1| + \frac{\varepsilon}{2|\phi(0)|}, \quad s \in \mathbb{R}.$$

For the case $\phi(0) = 0$, the result follows from Theorem 3.1. □

Corollary 3.3. *Let $\varepsilon \geqslant 0$ and $\vartheta : \mathbb{R}^4 \to [0, \infty)$ be a function. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying*

$$|D_\phi(s, t, u, v)| \leqslant \varepsilon\vartheta(s, t, u, v), \quad s, t, u, v \in \mathbb{R}.$$

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis in $\mathbb{R}^4$ and $\vartheta_i : \mathbb{R} \to \mathbb{R}$ be a function defined by $\vartheta_i(s) = \vartheta(se_i)$ for a given $i \in \{1, 2, 3, 4\}$. If $\phi(0) \neq 0$ and ϑ_i is bounded for some i, then ϕ is bounded.

Corollary 3.4. *Fix $\varepsilon \geqslant 0$ and $p \in (0, 2)$. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying*

$$|D_\phi(s, t, u, v)| \leqslant \varepsilon(|s|^p + |t|^p + |u|^p + |v|^p), \quad s, t, u, v \in \mathbb{R}.$$

Then, we have the following assertions:
(*i*) *If* $\phi(0) \neq 0$, *then*

$$\left| \phi(s) - \frac{1}{2} \right| \leqslant \varepsilon |s|^p, \quad s \in \mathbb{R}.$$

(*ii*) *If* $\phi(0) = 0$, *then there exists a unique quadratic function* $Q : \mathbb{R} \to \mathbb{R}$ *which satisfies* (*1.3*) *and*

$$|\phi(s) - Q(s)| \leqslant \frac{6\varepsilon}{4 - 2^p} |s|^p + 2\varepsilon, \quad s \in \mathbb{R},$$

where $Q(s) = s^2$ *if* $Q(1) = 1$ *and* $Q \equiv 0$ *if* $Q(1) \neq 1$.

Theorem 3.5. *Fix* $\varepsilon \geqslant 0$ *and* $p > 4$. *Suppose that* $\phi : \mathbb{R} \to \mathbb{R}$ *is a function satisfying*

$$|D_\phi(s, t, u, v)| \leqslant \varepsilon(|s|^p + |t|^p + |u|^p + |v|^p), \quad s, t, u, v \in \mathbb{R}. \tag{3.7}$$

Then, we have the following assertions:
(*i*) *If* $\phi(0) \neq 0$, *then*

$$\left| \phi(s) - \frac{1}{2} \right| \leqslant \varepsilon |s|^p, \quad s \in \mathbb{R}.$$

(*ii*) *If* $\phi(0) = 0$, *then there exists a unique quadratic function* $Q : \mathbb{R} \to \mathbb{R}$, *which satisfies* (1.3) *and*

$$|\phi(s) - Q(s)| \leqslant \frac{6\varepsilon}{\sqrt{2}^p - 4} (1 + |s|^p), \quad s \in \mathbb{R}, \tag{3.8}$$

where $Q(s) = s^2$ *if* $Q(1) = 1$ *and* $Q \equiv 0$ *if* $Q(1) \neq 1$.

Proof. It suffices to prove (ii). Letting $u = v = 0$ in (3.7), we get

$$|\phi(st) - \phi(s)\phi(t)| \leqslant \varepsilon(|s|^p + |t|^p), \quad s, t \in \mathbb{R}. \tag{3.9}$$

Putting $v = t$ and $u = s$ in (3.7), we obtain

$$|\phi(2st) - 4\phi(s)\phi(t)| \leqslant 2\varepsilon(|s|^p + |t|^p), \quad s, u, v \in \mathbb{R}. \tag{3.10}$$

Using (3.9) and (3.10), we obtain

$$|\phi(2st) - 4\phi(st)| \leqslant 6\varepsilon(|s|^p + |t|^p), \quad s, u, v \in \mathbb{R}.$$

Replacing s and t by $\frac{2}{2^{(n+1)/2}}$ and $\frac{1}{2^{(n+1)/2}}$, respectively, in the last inequality and multiplying the resulting inequality by 4^n, we obtain

$$\left| 4^n \phi\left(\frac{s}{2^n}\right) - 4^{n+1} \phi\left(\frac{s}{2^{n+1}}\right) \right| \leqslant \frac{3}{2} \left(\frac{4}{2^{p/2}}\right)^{n+1} \varepsilon(1 + |s|^p), \quad s \in \mathbb{R}. \tag{3.11}$$

Then,

$$\left| 4^n \phi\left(\frac{s}{2^n}\right) - 4^m \phi\left(\frac{s}{2^m}\right) \right| = \left| \sum_{i=n}^{m-1} \left[4^i \phi\left(\frac{s}{2^i}\right) - 4^{i+1} \phi\left(\frac{s}{2^{i+1}}\right) \right] \right| \leqslant \frac{3}{2} \varepsilon(1 + |s|^p) \sum_{i=n+1}^{m} \left(\frac{4}{2^{p/2}}\right)^i, \quad s \in \mathbb{R}, \ m > n \geqslant 0. \tag{3.12}$$

This implies that the sequence $\{4^n \phi(\frac{s}{2^n})\}$ is a Cauchy sequence and so the sequence is convergent for all $s \in \mathbb{R}$. We define a function $Q : \mathbb{R} \to \mathbb{R}$ by

$$Q(s) = \lim_{n \to \infty} 4^n \phi\left(\frac{s}{2^n}\right).$$

Since $\phi(0) = 0$, we get $Q(0) = 0$. By Assumption (3.7) and Inequality (3.9), we obtain that Q satisfies (1.3) and $Q(st) = Q(s)Q(t)$ for all $s, t \in \mathbb{R}$. Hence by Theorem 2.3, we get $Q(s) = s^2$ $(s \in \mathbb{R})$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) = 0$. Letting $n = 0$ and $m \to \infty$ in (3.12), we obtain (3.8). □

4 Hyers-Ulam stability of the functional equation (1.3): fixed point method

In this section, we will adopt the idea of [16] to prove the Hyers-Ulam stability of the functional equation (1.3) by using the fixed point method.

Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is called a generalized metric on $\mathcal{X}$ if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

It should be noted that the only difference between the generalized metric and the metric is that the generalized metric accepts the infinity.

We will use the following fundamental result in the fixed point theory.

Theorem 4.1. [17] *Let $(\mathcal{X}, d)$ be a generalized complete metric space and $\Lambda : \mathcal{X} \to \mathcal{X}$ be a strictly contractive function with the Lipschitz constant $L < 1$. Suppose that for a given element $a \in \mathcal{X}$ there exists a nonnegative integer k such that $d(\Lambda^{k+1}a, \Lambda^k a) < \infty$. Then,*

(i) *the sequence $\{\Lambda^n a\}_{n=1}^{\infty}$ converges to a fixed point $b \in \mathcal{X}$ of Λ;*
(ii) *b is the unique fixed point of Λ in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\Lambda^k a, y) < \infty\}$;*
(iii) *$d(y, b) \leqslant \frac{1}{1-L} d(y, \Lambda y)$ for all $y \in \mathcal{Y}$.*

Theorem 4.2. *Let $0 < L < 1$ and $\vartheta : \mathbb{R}^4 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\vartheta(2^n s, 2^n t, 2^n u, 2^n v)}{4^n} = 0, \quad \psi(2s, 2t) \leqslant 4L\psi(s, t), \quad s, t, u, v \in \mathbb{R}, \tag{4.1}$$

where $\psi(s, t) := \vartheta(s, 1, t, 1) + 2\vartheta(s, 1, 0, 0) + 2\vartheta(t, 1, 0, 0)$. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\phi(0) = 0$ and

$$|D_\phi(s, t, u, v)| \leqslant \vartheta(s, t, u, v), \quad s, t, u, v \in \mathbb{R}. \tag{4.2}$$

Then, there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$, which satisfies (1.3) and

$$|\phi(s) - Q(s)| \leqslant \frac{1}{4(1-L)} \psi(s, s), \quad s \in \mathbb{R}, \tag{4.3}$$

where $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) \neq 1$.

Proof. It is clear that $\lim_{n\to\infty} \frac{\psi(2^n s, 2^n t)}{4^n} = 0$ for all $s, t \in \mathbb{R}$. Using a similar argument of the proof of Theorem 3.1, we obtain

$$|\phi(s + u) + \phi(s - u) - 2[\phi(s) + \phi(u)]| \leqslant \psi(s, u), \quad s, u \in \mathbb{R},$$

where $\psi(s, u) := \vartheta(s, 1, u, 1) + 2\vartheta(s, 1, 0, 0) + 2\vartheta(u, 1, 0, 0)$. Hence by [18, Theorem 4], there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$ such that

$$Q(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{4^n}, \quad |\phi(s) - Q(s)| \leqslant \frac{1}{4(1-L)} \psi(s, s), \quad s \in \mathbb{R}. \tag{4.4}$$

From the definition of Q and using (4.1) and (4.2), we obtain that Q satisfies (1.3). Since $Q(0) = 0$, it follows from Theorem 2.3 that $Q(s) = s^2$ ($s \in \mathbb{R}$) if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) = 0$. Hence, we get (4.3), which ends the proof. □

Obviously, Corollary 3.2 can be obtained by using Theorem 4.2. By a similar argument as in the proof of Theorem 4.2, we can obtain the following result.

Theorem 4.3. *Let $0 < L < 1$ and $\vartheta : \mathbb{R}^4 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 4^n \vartheta\left(\frac{s}{2^n}, \frac{t}{2^n}, \frac{u}{2^n}, \frac{v}{2^n}\right) = 0, \quad \psi(s, t) \leqslant \frac{L}{4}\psi(2s, 2t), \quad s, t, u, v \in \mathbb{R},$$

where $\psi(s, t) := \vartheta(s, 1, t, 1) + 2\vartheta(s, 1, 0, 0) + 2\vartheta(t, 1, 0, 0)$. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\phi(0) = 0$ and

$$|D_\phi(s, t, u, v)| \leqslant \vartheta(s, t, u, v), \quad s, t, u, v \in \mathbb{R}.$$

Then, there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$, which satisfies (1.3) and

$$|\phi(s) - Q(s)| \leqslant \frac{L}{4(1-L)}\psi(s, s), \quad s \in \mathbb{R},$$

where $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) \neq 1$.

Theorem 4.4. *Let $d > 0$ and $\varepsilon \geqslant 0$. Assume that a function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ satisfies the inequality*

$$|D_\phi(s, t, u, v)| \leqslant \varepsilon \tag{4.5}$$

for all $s, t, u, v \in \mathbb{R}$ with $\max\{|s|, |t|, |u|, |v|\} \geqslant d$. Then, there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$, which satisfies (1.3) and

$$|\phi(s) - Q(s)| \leqslant \varepsilon, \quad s \in \mathbb{R},$$

where $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) \neq 1$.

Proof. Letting $u = v = 0$ in (4.5), we get

$$|\phi(st) - \phi(s)\phi(t)| \leqslant \varepsilon, \quad s, t \in \mathbb{R}, \ |t| \geqslant d. \tag{4.6}$$

Setting $v = t$ in (4.5), we obtain

$$|\phi(st - tu) + \phi(st + tu) - 2\phi(t)[\phi(s) + \phi(u)]| \leqslant \varepsilon, \quad s, t, u \in \mathbb{R}, \ |t| \geqslant d. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$|\phi(st - tu) + \phi(st + tu) - 2\phi(st) - 2\phi(tu)| \leqslant 3\varepsilon, \quad s, t, u \in \mathbb{R}, \ |t| \geqslant d. \tag{4.8}$$

Letting $t = d$ and replacing s and u by $\frac{s}{d}$ and $\frac{u}{d}$ in (4.8), respectively, we get

$$|\phi(s - u) + \phi(s + u) - 2\phi(s) - 2\phi(u)| \leqslant 3\varepsilon, \quad s, u \in \mathbb{R}.$$

Hence by [12, Theorem 1], there is a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$ such that

$$Q(s) = \lim_{n\to\infty} \frac{\phi(2^n s)}{4^n}, \quad |\phi(s) - Q(s)| \leqslant \varepsilon, \quad s \in \mathbb{R}. \tag{4.9}$$

Applying (4.5) and (4.9), we obtain

$$|D_Q(s, t, u, v)| = \lim_{n\to\infty} \frac{1}{16^n}|D_\phi(2^n s, 2^n t, 2^n u, 2^n v)| \leqslant \lim_{n\to\infty} \frac{1}{16^n}\varepsilon = 0, \quad s, t, u, v \in \mathbb{R}.$$

Since $Q(0) = 0$, by Theorem 2.3, we get $Q(s) = s^2$ if $Q(1) = 1$ and $Q \equiv 0$ if $Q(1) = 0$. □

Jung [19] proved an asymptotic property of the quadratic mapping. It is a natural thing to expect such a property also for the functional equation (1.3).

Corollary 4.5. *Assume that a nonzero function* $\phi : \mathbb{R} \to \mathbb{R}$ *with* $\phi(0) = 0$ *satisfies*

$$\limsup_{\max\{|s|,|t|,|u|,|v|\}\to+\infty} |D_\phi(s,t,u,v)| = 0. \tag{4.10}$$

Then, $\phi(s) = s^2$ *for all* $s \in \mathbb{R}$.

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers which decreases monotonically to zero. Then, by (4.10) there exists an increasing sequence $\{d_n\}$ of positive real numbers such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$|D_\phi(s,t,u,v)| \leqslant \varepsilon_n, \quad \max\{|s|,|t|,|u|,|v|\} \geqslant d_n.$$

According to Theorem 4.4, there exists a unique quadratic function $Q_n : \mathbb{R} \to \mathbb{R}$ satisfying (1.3) and $|\phi(s) - Q_n(s)| \leqslant \varepsilon_n$ for all $s \in \mathbb{R}$. Then, the quadratic function $Q_n : \mathbb{R} \to \mathbb{R}$ satisfies $|\phi(s) - Q_n(s)| \leqslant \varepsilon_1$ for all $s \in \mathbb{R}$ and all $n \geqslant 1$. Hence, the uniqueness implies $Q_n = Q_1$ for all n. Therefore, $|\phi(s) - Q_1(s)| \leqslant \varepsilon_n$ for all $s \in \mathbb{R}$ and all $n \geqslant 1$. Letting $n \to +\infty$, we get $\phi(s) = Q_1(s)$ for all $s \in \mathbb{R}$. So ϕ satisfies (1.3) and $\phi(0) = 0$. According to Corollary 2.4, we get $\phi(s) = s^2$ for all $s \in \mathbb{R}$. □

Applying the same argument given in the proof of Theorem 4.4, we obtain the following result.

Theorem 4.6. *Let* $d > 0$ *and* $\varepsilon \geqslant 0$. *Assume that a function* $\phi : \mathbb{R} \to \mathbb{R}$ *with* $\phi(0) = 0$ *satisfies one of the following inequalities:*

1. $|D_\phi(s,t,u,v)| \leqslant \varepsilon$, $s,t,u,v \in \mathbb{R}$ *with* $|s| \geqslant d$;
2. $|D_\phi(s,t,u,v)| \leqslant \varepsilon$, $s,t,u,v \in \mathbb{R}$ *with* $|t| \geqslant d$;
3. $|D_\phi(s,t,u,v)| \leqslant \varepsilon$, $s,t,u,v \in \mathbb{R}$ *with* $|u| \geqslant d$;
4. $|D_\phi(s,t,u,v)| \leqslant \varepsilon$, $s,t,u,v \in \mathbb{R}$ *with* $|v| \geqslant d$;
5. $|D_\phi(s,t,u,v)| \leqslant \varepsilon$, $s,t,u,v \in \mathbb{R}$ *with* $|s| + |t| + |u| + |v| \geqslant d$.

Then, there exists a unique quadratic function $Q : \mathbb{R} \to \mathbb{R}$, *which satisfies* (1.3) *and*

$$|\phi(s) - Q(s)| \leqslant \varepsilon, \quad s \in \mathbb{R},$$

where $Q(s) = s^2$ *if* $Q(1) = 1$ *and* $Q \equiv 0$ *if* $Q(1) \neq 1$.

Corollary 4.7. *Assume that a nonzero function* $\phi : \mathbb{R} \to \mathbb{R}$ *with* $\phi(0) = 0$ *satisfies one of the following conditions:*

1. $\limsup_{|s|\to+\infty} |D_\phi(s,t,u,v)| = 0, \quad s,t,u,v \in \mathbb{R}$;
2. $\limsup_{|t|\to+\infty} |D_\phi(s,t,u,v)| = 0, \quad s,t,u,v \in \mathbb{R}$;
3. $\limsup_{|u|\to+\infty} |D_\phi(s,t,u,v)| = 0, \quad s,t,u,v \in \mathbb{R}$;
4. $\limsup_{|v|\to+\infty} |D_\phi(s,t,u,v)| = 0, \quad s,t,u,v \in \mathbb{R}$;
5. $\limsup_{|s|+|t|+|u|+|v|\to+\infty} |D_\phi(s,t,u,v)| = 0, \quad s,t,u,v \in \mathbb{R}$.

Then, $\phi(s) = s^2$ *for all* $s \in \mathbb{R}$.

5 Conclusions

Using the direct method and the fixed point method, we have obtained the general solution and have proved the Hyers-Ulam stability of the quadratic-multiplicative functional equation:

$$\phi(st - uv) + \phi(sv + tu) = [\phi(s) + \phi(u)][\phi(t) + \phi(v)].$$

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