



Bridging the chiral symmetry and confinement with singularity

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ARTICLE INFO

Article history:

Received 9 January 2020
 Received in revised form 13 April 2020
 Accepted 13 April 2020
 Available online 20 April 2020
 Editor: B. Grinstein

Keywords:

Chiral symmetry
 Confinement
 Heun's equation
 Bag-Model
 Holography

ABSTRACT

We consider a holographic quark model where the confinement is a consequence of the quark condensate. Surprisingly, the equation of motion of our holographic model can be mapped to the old spin-less bag model. Both models correctly reproduce the linear Regge trajectory of hadrons for zero quark mass. For the case of non-zero quark mass, the model lead us to Heun's equation. The mass term is precisely the origin of the higher singularity, which changes the system behavior drastically. Our result can shed some light on why the chiral transition is so close to the confinement transition. In the massive case, the Schroedinger equation is exactly solvable, but only if a surprising new quantization condition, additional to the energy quantization, is applied.

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1. Introduction

A frequent question for the phase diagram of the quantum chromo dynamics (QCD) [1,2] is why the chiral and the deconfinement transitions are so close, while two are separate concepts. The former is defined when the current quark mass is zero, while the latter is due to the infrared dynamics QCD which is often summarized by the QCD string [3,4] and its spectrum called Regge trajectory,

$$\alpha' m^2 = n + \beta. \quad (1)$$

The (approximate) coincidence of the phase boundaries will be explained if one can show that one is a consequence of the other.

Some time ago, Gürsey [7] showed that the spectrum of semi-classical bag model introduced by Lichtenberg et al. [6] for the meson follows the Regge trajectory if the quark mass vanishes. In a recent paper [5], the model for non-vanishing quark mass was studied numerically and the spectrum turns out to be highly non-linear. Since the linear confinement appears only for the vanishing quark mass, which allows to define the chiral symmetry, one may wonder if the chiral symmetry is a consequence of the confinement dynamics.

In this paper, we consider a holographic model which gives the linear confinement in the presence of the quark condensate,

so that the confinement is consequence of the quark condensate in this model. This is a holographic fermion model coupled with a neutral scalar. We show that the equation of motion of this model can be mapped to the Lichtenberg model. Considering that two models are based on completely different idea, presence of such mapping is quite remarkable. In both models the presence of current quark mass is inconsistent with Regge trajectory, and the quark mass triggers the change of the singularity type from the hypergeometric type to Heun's type. The linear trajectory appears at a limit where higher singularity disappears.

We will develop polynomials whose roots gives quantized value of hadron spectrum and we explicitly calculated the hadron spectrum in the presence of the quark mass to understand the spectrum analytically. It turns out that the drastic change of spectrum in the presence of quark mass is a consequence of change of the singularity type, which requests an extra quantization: apart from the energy, one more parameter in the potential should be quantized, which is rather a surprising phenomena. One should notice that this is relevant to general situations: whenever Schrödinger equation has the potential with both even and odd powers of radial coordinate, there are extra quantizations apart from that of energy.

Finally, we also emphasize that the massless limit of the spectrum is singular. Such inconsistencies of the spectrum of hadron in the presence of the quark mass suggests that the chiral symmetry should be tied with the color confinement, although the dynamical mechanism of suppressing the quark mass is still an open question.

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2. Holographic fermion as a constituent quark

To consider the hadron mass problem in terms of effective theory, it is convenient to consider a model of constituent quark, where all the correlation by the gluons are encoded into the constituent quark mass. Namely, we consider a fermion χ in a bag which is dual to the fermion ψ living outside the central region of the AdS. The dynamics of ψ in the warped space determines the mass of the excitation, which we interpret as the constituent quark. Such mass of the constituent quark can be used to describe meson mass as well as baryon mass, by assuming that there is no interaction between constituent quarks.

For this, we consider following fermion action in AdS space with coupling to the scalar describing the bare quark mass and chiral condensation M .

$$S_\psi = \int d^{d+1}x \sqrt{-g} i \bar{\psi} (\Gamma^\mu \mathcal{D}_\mu - m - \Phi) \psi + S_\Phi, \quad (2)$$

where $\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \omega_{ab\mu} \Gamma^{ab}$. One may simply consider this as a model for a Baryon instead of a constituent quark. We consider only $d=3$ for the analytical simplicity. The dynamics of the boson Φ is given by

$$S_\Phi = \int d^{d+1}x \sqrt{-g} \left(-|\partial_\mu \Phi|^2 - m_\Phi^2 |\Phi|^2 \right). \quad (3)$$

We treat all the fields in the probe limit where the metric is fixed as that of AdS₄:

$$ds^2 = (dz^2 + \eta^{\mu\nu} dx^\mu dx^\nu) / z^2, \quad \text{with } \eta^{00} = -1. \quad (4)$$

Bulk mass of the boson, m_Φ^2 , is given in terms of the conformal dimension of the dual operator: $m_\Phi^2 = \Delta(\Delta - d)$. We will fix it such that $\Delta = 2$, so that $m_\Phi^2 = -2$ in $d = 2 + 1$ and $m_\Phi^2 = -4$ for $d = 3 + 1$. Although $\Delta = 2$ for the operator $\bar{q}q$ is realized in 4 dimension at the lower boundary of conformal window of N_f/N_c [8], here we consider 2+1 case only. The field equation then gives

$$\Phi = M_0 z + M z^2, \quad \text{in AdS}_4, \quad (5)$$

which is an exact solution of the scalar field equation *in the probe limit*.

The equation of motion of (2) is given by

$$(\Gamma^\mu \mathcal{D}_\mu - m - q\Phi) \psi = 0, \quad (6)$$

which can be written as a Schrödinger equation

$$-\Psi_n''(z) + V(z) \Psi_n(z) = E_n \Psi_n(z), \quad (7)$$

$$\text{with } V(z) = \frac{m(m-1) + \Phi^2}{z^2}, \quad (8)$$

$$= \frac{m(m-1)}{z^2} + q^2 (Mz + M_0)^2, \quad (9)$$

$$E_n := m_n^2 - 2qM(m + \frac{1}{2}). \quad (10)$$

We interpret m_n^2 as the constituent quark mass inside a Hadron and it was shown that for $M_0 = 0$, spectrum is linear [9–11]

$$m_n^2 = 4qM(n + m + 1/2). \quad (11)$$

Notice that when $M = 0$ we have $m_n^2 = \omega^2 - k^2 = 0$ so that the only spectrum is massless one with $\omega = k$, which can not be the spectrum of a confined object: the energy of the confined massless quark contribute to the mass of the hadron containing it. That is what we mean by constituent quark mass. Therefore we can say that M is the order parameter of the confinement transition as

well as that of the chiral transition. Notice that in 2+1 dimension there is no chiral symmetry. By chiral symmetry breaking (CSB) in this paper, what we meant is ‘non-zero quark condensate’ M , which is equivalent to CSB in 3+1 dimensional case. The chiral symmetry itself is not relevant to our discussion.

We also emphasize that since $M \sim \langle \bar{q}q \rangle$ is the slope of the Regge trajectory, the linear confinement is consequence of the non-zero quark condensate. Therefore two transitions must be identical in this model. Although it is not clear whether this is a model specific property or a generic property of QCD, above argument explain the coincidence of two transition at least partially in the context of this specific model.

When the quark mass $M_0 \neq 0$, we will show shortly that it will lead to a type of Heun’s equation.

3. Heun’s equation

If we formally replace $z \rightarrow r$,

$$m \rightarrow -L, \quad qM_0 \rightarrow m_q, \quad qM \rightarrow \frac{b}{2}, \quad \text{and } \Psi_n \rightarrow u, \quad (12)$$

then eq. (7) defined in AdS₄ space becomes

$$\left[-\frac{d^2}{dr^2} + V(r) \right] u(r) = E u(r), \quad (13)$$

$$V(r) = \left(m_q + \frac{1}{2} b_\tau r \right)^2 + \frac{L(L+1)}{r^2}, \quad (14)$$

which is a Heun’s equation [12–14] with 4-singularities. One interesting observation is that above equation is precisely the radial equation coming from the bag model [6,7,5] for a meson, whose mass squared is given by $4E$. We emphasize that the physical ideas and the spaces in which they are defined are completely different: one in AdS₄ and the other in a flat space \mathbf{R}^3 .

To reveal the mathematical structure more clearly, we consider slightly generalized one defined by the potential

$$V(r) = c^2 r^2 + br - \frac{a}{r} + \frac{L(L+1)}{r^2}, \quad (15)$$

which is obtained from the potential eq. (14) by shifting $V \rightarrow V - a/r - m_q^2$ and redefining $c = b_\tau/2$ and $b = m_q b_\tau$. Factoring out the behavior near $r = 0$ by $u(r) = r^{L+1} f(r)$, above equation becomes

$$\frac{d^2 f(r)}{dr^2} + \frac{2(L+1)}{r} \frac{df(r)}{dr} + \left(E - c^2 r^2 - br + \frac{a}{r} \right) f(r) = 0. \quad (16)$$

Factoring out near ∞ behavior by $f(r) = \exp\left(-\frac{\varepsilon}{2} r^2 - \frac{b}{2c} r\right) y(r)$ and introducing $\rho = \sqrt{c}r$, $a_0 = a/c^{1/2}$, $b_0 = b/c^{3/2}$, $\mathcal{E} = E/c$, we get the bi-confluent Heun’s equation:

$$\rho \frac{d^2 y}{d\rho^2} + \left(\mu \rho^2 + \varepsilon \rho + \nu \right) \frac{dy}{d\rho} + (\Omega \rho + \varepsilon \omega) y = 0, \quad (17)$$

with $\mu = -2$, $\varepsilon = -b_0$, $\nu = 2L + 2$ and

$$\Omega = \mathcal{E} + b_0^2/4 - (2L + 3), \quad \text{and } \omega = L + 1 - a_0/b_0. \quad (18)$$

It has a regular singularity at the origin and an irregular singularity of rank two [12–14] at the infinity.

Substituting $y(\rho) = \sum_{n=0}^{\infty} d_n \rho^n$ into (17), we obtain the recurrence relation:

$$d_{n+1} = A_n d_n + B_n d_{n-1} \quad \text{for } n \geq 1, \quad \text{with} \quad (19)$$

$$A_n = -\frac{\varepsilon(n+\omega)}{(n+1)(n+\nu)}, \quad B_n = -\frac{\Omega + \mu(n-1)}{(n+1)(n+\nu)}. \quad (20)$$

The first two d_n ’s are given by $d_1 = A_0 d_0$ and $d_{-1} = 0$.

It is essential to notice that when $m_q = 0$, we have

$$A_n = \varepsilon = b_0 = 0, \tag{21}$$

so that the three term recurrence relation given in eq. (19) is reduced to two term recurrence relation between d_{n+1} and d_{n-1} and the Heun's equation is reduced to hypergeometric one. That is, the quark mass is precisely the term increasing the singularity order.

Now, unless $y(\rho)$ is a polynomial, $u(r)$ is divergent as $\rho \rightarrow \infty$. Therefore we need to impose regularity conditions by which the solution is normalizable. If we impose two conditions [12–14],

$$B_{N+1} = d_{N+1} = 0 \quad \text{where } N \in \mathbb{N}_0, \tag{22}$$

the series expansion becomes a polynomial of degree N : as one can see from eq. (19), eq. (22) is sufficient to give $d_{N+2} = d_{N+3} = \dots = 0$ recursively. Then the solution is a polynomial of order N , $y_N(\rho) = \sum_{i=0}^N d_i \rho^i$. The question whether imposing both equations in eq (22) is really necessary was studied numerically in our earlier work [5]. In general, $d_{N+1} = 0$ will define a $N + 1$ -th order polynomial \mathcal{P}_{N+1} in a_0, b_0 , so that Eq. (22) gives

$$\mathcal{E}_{N,L} = 2N + 2L + 3 - b_0^2/4, \quad \mathcal{P}_{N+1}(a_0, b_0) = 0, \tag{23}$$

where the first comes from $B_{N+1} = 0$, and it is nothing but the usual energy quantization condition. Below we will examine the meaning of the second equation. To do that we need explicit expressions of a few lower order polynomial \mathcal{P}_{N+1} :

$$\begin{aligned} \mathcal{P}_1(a_0, b_0) &= b_0(L + 1) - a_0, \\ \mathcal{P}_2(a_0, b_0) &= (b_0(L + 1) - a_0)(b_0(L + 2) - a_0) - 4(L + 1), \\ \mathcal{P}_3(a_0, b_0) &= (L + 1)(L + 2)(L + 3)b_0^3 - (3L(L + 4) + 11)a_0b_0^2 \\ &\quad + (3(L + 2)a_0^2 - 4(L + 1)(4L + 9))b_0 - a_0^3 + 4(4L + 5)a_0, \\ \mathcal{P}_4(a_0, b_0) &= (L + 1)(L + 2)(L + 3)(L + 4)b_0^4 \\ &\quad - 2(2L + 5)(L(L + 5) + 5)a_0b_0^3 \\ &\quad + ((6L(L + 5) + 35)a_0^2 - 4(L + 1)(5L(2L + 11)) + 72)b_0^2 \\ &\quad - (2(2L + 5)a_0^3 + 4(20L(L + 4) + 69)a_0)b_0 - 20(2L + 3)a_0^2 \\ &\quad + 144(L + 1)(L + 2) + a_0^4, \end{aligned} \tag{24}$$

4. Extra quantization

We have seen that a_0, b_0 should be related by $\mathcal{P}_{N+1}(a_0, b_0) = 0$. This means that if we fix one of them, the other should be a solution of a polynomial equation. Let's examine a few low orders in N . We normalize the solution using $d_0 = 1$ for simplicity.

1. For $N = 0$: $\mathcal{P}_1(a_0, b_0) = b_0(L + 1) - a_0 = 0$. The eigenfunction is $y_0(\rho) = 1$.
2. For $N = 1$, $\mathcal{P}_2(a_0, b_0) = 0$ defines a hyperbola in a_0, b_0 such that there are always two branches because the discriminant $D = b_0^2 + 16(L + 1) > 0$. That is, for a given b_0, a_0 always has real solutions. $2a_0 = b_0(2L + 3) \pm \sqrt{b_0^2 + 16(L + 1)}$. In this case, $y_1(\rho) = 1 + d_1\rho$ with $d_1 = (-b_0 \pm \sqrt{b_0^2 + 16(L + 1)}) / (4L + 4)$.
3. For $2 \leq N$, $\mathcal{P}_N = 0$ has N branches. $\mathcal{P}_4 = 0$ is plotted in Fig. 1. Apart from the central region where $a_0, b_0 \sim \mathcal{O}(1)$ which is shown in Fig. 1(a) the curves are approximately linear. Such linearity can be confirmed by drawing the same figure in large scale as in Fig. 1(b), where we used $L = 0$. Note that the slopes of the lines: $a_0/b_0 = 1, 2, 3, 4$.

Table 1
Roots of a_0 for $b_0 = 1, N = 4$.

	$a_{00}^{(N=4)}$	$a_{01}^{(N=4)}$	$a_{02}^{(N=4)}$	$a_{03}^{(N=4)}$	$a_{04}^{(N=4)}$
$L = 0$	-7.50342	-2.26852	2.5487	7.93985	14.2834
$L = 1$	-9.22584	-2.68053	3.72372	10.4374	17.7452
$L = 2$	-10.4722	-2.80774	4.79946	12.6207	20.8598
$L = 3$	-11.4284	-2.78208	5.84226	14.6311	23.7371
$L = 4$	-12.1842	-2.65493	6.8699	16.5287	26.4406

Table 2
Roots of a_0 for $b_0 = 1, N = 5$.

	$a_{00}^{(N=5)}$	$a_{01}^{(N=5)}$	$a_{02}^{(N=5)}$	$a_{03}^{(N=5)}$	$a_{04}^{(N=5)}$	$a_{05}^{(N=5)}$
$L = 0$	-10.5701	-4.75187	0.363597	5.60184	11.6841	18.6724
$L = 1$	-12.7643	-5.82539	0.801156	7.5262	14.7189	22.5434
$L = 2$	-14.4605	-6.49042	1.30825	9.19107	17.3924	26.0593
$L = 3$	-15.834	-6.93228	1.86483	10.7358	19.8467	29.319
$L = 4$	-16.9777	-7.22567	2.45866	12.2089	22.1509	32.3849
$L = 5$	-17.9475	-7.4102	3.08179	13.6334	24.3443	35.2982

For general N , we can show that for large enough a_0, b_0 , $\mathcal{P}_{L+1}(a_0, b_0) = 0$ gives following relation.

$$\frac{a_0}{b_0} = \frac{ac}{b} \simeq L + 1 + K, \text{ for } K = 0, 1, \dots, N. \tag{25}$$

This means that for a given b_0 , there are $N + 1$ a_0 's for any L . This is also true for $|a_0|, |b_0| \leq \mathcal{O}(1)$ although we can not write down the explicitly. Similarly if we set $a_0 = 0$, the allowed values of b_0 are given by the crossing points of $N + 1$ branches of the $\mathcal{P}_{N+1} = 0$ with the vertical line $a_0 = 0$. We call such fixing b -quantization. See Fig. 1a.

Such extra quantization is a consequence of the Heun's equation. As we have seen before explicitly, for the hypergeometric equations, the three term recurrence relation is reduced to two term one after factoring out the asymptotic form so that we need to fine tune only one parameter, the energy, to have a polynomial solution. For the Heun's equation, its higher singularity requests higher regularity: the three term recurrence relation is not reduced to the two term one, which in turn request an extra quantization of system parameter apart from the energy eigenvalue.

Notice that due to the N, L dependence of b_0 , the spectrum \mathcal{E} is NOT linear in N anymore. Notice also that for a -quantization, \mathcal{E} is linear in N, L and does not depend on a quantized value of a_0 as far as a_0 is actually given by one of those quantized value that depends on N, L and b_0 . Table 1 tells us all possible roots of a_0 's for each L when $N = 4$ and $b_0 = 1$. Similarly, Table 2 shows us all possible roots of a_0 's for each L when $N = 5$ and $b_0 = 1$. As you can see easily from the table, most of the quantized values are in the linear regime where $a_0 \approx (L + 1 + K)b_0$.

From the explicit calculation, we found the following pattern: List $N+1$ a_0 in the increasing order such that a_{0K} is K -th one, $K = 0, 1, \dots, N$. Then the polynomial solution for the a_{0K} has K nodes. The number of nodes does not depend on L . Figs. 2(a) shows that polynomial y_4 with $a_{0K}, K = 0, 1, 2, 3, 4$ has K nodes in $N = 4$ in the region $\rho > 0$. We fixed $L = 0$ and $b_0 = 1$. Figs. 2(b) shows that polynomials y_5 with a_{03} has 3 nodes in $N = 5$ independent of the value of $L = 0, 1, 2, 3, 4, 5$. There are two nodes in the unphysical region $\rho < 0$.

5. Spectrum for nonzero quark mass

We have seen that two very different models lead to the same Heun's equation. The spectrum of the Lichtenberg bag model for $m_q = 0$ was obtained in [7] and it is linear:

$$E_{N,L}^2 = 4b_\tau(N + L + 3/2). \tag{26}$$

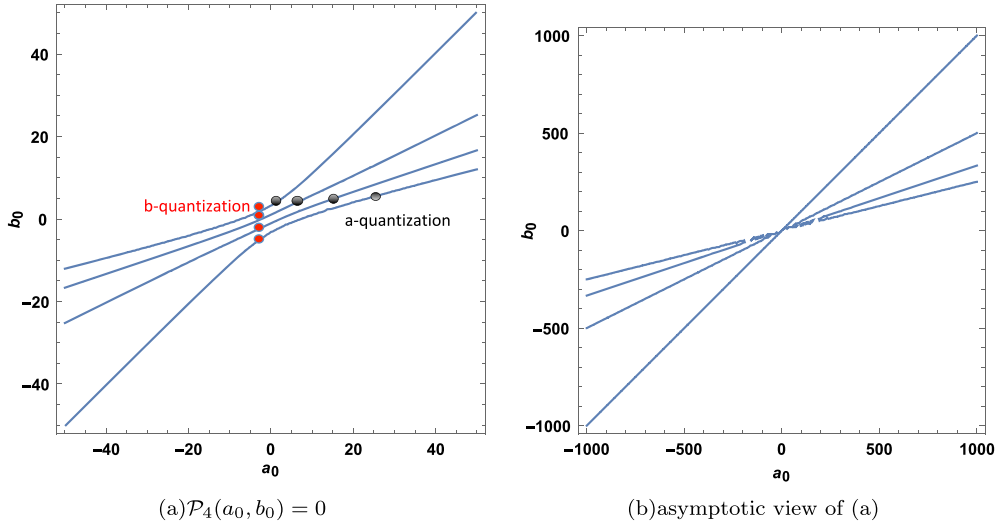


Fig. 1. Contour of $\mathcal{P}_4(a_0, b_0) = 0$ and its asymptotic view. Fig. (a) also shows definition of a_0 (or b_0)-quantization depending on we fix b_0 or a_0 .

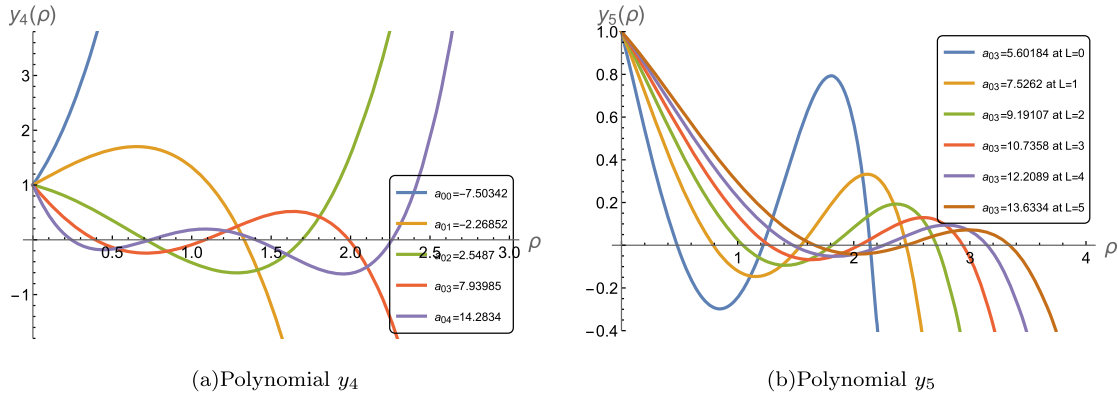


Fig. 2. (a) Polynomial y_4 for various a_{0K} for each $a_{0K}, K = 0, \dots, 4$. (b) y_5 for various a_{03} corresponding to $L = 0, 1, \dots, 5$.

On the other hand, for $m_q \neq 0$, b can not be an arbitrary value. It is determined by b -quantization because the parameter $a = 0$. The value of b for given N, L was determined numerically [5] and given by:

For odd N ,

$$b_\tau \approx \frac{8}{5} \left(N + \frac{6}{5}L + \frac{5}{3} \right) m_q^2 \tag{27}$$

For even N ,

$$b_\tau \approx \frac{2}{5} \left(N + \frac{6}{5}L + \frac{5}{3} \right) m_q^2 \tag{28}$$

By inserting eq (27) and eq (28) to eq (26), we see that not only the spectrum is highly nonlinear in N, L but also the string tension is vanishingly small in the limit of $m_q \rightarrow 0$, which is inconsistent with the nature.

From the correspondence of two system given in eq (12), we can read off the spectrum of holographic model from that of the bag model by replacing

$$m_q \rightarrow qM_0, b_\tau/2 \rightarrow qM, L \rightarrow |m|, E_{N,L}^2 \rightarrow m_n^2. \tag{29}$$

Exactly parallel comment for the bag model can be applied to the spectrum.

6. Summary and discussion

In this paper, we considered a holographic model where the confinement is consequence of the non-zero quark condensate M , so that M play the role of the order parameter for the confinement transition as well as that for the chiral transition. We also showed that the equation of motion of our holographic model can be mapped to the Lichtenberg model. The quark mass triggers the change of the singularity type from the hypergeometric type to Heun's type. The Regge trajectory appears only at the zero current quark mass limit where higher singularity disappears.

Before we finish, we discuss a similar model in AdS_5 . The scalar solution in AdS_5 with $m_\Phi^2 = -4$ is

$$\Phi = M_0 z^2 \ln z^{-1} + Mz^2, \text{ in } AdS_5. \tag{30}$$

If the quark mass M_0 vanishes, we still have $\Phi = Mz^2$, which is necessary power in z to give linear confinement. Therefore in this model, exactly the same calculation leads to the same result of AdS_4 model. However, there is one subtlety here. $m_\Phi^2 = 2(2 - 4) = -4$ follows from the assumption that the dimension of $\langle \bar{q}q \rangle$ is $\Delta = 2$, while its value for the free theory is 3 for 3+1 dimensional boundary theory. Therefore for our scenario to work, we need anomalous dimension $\gamma = -1$ In 2+1 dimension, on the other hand, we can simply use the $m_\Phi^2 = -2$ for $\Delta = 2$, which is the reason why we used AdS_4 model in the main text.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We appreciate the useful discussion with Eunseok Oh and the hospitality of the APCTP during the workshop “Quantum Matter from the Entanglement and Holography”. This work is supported by Mid-career Researcher Program through the National Research Foundation of Korea grant No. NRF-2016R1A2B3007687.

References

- [1] E.V. Shuryak, The QCD vacuum, hadrons and the superdense matter, *World Sci. Lect. Notes Phys.* 71 (2004) 1, <https://doi.org/10.1142/5367>.
- [2] K. Yagi, T. Hatsuda, Y. Miake, Quark-gluon plasma: from Big Bang to little bang, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* 23 (2005) 1.
- [3] S. Mandelstam, Vortices and quark confinement in nonabelian gauge theories, *Phys. Rep.* 23 (1976) 245–249.
- [4] G. 't Hooft, Gauge theory for strong interactions, in new phenomena in subnuclear physics, in: *Proceedings, International School of Subnuclear Physics, Part A, Erice, Sicily, Jul 11-Aug 1, 1975, 1975*, p. 0261.
- [5] Y.S. Choun, S.J. Sin, Why the quark mass is so small: the chiral symmetry and Heun's equation, arXiv:1909.07215 [hep-ph].
- [6] D.B. Lichtenberg, W. Namgung, E. Predazzi, J.G. Wills, Baryon masses in a relativistic quark-diquark model, *Phys. Lett.* 48 (1982) 1653.
- [7] F. Gürsey, Comments on hardronic mass formulae, in: A. Das (Ed.), *From Symmetries to Strings: Forty Years of Rochester Conferences*, World Scientific, Singapore, 1990.
- [8] D.B. Kaplan, J.-W. Lee, D.T. Son, M.A. Stephanov, Conformality lost, *Phys. Rev. D* 80 (2009) 125005, arXiv:0905.4752.
- [9] E. Oh, S.J. Sin, Holographic Abelian Higgs model and the linear confinement, arXiv:1909.13801 [hep-ph].
- [10] A. Karch, E. Katz, D.T. Son, M.A. Stephanov, Linear confinement and AdS/QCD, *Phys. Rev. D* 74 (2006) 015005, arXiv:hep-ph/0602229.
- [11] R. Argurio, A. Marzolla, A. Mezzalana, D. Musso, J. High Energy Phys. 1603 (2016) 012, [https://doi.org/10.1007/JHEP03\(2016\)012](https://doi.org/10.1007/JHEP03(2016)012), arXiv:1512.03750 [hep-th].
- [12] N.I.S.T. Digital, Library of mathematical functions, “confluent forms of Heun equation”, <http://dlmf.nist.gov/31.12>.
- [13] A. Ronveaux, *Heun Differential Equations*, Oxford University Press, 1995.
- [14] S.Yu. Slavyanov, W. Lay, *Special Functions: A Unified Theory Based on Singularities*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.