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Research Article

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Existence of a common solution to systems of integral equations via fixed point results

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Abstract: The goal of this article is to prove some coupled common fixed point results by using weakly increasing mappings with two variables. Several examples indicating the usability are provided. Also, we use the results obtained to demonstrate the existence of a common solution to a system of integral equations.

Keywords: coupled common fixed point, weakly increasing mapping, partially ordered metric space, integral equation

MSC 2010: 47H10, 54H25

1 Introduction and preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity; see [1–4]. The term coupled fixed point that was familiarized and studied by Opoitsev [5,6] and then by Guo and Lakhsmikantham [7] has been a center of attraction by many authors regarding the application potential of it [8–16]. Recently, the studies on the coupled common fixed point theory and its applications appeared in [17–21].

In this article, we establish some coupled common fixed point results by using weakly increasing mappings with two variables. Several examples and an application to integral equations indicating the usability of the new theory are also provided.

Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 1.1. [7] An element $(u, v) \in X^2$ is said to be a coupled fixed point of a mapping $A: X^2 \to X$ if $u = A(u, v)$ and $v = A(v, u)$.

Definition 1.2. [17] An element $(u, v) \in X^2$ is called a coupled common fixed point of mappings $A, B: X^2 \to X$ if $A(u, v) = B(u, v) = u$ and $A(u, v) = B(u, v) = v$. The set of all coupled common fixed points of A and B is indicated by $\mathcal{F}(A, B)$.

Definition 1.3. [17] Let $(X, \preceq)$ be an ordered set. Two mappings $A, B: X^2 \to X$ are said to be weakly increasing with respect to $\preceq$ if

$$A(u, v) \preceq B(A(u, v), A(v, u)) \text{ and } B(u, v) \preceq A(B(u, v), B(v, u))$$

hold for all $(u,v) \in X^2$.

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Following Su [22], we define the set $\Psi = \{\psi:[0, +\infty) \to [0, +\infty):\psi]$ that satisfies the conditions (i)–(iii)}, where
(i) is nondecreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$,
(iii) is subadditive, that is, $\psi(t + s) \le \psi(t) + \psi(s)$ for all $t, s \in [0, +\infty)$.

The set $\Phi = \{\phi:[0, +\infty) \to [0, +\infty):\phi$ is a nondecreasing and right upper semi-continuous function with $\psi(t) > \phi(t)$ for all $t > 0$, where $\psi \in \Psi\}$.

Throughout the article, $(X, d, \preceq)$ states an ordered metric space where d is a metric on X and $\preceq$ is a partial order on the set X. In addition, we say that $(x,y) \in X^2$ is comparable to $(u,v) \in X^2$ if $x \preceq u$ and $y \preceq v$, or $u \preceq x$ and $v \preceq y$. For brevity, we denote by $(x, y) \preceq (u, v)$ or $(x, y) \succeq (u, v)$.

If d is a metric on X, then δ: $X^2 \times X^2 \to [0, +\infty)$, defined by $\delta((x,y),(u,v)) = d(x,u) + d(y,v)$ for all $(x,y),(u,v) \in X^2$, is also a metric on X^2.

Now, we define Su type contractive pairs, which will be utilized in our main results.

Definition 1.4. Let $(X, d, \preceq)$ be an ordered metric space and A, B: $X^2 \to X$ be given mappings. We say that (A, B) is a Su type contractive pair if, for all comparable pairs $(x, y),(u, v) \in X^2$,

$$\psi(d(A(x, y), B(u, v))) \le \frac{1}{2}\phi(Q(x, y, u, v)) \tag{1}$$

holds, where

$$Q(x, y, u, v) = \max\begin{cases} \delta((x, y), (u, v)), \delta((x, y), (A(x, y), A(y, x))), \\ \delta((u, v), (B(u, v), B(v, u))), \\ \frac{1}{2}\left[\begin{array}{l} \delta((x, y), (B(u, v), B(v, u))) \\ +\delta((u, v), (A(x, y), A(y, x))) \end{array}\right]. \end{cases}$$

Remark 1.5. By the definition of $Q(x, y, u, v)$, it is obvious that

$$Q(x, y, u, v) = Q(y, x, v, u).$$

2 Existence of a common solution to systems of integral equations

The following is one of the main results.

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered complete metric space, A, B: $X^2 \to X$ weakly increasing mappings with respect to $\preceq$ and (A, B) be a Su type contractive pair. If A (or B) is continuous, then $\mathcal{F}(A, B) \neq \varnothing$.*

Proof. Let $u_0, v_0, \in X$ Define sequences $\{u_n\}$ and $\{v_n\}$ in X by

$$u_{2n+1} = A(u_{2n}, v_{2n}), \quad u_{2n+2} = B(u_{2n+1}, v_{2n+1})$$

and

$$v_{2n+1} = A(v_{2n}, u_{2n}), \quad v_{2n+2} = B(v_{2n+1}, v_{2n+1})$$

for all $n \ge 0$. Since A and B are weakly increasing, we have

$$u_n \preceq u_{n+1} \text{ and } v_n \preceq v_{n+1}, \quad n \ge 1. \tag{2}$$

Suppose that $u_n \neq u_{n+1}$ and $v_n \neq v_{n+1}$ for all $n \geq 0$ Then, for $n = 2m + 1$ using (1) and (2), we have

$$\begin{aligned}\psi(d(u_n, u_{n+1})) &= \psi(d(u_{2m+1}, u_{2m+2}))\\ &= \psi(d(A(u_{2m}, v_{2m}), B(u_{2m+1}, v_{2m+1})))\\ &\leq \frac{1}{2}\phi(Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1})),\end{aligned} \tag{3}$$

where

$$\begin{aligned}Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1}) &= \max\{\delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1})),\\ &\quad \delta((u_{2m}, v_{2m}), (A(u_{2m}, v_{2m}), A(v_{2m}, u_{2m})))\\ &\quad \delta((u_{2m+1}, v_{2m+1}), (B(u_{2m+1}, v_{2m+1}), B(v_{2m+1}, u_{2m+1}))),\\ &\quad \frac{1}{2}[\delta((u_{2m}, v_{2m}), (B(u_{2m+1}, v_{2m+1}), B(v_{2m+1}, u_{2m+1})))\\ &\quad + \delta((u_{2m+1}, v_{2m+1}), (A(u_{2m}, v_{2m}), A(v_{2m}, u_{2m})))]\}\\ &= \max\{\delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1})), \delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1})),\\ &\quad \delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2})),\\ &\quad \frac{1}{2}[\delta((u_{2m}, v_{2m}), (u_{2m+2}, v_{2m+2}))\\ &\quad + \delta((u_{2m+1}, v_{2m+1}), (u_{2m+1}, v_{2m+1}))]\}.\end{aligned}$$

Since

$$\delta((u_{2m+1}, v_{2m+1}), (u_{2m+1}, v_{2m+1})) = d(u_{2m+1}, v_{2m+1}) + d(v_{2m+1}, u_{2m+1}) = 0$$

and

$$\begin{aligned}\delta((u_{2m}, v_{2m}), (u_{2m+2}, v_{2m+2})) &= d(u_{2m}, u_{2m+2}) + d(v_{2m}, v_{2m+2})\\ &\leq d(u_{2m}, u_{2m+1}) + d(u_{2m+1}, u_{2m+2})\\ &\quad + d(v_{2m}, v_{2m+1}) + d(v_{2m+1}, v_{2m+2})\\ &= \delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1}))\\ &\quad + \delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2})),\end{aligned}$$

we obtain

$$Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1}) = \max\{\delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1})), \delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2}))\}.$$

Similarly, by (1) and (2), we obtain

$$\begin{aligned}\psi(d(v_n, v_{n+1})) &= \psi(d(v_{2m+1}, v_{2m+2}))\\ &= \psi(d(A(v_{2m}, u_{2m}), B(v_{2m+1}, u_{2m+1})))\\ &\leq \frac{1}{2}\phi(Q(v_{2m}, u_{2m}, v_{2m+1}, u_{2m+1}))\\ &= \frac{1}{2}\phi(Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1})).\end{aligned} \tag{4}$$

Summing the inequalities (3) and (4) and using the subadditivity property of ψ, we obtain

$$\psi(d(u_{2m+1}, u_{2m+2}) + d(v_{2m+1}, v_{2m+2})) \leq \phi(Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1})). \tag{5}$$

If $Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1}) = \delta((u_{2m+1}, v_{2m+1}),(u_{2m+2}, v_{2m+2})$ for some m, then by (5), we obtain

$$\begin{aligned}\psi(\delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2}))) &\le \phi(\delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2})))\\ &< \psi(\delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2}))).\end{aligned}$$

Since ψ is nondecreasing,

$$\delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2})) < \delta((u_{2m+1}, v_{2m+1}), (u_{2m+2}, v_{2m+2})),$$

which is a contradiction. Then,

$$Q(u_{2m}, v_{2m}, u_{2m+1}, v_{2m+1}) = \delta((u_{2m}, v_{2m}), (u_{2m+1}, v_{2m+1}))$$

and so, by (5),

$$\psi(\delta((u_n, v_n), (u_{n+1}, v_{n+1}))) \le \phi(\delta((u_{n-1}, v_{n-1}), (u_n, v_n))). \tag{6}$$

Set $\delta_n := \{\delta((u_n,v_n),(u_{n+1},v_{n+1}))\}$. Then, the sequence $\{\delta_n\}$ is decreasing. Thus, there exists $r \ge 0$ such that $\lim_{n\to\infty}\delta_n = r$. Suppose that $r > 0$. Letting $n \to \infty$ in (6), we deduce

$$\begin{aligned}\psi(r) &\le \lim_{n\to\infty} \psi(\delta((u_n, v_n), (u_{n+1}, v_{n+1})))\\ &\le \lim_{n\to\infty} \phi(\delta((u_{n-1}, v_{n-1}), (u_n, v_n))) \le \phi(r),\end{aligned}$$

a contradiction, and hence, $r = 0$, that is,

$$\lim_{n\to\infty} \delta((u_n, v_n), (u_{n+1}, v_{n+1})) = \lim_{n\to\infty} [d(u_n, u_{n+1}) + d(v_n, v_{n+1})] = 0. \tag{7}$$

To prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences, it is sufficient to show that $\{u_{2n}\}$ and $\{v_{2n}\}$ are Cauchy sequences in (X, d). Suppose, to the contrary, that at least one of $\{u_{2n}\}$ or $\{v_{2n}\}$ is not Cauchy sequence. Then, there exists an $\varepsilon > 0$ for which we can find subsequences $\{u_{2m_k}\}$, $\{u_{2n_k}\}$ of $\{u_{2n}\}$ and $\{v_{2m_k}\}$, $\{v_{2n_k}\}$ of $\{v_{2n}\}$, such that n_k is the smallest index for which $n_k > m_k > k$ and

$$\begin{aligned}&d(u_{2n_k}, u_{2m_k}) + d(v_{2n_k}, v_{2m_k}) \ge \varepsilon,\\ &d(u_{2n_k-1}, u_{2m_k}) + d(v_{2n_k-1}, v_{2m_k}) < \varepsilon.\end{aligned} \tag{8}$$

By using the triangle inequality and (8), we obtain

$$\begin{aligned}\varepsilon &\le d(u_{2n_k}, u_{2m_k}) + d(v_{2n_k}, v_{2m_k})\\ &\le d(u_{2m_k}, u_{2n_k-1}) + d(u_{2n_k-1}, u_{2n_k}) + d(v_{2m_k}, v_{2n_k-1}) + d(v_{2n_k-1}, v_{2n_k})\\ &< \varepsilon + \delta_{2n_k-1}.\end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (7), we deduce

$$\lim_{k\to\infty} [d(u_{2n_k}, u_{2m_k}) + d(v_{2n_k}, v_{2m_k})] = \varepsilon. \tag{9}$$

Again, from the triangle inequality, we have

$$
\begin{aligned}
& d\left(u_{2n_k}, u_{2m_k}\right) + d\left(v_{2n_k}, v_{2m_k}\right) \\
& \quad \le d\left(u_{2n_k}, u_{2n_k+1}\right) + d\left(u_{2n_k+1}, u_{2n_k+2}\right) + d\left(u_{2n_k+2}, u_{2m_k+1}\right) + d\left(u_{2m_k+1}, u_{2m_k}\right) \\
& \qquad + d\left(v_{2n_k}, v_{2n_k+1}\right) + d\left(v_{2n_k+1}, v_{2n_k+2}\right) + d\left(v_{2n_k+2}, v_{2m_k+1}\right) + d\left(v_{2m_k+1}, v_{2m_k}\right) \\
& \quad \le \delta_{2n_k} + \delta_{2n_k+1} + \delta_{2m_k} + d\left(u_{2n_k+2}, u_{2n_k+1}\right) + d\left(u_{2n_k+1}, u_{2m_k+1}\right) \\
& \qquad + d\left(v_{2n_k+2}, v_{2n_k+1}\right) + d\left(v_{2n_k+1}, v_{2m_k+1}\right) \\
& \quad \le \delta_{2n_k} + 2\delta_{2n_k+1} + \delta_{2m_k} + d\left(u_{2n_k+1}, u_{2m_k}\right) + d\left(u_{2m_k}, u_{2m_k+1}\right) \\
& \qquad + d\left(v_{2n_k+1}, v_{2m_k}\right) + d\left(v_{2m_k}, v_{2m_k+1}\right) \\
& \quad \le \delta_{2n_k} + 2\delta_{2n_k+1} + 2\delta_{2m_k} + d\left(u_{2n_k+1}, u_{2n_k+2}\right) + d\left(u_{2n_k+2}, u_{2m_k}\right) \\
& \qquad + d\left(v_{2n_k+1}, v_{2n_k+2}\right) + d\left(v_{2n_k+2}, v_{2m_k}\right) \\
& \quad \le 2\delta_{2n_k} + 2\delta_{2m_k} + 4\delta_{2n_k+1} + d\left(u_{2n_k}, u_{2m_k}\right) + d\left(v_{2n_k}, v_{2m_k}\right).
\end{aligned}
$$

Letting $k \to \infty$ in the above inequality and using (7) and (9), we have

$$
\begin{aligned}
& \lim_{k\to\infty} \left[d\left(u_{2n_k+2}, u_{2m_k+1}\right) + d\left(v_{2n_k+2}, v_{2m_k+1}\right)\right] = \varepsilon, \\
& \lim_{k\to\infty} \left[d\left(u_{2n_k+1}, u_{2m_k+1}\right) + d\left(v_{2n_k+1}, v_{2m_k+1}\right)\right] = \varepsilon, \\
& \lim_{k\to\infty} \left[d\left(u_{2n_k+1}, u_{2m_k}\right) + d\left(v_{2n_k+1}, v_{2m_k}\right)\right] = \varepsilon, \\
& \lim_{k\to\infty} \left[d\left(u_{2n_k+2}, u_{2m_k}\right) + d\left(v_{2n_k+2}, v_{2m_k}\right)\right] = \varepsilon.
\end{aligned} \tag{10}
$$

Since $(u_{2m_k}, v_{2m_k}) \preceq (u_{2n_k+1}, v_{2n_k+1})$ for $n_k > m_k$, using (1), we obtain

$$
\begin{aligned}
\psi\left(d\left(u_{2m_k+1}, u_{2n_k+2}\right)\right) &= \psi\left(d\left(A\left(u_{2m_k}, v_{2m_k}\right), B\left(u_{2n_k+1}, v_{2n_k+1}\right)\right)\right) \\
&\le \frac{1}{2}\phi\left(Q\left(u_{2m_k}, v_{2m_k}, u_{2n_k+1}, v_{2n_k+1}\right)\right),
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
& Q\left(u_{2m_k}, v_{2m_k}, u_{2n_k+1}, v_{2n_k+1}\right) \\
& \quad = \max\{\delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(u_{2n_k+1}, v_{2n_k+1}\right)\right), \\
& \qquad \delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(A\left(u_{2m_k}, v_{2m_k}\right), A\left(v_{2m_k}, u_{2m_k}\right)\right)\right), \\
& \qquad \delta\left(\left(u_{2n_k+1}, v_{2n_k+1}\right), \left(B\left(u_{2n_k+1}, v_{2n_k+1}\right), B\left(v_{2n_k+1}, u_{2n_k+1}\right)\right)\right), \\
& \qquad \frac{1}{2}[\delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(B\left(u_{2n_k+1}, v_{2n_k+1}\right), B\left(v_{2n_k+1}, u_{2n_k+1}\right)\right)\right) \\
& \qquad + \delta\left(\left(u_{2n_k+1}, v_{2n_k+1}\right), \left(A\left(u_{2m_k}, v_{2m_k}\right), A\left(v_{2m_k}, u_{2m_k}\right)\right)\right)]\} \\
& \quad = \max\{\delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(u_{2n_k+1}, v_{2n_k+1}\right)\right), \delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(u_{2m_k+1}, v_{2m_k+1}\right)\right), \\
& \qquad \delta\left(\left(u_{2n_k+1}, v_{2n_k+1}\right), \left(u_{2n_k+2}, v_{2n_k+2}\right)\right), \\
& \qquad \frac{1}{2}[\delta\left(\left(u_{2m_k}, v_{2m_k}\right), \left(u_{2n_k+2}, v_{2n_k+2}\right)\right) \\
& \qquad + \delta\left(\left(u_{2n_k+1}, v_{2n_k+1}\right), \left(u_{2m_k+1}, v_{2m_k+1}\right)\right)]\}.
\end{aligned}
$$

Again, since $(v_{2m_k}, u_{2m_k}) \preceq (v_{2n_k+1}, u_{2n_k+1})$, by (1), we also have

$$
\begin{aligned}
\psi\left(d\left(v_{2m_k+1}, v_{2n_k+2}\right)\right) &= \psi\left(d\left(A(v_{2m_k}, u_{2m_k}), B\left(v_{2n_k+1}, u_{2n_k+1}\right)\right)\right) \\
&= \frac{1}{2}\phi\left(Q\left(u_{2m_k}, v_{2m_k}, u_{2n_k+1}, v_{2n_k+1}\right)\right).
\end{aligned} \tag{12}
$$

Summing the inequalities (11) and (12) and using the subadditivity property of ψ, we obtain

$$\psi\left(d\left(u_{2m_k+1}, u_{2n_k+2}\right) + d\left(v_{2m_k+1}, v_{2n_k+2}\right)\right) \leq \phi\left(Q\left(u_{2m_k}, v_{2m_k}, u_{2n_k+1}, v_{2n_k+1}\right)\right).$$

Now, by using (7), (9) and (10) and letting $k \to \infty$ in the above inequality, we deduce

$$\begin{aligned}\psi(\varepsilon) &\leq \lim_{k\to\infty} \psi\left(d\left(u_{2m_k+1}, u_{2n_k+2}\right) + d\left(v_{2m_k+1}, v_{2n_k+2}\right)\right)\\ &\leq \lim_{k\to\infty} \phi\left(Q\left(u_{2m_k}, v_{2m_k}, u_{2n_k+1}, v_{2n_k+1}\right)\right)\\ &\leq \phi(\max\{\varepsilon, 0, 0, \varepsilon) = \phi(\varepsilon),\end{aligned}$$

which implies $\varepsilon = 0$ a contradiction with $\varepsilon > 0$. Therefore, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X.

Now, we prove the existence of coupled common fixed point of A and B.

Owing to the completeness of (X, d), there exist $u, v \in X$ such that

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} v_n = v. \tag{13}$$

Without loss of generality, we assume that A is continuous. Now we have

$$u = \lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} A\ (u_{2n}, v_{2n}) = A\left(\lim_{n\to\infty} u_{2n}, \lim_{n\to\infty} v_{2n}\right) = A(u, v)$$

and

$$v = \lim_{n\to\infty} v_{2n+1} = \lim_{n\to\infty} A(v_{2n}, u_{2n}) = A\left(\lim_{n\to\infty} v_{2n}, \lim_{n\to\infty} u_{2n}\right) = A(v, u).$$

We now assert that $d(u, B(u, v)) = d(v, B(v, u)) = 0$. To establish the claim, assume that $d(u, B(u, v)) > 0$ and $d(v, B(v, u)) > 0$. Since $(u, v) \in X^2$ is comparable to its own, from (1), we obtain

$$\psi(d(u, B(u, v))) = \psi(d(A(u, v), B(u, v))) \leq \frac{1}{2}\phi(Q(u, v, u, v)), \tag{14}$$

where

$$\begin{aligned}Q(u, v, u, v) &= \max\{\delta((u, v), (u, v)), \delta((u, v), (A(u, v), A(v, u))),\\ &\qquad \delta((u, v), (B(u, v), B(v, u))), \frac{1}{2}[\delta((u, v), (B(u, v), B(v, u)))\\ &\qquad + \delta((u, v), (A(u, v), A(v, u)))]\}\\ &= \delta((u, v), (B(u, v), B(v, u))).\end{aligned}$$

Again, since $(v, u) \preceq (v, u)$, by (1), we have

$$\psi(d(v, B(v, u))) = \psi(d(A(v, u), B(v, u))) \leq \frac{1}{2}\phi(Q(u, v, u, v)). \tag{15}$$

Thus, it follows from (14) and (15) that

$$\begin{aligned}\psi(d(u,B(u,v))+d(v,G(v,u))) &\le \phi(Q(u,v,u,v))\\ &= \phi(\delta((u,v),(B(u,v),B(v,u))))\\ &= \phi(d(u,B(u,v))+d(v,B(v,u))),\end{aligned}$$

which implies $d(u, B(u, v)) = d(v, B(v, u)) = 0$

Therefore, $u = A(u, v) = B(u, v)$ and $v = A(v, u) = B(v, u)$ □

Example 2.2. Let $X = [0,1]$ be equipped with the usual metric and the partial order defined by

$$x \preceq y \text{ if and only if } y \le x.$$

Define mappings A, B: $X^2 \to X$ by $A(u,v) = \frac{u+v}{4}$ and $B(u,v) = \frac{u+v}{3}$. Then, A and B are weakly increasing with respect to $\preceq$ and continuous.

Also, (A, B) is a Su type contractive pair. Indeed, for all comparable (x, y), $(u, v) \in X^2$,

$$\begin{aligned}\psi(d(A(x,y),B(u,v))) &= \left|\frac{x+y}{4}-\frac{u+v}{3}\right| \le \frac{1}{4}(|x-u|+|y-v|) = \frac{1}{2}\phi(\delta((x,y),(u,v)))\\ &\le \frac{1}{2}\phi\left(\max\left\{\begin{matrix}\delta((x,y),(u,v)),\delta((x,y),(A(x,y),A(y,x))),\\ \delta((u,v),(B(u,v),B(v,u))),\\ \frac{1}{2}\begin{bmatrix}\delta((x,y),(B(u,v),B(v,u)))\\ +\delta((u,v),(A(x,y),A(y,x)))\end{bmatrix}\end{matrix}\right\}\right)\\ &= \frac{1}{2}\phi(Q(x,y,u,v)),\end{aligned}$$

where $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$. Thus, all the hypotheses of Theorem 2.1 are fulfilled. Therefore, A and B have a coupled common fixed point, which is (0,0).

Definition 2.3. Let $(X, d, \preceq)$ be an ordered metric space. We say that $(X, d, \preceq)$ is regular if each nondecreasing sequence $\{x_n\}$ with $d(x_n,x) \to 0$ implies that $x_n \preceq x$ for all n.

We replace the continuity of A (or B) with the regularity of $(X, d, \preceq)$ in the following theorem.

Theorem 2.4. *Let $(X, d, \preceq)$ be an ordered complete metric space, A, B: $X^2 \to X$ weakly increasing mappings with respect to $\preceq$ and (A, B) be a Su type contractive pair. If $(X, d, \preceq)$ is regular, then A and B have a coupled common fixed point.*

Proof. Let u_0, $v_0 \in X$. Define sequences $\{u_n\}$ and $\{v_n\}$ in X by

$$u_{2n+1} = A(u_{2n}, v_{2n}),\ u_{2n+2} = B(u_{2n+1}, v_{2n+1})$$

and

$$v_{2n+1} = A(v_{2n}, u_{2n}),\ v_{2n+2} = B(v_{2n+1}, v_{2n+1})$$

for all $n \ge 0$. Following the proof of Theorem 2.1, we can show that the sequences $\{u_n\}$ and $\{v_n\}$ are nondecreasing, $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Since $(X, d, \preceq)$ is regular, we deduce that (u_n, v_n) is comparable to (u, v) for all n. From (1), we obtain

$$\psi(d(u_{2n+1}, B(u,v))) = \psi(d(A(u_{2n}, v_{2n}), B(u,v))) \le \frac{1}{2}\phi(Q(u_{2n}, v_{2n}, u, v)), \tag{16}$$

where

$$\begin{aligned} Q(u_{2n}, v_{2n}, u, v) = \max\{&\delta((u_{2n}, v_{2n}), (u, v)), \delta((u_{2n}, v_{2n}), ((u_{2n+1}, v_{2n+1}))), \\ &\delta((u, v), (B(u, v), B(v, u))), \frac{1}{2}[\delta((u_{2n}, v_{2n}), (B(u, v), G(v, u))) \\ &+ \delta((u, v), (u_{2n+1}, v_{2n+1}))]\}. \end{aligned}$$

Again, by (1), we obtain

$$\psi(d(v_{2n+1}, B(v, u))) = \psi(d(A(v_{2n}, u_{2n}), B(v, u))) \le \frac{1}{2}\phi(Q(u_{2n}, v_{2n}, u, v)). \tag{17}$$

Thus, it follows from (16), (17) and the subadditivity property of ψ that

$$\psi(d(u_{2n+1}, B(u, v)) + d(v_{2n+1}, B(v, u))) \le \phi(Q(u_{2n}, v_{2n}, u, v)).$$

Taking $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} &\psi(d(u, B(u, v)) + d(v, B(v, u))) \\ &\quad \le \lim_{n\to\infty} \psi(d(u_{2n+1}, B(u, v)) + d(v_{2n+1}, B(v, u))) \\ &\quad \le \lim_{n\to\infty} \phi(Q(u_{2n}, v_{2n}, u, v)) \\ &\quad \le \phi(\delta((u, v), (B(u, v), B(v, u)))) \\ &\quad = \phi(d(u, B(u, v)) + d(v, B(v, u))), \end{aligned}$$

which implies that $d(u, B(u, v)) + d(v,B(v, u)) = 0$, that is, $u = B(u, v)$ and $v = B(v, u)$.

Since $(u, v) \preceq (u, v)$, by (1), we deduce

$$\psi(d(A(u, v), u)) = \psi(d(A(u, v), B(u, v))) \le \frac{1}{2}\phi(Q(u, v, u, v)), \tag{18}$$

where

$$Q(u, v, u, v) = \delta((u, v), (A(u, v), A(v, u))).$$

Again, by (1),

$$\psi(d(A(v, u), v)) = \phi(d(A(v, u), B(v, u))) \le \frac{1}{2}\phi(Q(u, v, u, v)). \tag{19}$$

Hence, it follows from (18), (19) and the subadditivity of ψ that

$$\psi(d(A(u, v), u) + d(A(v, u), v)) \le \phi(Q(u, v, u, v)) = \phi(d(u, A(u, v)) + d(v, A(v, u))),$$

which implies $d(u, A(u, v)) = d(v, A(v, u)) = 0$. This completes the proof. ☐

Example 2.5. Let $X = [0,+\infty)$ be equipped with the usual metric and the partial order defined by

$$x \preceq y \text{ if and only if } y \le x.$$

Define mappings A, B: $X^2 \to X$ by

$$A(u, v) = \begin{cases} \dfrac{u + v}{6}, & \text{if } u \ge v, \\ 0, & \text{if } u < v, \end{cases}$$

and

$$B(u,v) = \begin{cases} \dfrac{u+v}{5}, & \text{if } u \geq v, \\ 0, & \text{if } u < v. \end{cases}$$

Then, A and B are weakly increasing with respect to $\preceq$ and discontinuous.

Now we demonstrate that (A, B) is a Su type contractive pair. For all comparable $(x, y), (u, v) \in X^2$,

$$\begin{aligned} \psi(d(A(x,y), B(u,v))) &= \left| \frac{x+y}{6} - \frac{u+v}{5} \right| \leq \frac{1}{6}(|x-u| + |y-v|) = \frac{1}{2}\phi(\delta((x,y),(u,v))) \\ &\leq \frac{1}{2}\phi\left(\max\left\{ \begin{array}{c} \delta((x,y),(u,v)), \delta((x,y),(A(x,y),A(y,x))), \\ \delta((u,v),(B(u,v),B(v,u))), \\ \frac{1}{2}\left[\begin{array}{c} \delta((x,y),(B(u,v),B(v,u))) \\ +\delta((u,v),(A(x,y),A(y,x))) \end{array} \right] \end{array} \right\} \right) \\ &= \frac{1}{2}\phi(Q(x,y,u,v)), \end{aligned}$$

where $\psi(t) = t$ and $\phi(t) = \frac{t}{3}$. Thus, all the hypotheses of Theorem 2.4 are fulfilled. Therefore, A and B have a coupled common fixed point.

If we replace $Q(x, y, u, v)$ with $d(x, y) + d(u, v)$ in Theorem 2.1 (or Theorem 2.4), then we obtain the following corollary, which is an extended version of the main result of Işık and Turkoglu [23].

Corollary 2.6. *Let $(X, d, \preceq)$ be an ordered complete metric space and $A, B: X^2 \to X$ be weakly increasing mappings with respect to $\preceq$ such that*

$$\psi(d(A(x,y), B(u,v))) \leq \frac{1}{2}\phi(d(x,y) + d(u,v)) \tag{20}$$

for all comparable $(x, y), (u, v) \in X^2$, where $\psi \in \Psi$ and $\phi \in \Phi$. Assume that one of the following conditions is satisfied:

(a) *A (or B) is continuous;*
(b) *$(X, d, \preceq)$ is regular.*

Then, A and B have a coupled common fixed point.

If we choose $\psi(t) = t$ and $\phi(t) = kt$ in Corollary 2.6 for $k \in [0,1)$, then we obtain the following result, which is an extended version of the main result of Bhaskar and Laksmikantham [24].

Corollary 2.7. *Let $(X, d, \preceq)$ be an ordered complete metric space and $A, B: X^2 \to X$ be weakly increasing mappings with respect to $\preceq$, such that*

$$d(A(x,y), B(u,v)) \leq \frac{k}{2}[d(x,y) + d(u,v)] \tag{21}$$

for all comparable $(x, y), (u, v) \in X^2$, where $k \in [0,1)$. Assume that one of the following conditions is satisfied:

(a) *A (or B) is continuous;*
(b) *$(X, d, \preceq)$ is regular.*

Then, A and B have a coupled common fixed point.

3 Applications

Consider the following coupled systems of integral equations:

$$\begin{cases} u(s) = \int_a^b H_1(s, r, u(r), v(r))\mathrm{d}r, \\ v(s) = \int_a^b H_1(s, r, v(r), u(r))\mathrm{d}r, \end{cases} \tag{22}$$

and

$$\begin{cases} u(s) = \int_a^b H_2(s, r, u(r), v(r))\mathrm{d}r, \\ v(s) = \int_a^b H_2(s, r, v(r), u(r))\mathrm{d}r, \end{cases} \tag{23}$$

where $s \in I = [a, b]$ $H_1, H_2 \colon I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $b > a \geq 0$.

In this section, we present an existence theorem for a common solution to (22) and (23) that belongs to $X := C(I, \mathbb{R})$ (the set of continuous functions defined on I) by using the obtained result in Corollary 2.6.

We consider the operators $A, B \colon X^2 \to X$ given by

$$A(u, v)(s) = \int_a^b H_1(s, r, u(r), v(r))\mathrm{d}r, u, v \in X, s \in I,$$

and

$$B(u, v)(s) = \int_a^b H_2(s, r, u(r), v(r))\mathrm{d}r, u, v \in X, s \in I.$$

Then, the existence of a common solution to the integral equations (22) and (23) is equivalent to the existence of a coupled common fixed point of A and B.

It is well known that X, endowed with the metric d defined by

$$d(u, v) = \sup_{s \in I} |u(s) - v(s)|$$

for all $u, v \in X$ is a complete metric space. X can also be equipped with the partial order $\preceq$ given by

$$u, v \in X, u \preceq v \text{ if and only if } u(s) \geq v(s), \text{ for all } s \in I. \tag{24}$$

Recall that it is proved that $(X, d, \preceq)$ is regular (see [25]).

Suppose that the following conditions hold:

(A) $H_1, H_2 \colon I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;

(B) for all $s, r \in I$, we have

$$H_1(s, r, u(r), v(r)) \geq H_2\left(s, r, \int_a^b H_1(r, \tau, u(\tau), v(\tau))\mathrm{d}\tau, \int_a^b H_1(r, \tau, v(\tau), u(\tau))\mathrm{d}\tau\right),$$

and

$$H_2(s, r, u(r), v(r)) \geq H_1\left(s, r, \int_a^b H_2(r, \tau, u(\tau), v(\tau))\mathrm{d}\tau, \int_a^b H_2(r, \tau, v(\tau), u(\tau))\mathrm{d}\tau\right);$$

(C) for all comparable (x, y), $(u, v) \in X^2$ and for every $s, r \in I$, we have

$$|H_1(s, r, x(r), y(r)) - H_2(s, r, u(r), v(r))|^2 \le \frac{1}{4}\gamma(s, r)(|x(r) - u(r)| + |y(r) - v(r)|)^2,$$

where $\gamma\colon I^2 \to \mathbb{R}^+$ is a continuous function satisfying

$$\sup_{s\in I} \int_a^b \gamma(s, r)\mathrm{d}r \le \frac{1}{b-a}.$$

Theorem 3.1. *Assume that the conditions* (A)–(C) *are satisfied. Then, the integral equations* (22) *and* (23) *have a common solution in X.*

Proof. From the condition (B), the mappings A and B are weakly increasing with respect to $\preceq$. Indeed, for all $s \in I$, we have

$$\begin{aligned} A(u, v)(s) &= \int_a^b H_1(s, r, u(r), v(r))\mathrm{d}r \\ &\ge \int_a^b H_2\left(s, r, \int_a^b H_1(r, \tau, u(\tau), v(\tau))\mathrm{d}\tau, \int_a^b H_1(r, \tau, v(\tau), u(\tau))\mathrm{d}\tau\right)\mathrm{d}r \\ &= \int_a^b H_2(s, r, A(u, v)(r), A(v, u)(r))\mathrm{d}r \\ &= B(A(u, v), A(v, u))(s), \end{aligned}$$

and so $A(u, v) \preceq B(A(u, v), A(v, u))$. Similarly, one can easily see that $B(u, v) \preceq A(B(u, v), B(v, u))$.

Let (x, y) be comparable to (u, v). Then, by (C), for all $s \in I$, we deduce

$$\begin{aligned} |A(x, y)(s) - B(u, v)(s)|^2 &\le \left(\int_a^b |H_1(s, r, x(r), y(r)) - H_2(s, r, u(r), v(r))|\,\mathrm{d}r\right)^2 \\ &\le \int_a^b 1^2\mathrm{d}r \int_a^b |H_1(s, r, x(r), y(r)) - H_2(s, r, u(r), v(r))|^2\mathrm{d}r \\ &\le (b-a)\int_a^b \frac{1}{4}\gamma(s, r)(|x(r) - u(r)| + |y(r) - v(r)|)^2\mathrm{d}r \\ &\le \frac{1}{4}(b-a)\int_a^b \gamma(s, r)(d(x, u) + d(y, v))^2\mathrm{d}r \\ &\le \frac{1}{4}(b-a)\sup_{s\in I}\left(\int_a^b \gamma(s, r)\mathrm{d}r\right)(d(x, u) + d(y, v))^2 \\ &\le \frac{1}{4}(d(x, u) + d(y, v))^2. \end{aligned}$$

Therefore, by the above inequality, we obtain

$$\left(\sup_{s\in I}|A(x, y)(s) - B(u, v)(s)|\right)^2 \le \frac{1}{4}(d(x, u) + d(y, v))^2.$$

Putting $\psi(t) = t^2$ and $\phi(t) = \frac{t^2}{2}$, we obtain

$$\psi(d(F(x, y), G(u, v))) \le \frac{1}{2}\phi(d(x, u) + d(y, v)) \tag{25}$$

for all comparable (x, y), $(u, v) \in X^2$. Hence, all the hypotheses of Corollary 2.6 are satisfied. So A and B have a coupled common fixed point, that is, the integral equations (22) and (23) have a common solution in X. □

Example 3.2. Consider the following systems of integral equations in $X = C(I = [0, 1], \mathbb{R})$

$$\begin{cases} u(s) = \int_0^1 \left(s^2 + \frac{r}{1+r} + \frac{1}{8}\frac{|u(r)|}{1+3|u(r)|} + \frac{1}{8}\frac{|v(r)|}{1+5|v(r)|}\right)\mathrm{d}r, \\ v(s) = \int_0^1 \left(s^2 + \frac{r}{1+r} + \frac{1}{8}\frac{|v(r)|}{1+3|v(r)|} + \frac{1}{8}\frac{|u(r)|}{1+5|u(r)|}\right)\mathrm{d}r, \end{cases} \tag{26}$$

and

$$\begin{cases} u(s) = \int_0^1 \left(s^2 + \frac{r}{1+r} + \frac{1}{9}\frac{|u(r)|}{1+7|u(r)|} + \frac{1}{9}\frac{|v(r)|}{1+9|v(r)|}\right)\mathrm{d}r, \\ v(s) = \int_0^1 \left(s^2 + \frac{r}{1+r} + \frac{1}{9}\frac{|v(r)|}{1+7|v(r)|} + \frac{1}{9}\frac{|u(r)|}{1+9|u(r)|}\right)\mathrm{d}r. \end{cases} \tag{27}$$

The systems (26) and (27) are particular cases of systems (22) and (23), respectively, where

$$H_1(s, r, u(r), v(r)) = s^2 + \frac{r}{1+r} + \frac{1}{8}\frac{|u(r)|}{1+3|u(r)|} + \frac{1}{8}\frac{|v(r)|}{1+5|v(r)|},$$

and

$$H_2(s, r, u(r), v(r)) = s^2 + \frac{r}{1+r} + \frac{1}{9}\frac{|u(r)|}{1+7|u(r)|} + \frac{1}{9}\frac{|v(r)|}{1+9|v(r)|}.$$

Clearly, H_1 and H_2 are continuous, that is, the condition (A) is satisfied. Also, one can easily prove that the condition (B) holds with respect to the relation $\preceq$ defined by (24).

For all (x, y), $(u, v) \in X^2$ with $x \geq u$, $y \geq v$ and for every $s, r \in I$, we obtain

$$\begin{aligned} &|H_1(s, r, x(r), y(r)) - H_2(s, r, u(r), v(r))|^2 \\ &\quad = \left|\frac{1}{8}\frac{|x(r)|}{1+3|x(r)|} + \frac{1}{8}\frac{|y(r)|}{1+5|y(r)|} - \frac{1}{9}\frac{|u(r)|}{1+7|u(r)|} - \frac{1}{9}\frac{|v(r)|}{1+9|v(r)|}\right|^2 \\ &\quad \leq \frac{1}{8}(|x(r) - u(r)| + |y(r) - v(r)|)^2 \\ &\quad = \frac{1}{4}\gamma(s, r)(|x(r) - u(r)| + |y(r) - v(r)|)^2, \end{aligned}$$

where $\gamma(s, r) = 1/2$, so that

$$\sup_{s \in I} \int_0^1 \gamma(s, r)\mathrm{d}r \leq 1.$$

Thus, all conditions of Theorem 3.1 are satisfied. Therefore, the coupled systems (26) and (27) have a common solution in X.

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