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Oscillation Results for Higher Order Differential Equations

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Abstract: The objective of our research was to study asymptotic properties of the class of higher order differential equations with a p-Laplacian-like operator. Our results supplement and improve some known results obtained in the literature. An illustrative example is provided.

Keywords: ocillation; higher-order; differential equations; p-Laplacian equations

1. Introduction

In this work, we are concerned with oscillations of higher-order differential equations with a p-Laplacian-like operator of the form

$$\left(r\left(t\right) \left| \left(y^{(n-1)}\left(t\right) \right) \right|^{p-2} y^{(n-1)}\left(t\right) \right)' + q\left(t\right) \left| y\left(\tau\left(t\right)\right) \right|^{p-2} y\left(\tau\left(t\right)\right) = 0. \tag{1}$$

We assume that $p > 1$ is a constant, $r \in C^1\left([t_0,\infty),\mathbb{R}\right)$, $r\left(t\right) > 0$, $q,\tau \in C\left([t_0,\infty),\mathbb{R}\right)$, $q > 0$, $\tau\left(t\right) \le t$, $\lim_{t\to\infty} \tau\left(t\right) = \infty$ and the condition

$$\eta\left(t_0\right) = \infty, \tag{2}$$

where

$$\eta\left(t\right) := \int_t^\infty \frac{ds}{r^{1/(p-1)}\left(s\right)}.$$

By a solution of (1) we mean a function $y \in C^{n-1}[T_y,\infty)$, $T_y \ge t_0$, which has the property $r\left(t\right) \left| \left(y^{(n-1)}\left(t\right) \right) \right|^{p-2} y^{(n-1)}\left(t\right) \in C^1[T_y,\infty)$, and satisfies (1) on $[T_y,\infty)$. We consider only those solutions y of (1) which satisfy $\sup\{|y\left(t\right)| : t \ge T\} > 0$, for all $T > T_y$. A solution of (1) is called oscillatory if it has arbitrarily large number of zeros on $[T_y,\infty)$, and otherwise it is called to be nonoscillatory; (1) is said to be oscillatory if all its solutions are oscillatory.

In recent decades, there has been a lot of research concerning the oscillation of solutions of various classes of differential equations; see [1–24].

It is interesting to study Equation (1) since the p-Laplace differential equations have applications in continuum mechanics [14,25]. In the following, we briefly review some important oscillation criteria obtained for higher-order equations, which can be seen as a motivation for this paper.

Elabbasy et al. [26] proved that the equation

$$\left(r\left(t\right) \left| \left(y^{(n-1)}\left(t\right) \right) \right|^{p-2} y^{(n-1)}\left(t\right) \right)' + q\left(t\right) f\left(y\left(\tau\left(t\right)\right)\right) = 0,$$

is oscillatory, under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r^{p-1}(t)} dt = \infty;$$

additionally,

$$\int_{\ell_0}^{\infty} \left(\psi(s) - \frac{1}{p^p} \phi^p(s) \frac{((n-1)!)^{p-1} \rho(s) a(s)}{((p-1)\mu s^{n-1})^{p-1}} - \frac{(p-1)\rho(s)}{a^{1/(p-1)}(s)\eta^p(s)} \right) ds = +\infty,$$

for some constant $\mu \in (0,1)$ and

$$\int_{\ell_0}^{\infty} kq(s) \frac{\tau(s)^{p-1}}{s^{p-1}} ds = \infty.$$

Agarwal et al. [2] studied the oscillation of the higher-order nonlinear delay differential equation

$$\left[\left| y^{(n-1)}(t) \right|^{\alpha-1} y^{(n-1)}(t) \right]' + q(t) |y(\tau(t))|^{\alpha-1} y(\tau(t)) = 0.$$

where α is a positive real number. In [27], Zhang et al. studied the asymptotic properties of the solutions of equation

$$\left[r(t) \left(y^{(n-1)}(t) \right)^{\alpha} \right]' + q(t) y^{\beta}(\tau(t)) = 0, \quad t \geq t_0.$$

where α and β are ratios of odd positive integers, $\beta \leq \alpha$ and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\, \mathrm{d}s < \infty. \tag{3}$$

In this work, by using the Riccati transformations, the integral averaging technique and comparison principles, we establish a new oscillation criterion for a class of higher-order neutral delay differential Equations (1). This theorem complements and improves results reported in [26]. An illustrative example is provided.

In the sequel, all occurring functional inequalities are assumed to hold eventually; that is, they are satisfied for all t large enough.

2. Main Results

In this section, we establish some oscillation criteria for Equation (1). For convenience, we denote that $F_+(t) := \max\{0, F(t)\}$,

$$B(t) := \frac{1}{(n-4)!} \int_t^{\infty} (\theta - t)^{n-4} \left(\frac{\int_\theta^\infty q(s) \left(\frac{\tau(s)}{s}\right)^{p-1} ds}{r(\theta)} \right)^{1/(p-1)} d\theta$$

and

$$D(s) := \frac{r(s)\delta(s)|h(t,s)|^p}{p^p \left[H(t,s) A(s) \mu \frac{s^{n-2}}{(n-2)!} \right]^{p-1}}.$$

We begin with the following lemmas.

Lemma 1 (Agarwal [1]). *Let $y(t) \in C^m[t_0,\infty)$ be of constant sign and $y^{(m)}(t) \neq 0$ on $[t_0,\infty)$ which satisfies $y(t)y^{(m)}(t) \leq 0$. Then,*

(**I**) *There exists a $t_1 \geq t_0$ such that the functions $y^{(i)}(t)$, $i = 1,2,...,m-1$ are of constant sign on $[t_0,\infty)$;*

(**II**) *There exists a number* $k \in \{1, 3, 5, ..., m-1\}$ *when* m *is even,* $k \in \{0, 2, 4, ..., m-1\}$ *when* m *is odd, such that, for* $t \geq t_1$,

$$y(t)\, y^{(i)}(t) > 0,$$

for all $i = 0, 1, ..., k$ *and*

$$(-1)^{m+i+1} y(t)\, y^{(i)}(t) > 0,$$

for all $i = k+1, ..., m$.

Lemma 2 (Kiguradze [15]). *If the function* y *satisfies* $y^{(j)} > 0$ *for all* $j = 0, 1, ..., m$, *and* $y^{(m+1)} < 0$, *then*

$$\frac{m!}{t^m} y(t) - \frac{(m-1)!}{t^{m-1}} y'(t) \geq 0.$$

Lemma 3 (Bazighifan [7]). *Let* $h \in C^m([t_0, \infty), (0, \infty))$. *Suppose that* $h^{(m)}(t)$ *is of a fixed sign, on* $[t_0, \infty)$, $h^{(m)}(t)$ *not identically zero, and that there exists a* $t_1 \geq t_0$ *such that, for all* $t \geq t_1$,

$$h^{(m-1)}(t)\, h^{(m)}(t) \leq 0.$$

If we have $\lim_{t\to\infty} h(t) \neq 0$, *then there exists* $t_\lambda \geq t_0$ *such that*

$$h(t) \geq \frac{\lambda}{(m-1)!} t^{m-1} \left| h^{(m-1)}(t) \right|,$$

for every $\lambda \in (0, 1)$ *and* $t \geq t_\lambda$.

Lemma 4. *Let* $n \geq 4$ *be even, and assume that* y *is an eventually positive solution of Equation (1). If (2) holds, then there exists two possible cases for* $t \geq t_1$, *where* $t_1 \geq t_0$ *is sufficiently large:*

$$\begin{array}{ll} (C_1) & y'(t) > 0, y''(t) > 0, \ y^{(n-1)}(t) > 0, \ y^{(n)}(t) < 0, \\ (C_2) & y^{(j)}(t) > 0, y^{(j+1)}(t) < 0 \text{ for all odd integer} \\ & j \in \{1, 2, ..., n-3\}, y^{(n-1)}(t) > 0, \ y^{(n)}(t) < 0. \end{array}$$

Proof. Let y be an eventually positive solution of Equation (1). By virtue of (1), we get

$$\left(r(t) \left| \left(y^{(n-1)}(t) \right) \right|^{p-2} y^{(n-1)}(t) \right)' < 0. \tag{4}$$

From ([11] Lemma 4), we have that $y^{(n-1)}(t) > 0$ eventually. Then, we can write (4) in the from

$$\left(r(t) \left(y^{(n-1)}(t) \right)^{p-1} \right)' < 0,$$

which gives

$$r'(t) \left(y^{(n-1)}(t) \right)^{p-1} + r(t)(p-1) \left(y^{(n-1)}(t) \right)^{p-2} y^{(n)}(t) < 0.$$

Thus, $y^{(n)}(t) < 0$ eventually. Thus, by Lemma 1, we have two possible cases (C_1) and (C_2). This completes the proof. □

Lemma 5. *Let* y *be an eventually positive solution of Equation (1) and assume that Case* (C_1) *holds. If*

$$\omega(t) := \delta(t) \left(\frac{r(t) \left| \left(y^{(n-1)}(t) \right) \right|^{p-1}}{y^{p-1}(t)} \right), \tag{5}$$

where $\delta \in C^1([t_0, \infty), (0, \infty))$, *then*

$$\omega'(t) \leq \frac{\delta'_+(t)}{\delta(t)}\omega(t) - \delta(t) q(t) \left(\frac{\tau^{n-1}(t)}{t^{n-1}}\right)^{p-1} - \frac{(p-1)\mu t^{n-2}}{(n-2)!(\delta(t) r(t))^{1/(p-1)}}\omega^{p/(p-1)}(t). \tag{6}$$

Proof. Let y be an eventually positive solution of Equation (1) and assume that Case (C_1) holds. From the definition of ω, we see that $\omega(t) > 0$ for $t \geq t_1$, and

$$\begin{aligned}\omega'(t) \quad \leq \quad & \delta'(t) \frac{r(t) \left|\left(y^{(n-1)}(t)\right)\right|^{p-1}}{y^{p-1}(t)} + \delta(t) \frac{\left(r(t) \left|\left(y^{(n-1)}(t)\right)\right|^{p-1}\right)'}{y^{p-1}(t)} \\ & -\delta(t) \frac{(p-1) y'(t) r(t) \left|\left(y^{(n-1)}(t)\right)\right|^{p-1}}{y^p(t)}.\end{aligned}$$

Using Lemma 3 with $m = n-1$, $h(t) = y'(t)$, we get

$$y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} y^{(n-1)}(t), \tag{7}$$

for every constant $\mu \in (0,1)$. From (5) and (7), we obtain

$$\begin{aligned}\omega'(t) \quad \leq \quad & \delta'(t) \frac{r(t)\left|\left(y^{(n-1)}(t)\right)\right|^{p-1}}{y^{p-1}(t)} + \delta(t) \frac{\left(r(t)\left|\left(y^{(n-1)}(t)\right)\right|^{p-1}\right)'}{y^{p-1}(t)} \\ & -\delta(t) \frac{(p-1)\mu t^{n-2}}{(n-2)!} \frac{r(t)\left|\left(y^{(n-1)}(t)\right)\right|^p}{y^p(t)}.\end{aligned} \tag{8}$$

By Lemma 2, we have

$$\frac{y(t)}{y'(t)} \geq \frac{t}{n-1}.$$

Integrating this inequality from $\tau(t)$ to t, we obtain

$$\frac{y(\tau(t))}{y(t)} \geq \frac{\tau^{n-1}(t)}{t^{n-1}}. \tag{9}$$

Combining (1) and (8), we get

$$\begin{aligned}\omega'(t) \quad \leq \quad & \delta'(t) \frac{r(t)\left|\left(y^{(n-1)}(t)\right)\right|^{p-1}}{y^{p-1}(t)} - \delta(t) \frac{q(t)\left(y^{(p-1)}(\tau(t))\right)}{y^{p-1}(t)} \\ & -\delta(t) \frac{(p-1)\mu t^{n-2}}{(n-2)!} \frac{r(t)\left|\left(y^{(n-1)}(t)\right)\right|^p}{y^p(t)}.\end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$\omega'(t) \leq \frac{\delta'_+(t)}{\delta(t)}\omega(t) - \delta(t) q(t) \left(\frac{\tau^{n-1}(t)}{t^{n-1}}\right)^{p-1} - \frac{(p-1)\mu t^{n-2}}{(n-2)!(\delta(t) r(t))^{1/(p-1)}}\omega^{p/(p-1)}(t). \tag{11}$$

It follows from (11) that

$$\delta(t) q(t) \left(\frac{\tau^{n-1}(t)}{t^{n-1}}\right)^{p-1} \leq \frac{\delta'_+(t)}{\delta(t)}\omega(t) - \omega'(t) - \frac{(p-1)\mu t^{n-2}}{(n-2)!(\delta(t) r(t))^{1/(p-1)}}\omega^{p/(p-1)}(t).$$

This completes the proof. □

Lemma 6. *Let y be an eventually positive solution of Equation (1) and assume that Case (C_2) holds. If*

$$\psi(t) := \sigma(t)\frac{y'(t)}{y(t)}, \tag{12}$$

where $\sigma \in C^1([t_0,\infty),(0,\infty))$, then

$$\sigma(t)B(t) \le -\psi'(t) + \frac{\sigma'(t)}{\sigma(t)}\psi(t) - \frac{1}{\sigma(t)}\psi^2(t). \tag{13}$$

Proof. Let y be an eventually positive solution of Equation (1) and assume that Case (C_2) holds. Using Lemma 2, we obtain

$$y(t) \ge ty'(t).$$

Thus we find that y/t is nonincreasing, and hence

$$y(\tau(t)) \ge y(t)\frac{\tau(t)}{t}. \tag{14}$$

Since $y > 0$, (1) becomes

$$\left(r(t)\left(y^{(n-1)}(t)\right)^{p-1}\right)' + q(t)y^{p-1}(\tau(t)) = 0.$$

Integrating that equation from t to ∞, we see that

$$\lim_{t\to\infty}\left(r(t)\left(y^{(n-1)}(t)\right)^{p-1}\right) - r(t)\left(y^{(n-1)}(t)\right)^{p-1} + \int_t^\infty q(s)y^{p-2}(\tau(s)) = 0. \tag{15}$$

Since the function $r\left(y^{(n-1)}\right)^{p-1}$ is positive $\left[r > 0 \text{ and } y^{(n-1)} > 0\right]$ and nonincreasing $\left(\left(r\left(y^{(n-1)}\right)^{p-1}\right)' < 0\right)$, there exists a $t_2 \ge t_0$ such that $r\left(y^{(n-1)}\right)^{p-1}$ is bounded above for all $t \ge t_2$, and so $\lim_{t\to\infty}\left(r(t)\left(y^{(n-1)}(t)\right)^{p-1}\right) = c \ge 0$. Then, from (15), we obtain

$$-r(t)\left(y^{(n-1)}(t)\right)^{p-1} + \int_t^\infty q(s)y^{p-2}(\tau(s)) \le -c \le 0.$$

From (14), we obtain

$$-r(t)\left(y^{(n-1)}(t)\right)^{p-1} + \int_t^\infty q(s)y(s)^{p-1}\frac{\tau(s)^{p-1}}{s^{p-1}}ds \le 0.$$

It follows from $y'(t) > 0$ that

$$-y^{(n-1)}(t) + \frac{y(t)}{r^{1/(p-1)}(t)}\left(\int_t^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p-1}ds\right)^{1/(p-1)} \le 0.$$

Integrating the above inequality from t to ∞ for a total of $(n-3)$ times, we get

$$y''(t) + \frac{\int_t^\infty(\theta-t)^{n-4}\left(\frac{\int_\theta^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p-1}ds}{r(\theta)}\right)^{1/(p-1)}d\theta}{(n-4)!}y(t) \le 0. \tag{16}$$

From the definition of $\psi(t)$, we see that $\psi(t) > 0$ for $t \geq t_1$, and

$$\psi'(t) = \sigma'(t)\frac{y'(t)}{y(t)} + \sigma(t)\frac{y''(t)\,y(t) - (y'(t))^2}{y^2(t)}. \tag{17}$$

It follows from (16) and (17) that

$$\sigma(t)\,B(t) \leq -\psi'(t) + \frac{\sigma'(t)}{\sigma(t)}\psi(t) - \frac{1}{\sigma(t)}\psi^2(t).$$

This completes the proof. □

Definition 1. *Let*

$$D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\} \text{ and } D_0 = \{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\}.$$

We say that a function $H \in C(D,\mathbb{R})$ belongs to the class $\Re$ if
($\mathbf{i}_1$) *$H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$, $(t,s) \in D_0$.*
($\mathbf{i}_2$) *H has a nonpositive continuous partial derivative $\partial H/\partial s$ on D_0 with respect to the second variable.*

Theorem 1. *Let $n \geq 4$ be even. Assume that there exist functions $H, H_* \in \Re$, $\delta, A, \sigma, A_* \in C^1([t_0,\infty),(0,\infty))$ and $h, h_* \in C(D_0,\mathbb{R})$ such that*

$$-\frac{\partial}{\partial s}\left(H(t,s)\,A(s)\right) = H(t,s)\,A(s)\frac{\delta'(t)}{\delta(t)} + h(t,s). \tag{18}$$

and

$$-\frac{\partial}{\partial s}\left(H_*(t,s)\,A_*(s)\right) = H_*(t,s)\,A_*(s)\frac{\sigma'(t)}{\sigma(t)} + h_*(t,s). \tag{19}$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1} - D(s)\right]ds = \infty, \tag{20}$$

for some constant $\mu \in (0,1)$ and

$$\limsup_{t\to\infty}\frac{1}{H_*(t,t_0)}\int_{t_0}^{t}\left(H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s) - \frac{\sigma(s)\,|h_*(t,s)|^2}{4H_*(t,s)\,A_*(s)}\right)ds = \infty, \tag{21}$$

then every solution of (1) is oscillatory.

Proof. Let y be a nonoscillatory solution of Equation (1) on the interval $[t_0,\infty)$. Without loss of generality, we can assume that y is an eventually positive. By Lemma 4, there exist two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Assume that (C_1) holds. From Lemma 5, we get that (6) holds. Multiplying (6) by $H(t,s)\,A(s)$ and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1} ds \\ \leq -\int_{t_1}^{t} H(t,s)\,A(s)\,\omega'(s)\,ds + \int_{t_1}^{t} H(t,s)\,A(s)\frac{\delta'(s)}{\delta(s)}\omega(s)\,ds \\ -\int_{t_1}^{t} H(t,s)\,A(s)\frac{(p-1)\,\mu s^{n-2}}{(n-2)!\,(\delta(s)\,r(s))^{1/(p-1)}}\omega^{p/(p-1)}(s)\,ds\end{aligned}$$

Thus

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1} ds\\&\quad\leq H(t,t_1)\,A(t_1)\,\omega(t_1)-\int_{t_1}^{t}\left(-\frac{\partial}{\partial s}\left(H(t,s)\,A(s)\right)-H(t,s)\,A(s)\,\frac{\delta'(t)}{\delta(t)}\right)\omega(s)\,ds\\&\qquad-\int_{t_1}^{t} H(t,s)\,A(s)\,\frac{(p-1)\,\mu s^{n-2}}{(n-2)!\,(\delta(s)\,r(s))^{1/(p-1)}}\omega^{p/(p-1)}(s)\,ds\end{aligned}$$

This implies

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1} ds\\&\quad\leq H(t,t_1)\,A(t_1)\,\omega(t_1)+\int_{t_1}^{t}|h(t,s)|\,\omega(s)\,d(s)\\&\qquad-\int_{t_1}^{t} H(t,s)\,A(s)\,\frac{(p-1)\,\mu s^{n-2}}{(n-2)!\,(\delta(s)\,r(s))^{1/(p-1)}}\omega^{p/(p-1)}(s)\,ds.\end{aligned}\tag{22}$$

Using the inequality

$$\beta U V^{\beta-1}-U^{\beta}\leq(\beta-1)\,V^{\beta},\quad \beta>1,\ U\geq 0 \text{ and } V\geq 0,\tag{23}$$

with $\beta=p/(p-1)$,

$$U=\left((p-1)\,H(t,s)\,A(s)\,\frac{\mu s^{n-2}}{(n-2)!}\right)^{(p-1)/p}\frac{\omega(s)}{(\delta(s)\,r(s))^{1/p}}$$

and

$$V=\left(\frac{p-1}{p}\right)^{p-1}|h(t,s)|^{p-1}\left(\frac{\delta(s)\,r(s)}{\left((p-1)\,H(t,s)\,A(s)\,\frac{\mu s^{n-2}}{(n-2)!}\right)^{p-1}}\right)^{(p-1)/p},$$

we get

$$\begin{aligned}&|h(t,s)|\,\omega(s)-H(t,s)\,A(s)\,\frac{(p-1)\,\mu s^{n-2}}{(n-2)!\,(\delta(s)\,r(s))^{1/(p-1)}}\omega^{p/(p-1)}\\&\qquad\leq\frac{\delta(s)\,r(s)}{\left(H(t,s)\,A(s)\,\frac{\mu s^{n-2}}{(n-2)!}\right)^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p},\end{aligned}$$

which with (23) gives

$$\begin{aligned}\int_{t_1}^{t}\left(H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1}-D(s)\right)ds&\leq H(t,t_1)\,A(t_1)\,\omega(t_1)\\&\leq H(t,t_0)\,A(t_1)\,\omega(t_1).\end{aligned}$$

Then

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1}-D(s)\right)ds$$
$$\leq A(t_1)\,\omega(t_1)+\int_{t_0}^{t_1}A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1}ds$$
$$<\infty,$$

for some $\mu\in(0,\ 1)$, which contradicts (20).

Assume that Case (C_2) holds. From Lemma 6, we get that (13) holds. Multiplying (13) by $H_*(t,s)\,A_*(s)$, and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned}\int_{t_1}^{t}H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s)\,ds &\leq -\int_{t_1}^{t}H_*(t,s)\,A_*(s)\,\psi'(s)\,ds+\int_{t_1}^{t}H_*(t,s)\,A_*(s)\,\frac{\sigma'(s)}{\sigma(s)}\psi(s)\,ds\\ &\quad -\int_{t_1}^{t}\frac{H_*(t,s)\,A_*(s)}{\sigma(s)}\psi^2(s)\,ds\\ &= H_*(t,t_1)\,A_*(t_1)\,\psi(t_1)-\int_{t_1}^{t}\frac{H_*(t,s)\,A_*(s)}{\sigma(s)}\psi^2(s)\,ds\\ &\quad -\int_{t_1}^{t}\left(-\frac{\partial}{\partial s}(H_*(t,s)\,A_*(s))-H_*(t,s)\,A_*(s)\,\frac{\sigma'(t)}{\sigma(t)}\right)\psi(s)\,ds.\end{aligned}$$

Then

$$\begin{aligned}\int_{t_1}^{t}H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s)\,ds &\leq H_*(t,t_1)\,A_*(t_1)\,\psi(t_1)+\int_{t_1}^{t}|h_*(t,s)|\,\psi(s)\,d(s)\\ &\quad -\int_{t_1}^{t}\frac{H_*(t,s)\,A_*(s)}{\sigma(s)}\psi^2(s)\,ds.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_{t_1}^{t}\left(H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s)-\frac{\sigma(s)\,|h_*(t,s)|^2}{4H_*(t,s)\,A_*}\right)ds &\leq H_*(t,t_1)\,A_*(t_1)\,\psi(t_1)\\ &\leq H_*(t,t_0)\,A_*(t_1)\,\psi(t_1).\end{aligned}$$

This implies

$$\frac{1}{H_*(t,t_0)}\int_{t_0}^{t}\left(H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s)-\frac{\sigma(s)\,|h_*(t,s)|^2}{4H_*(t,s)\,A_*}\right)ds$$
$$\leq A_*(t_1)\,\psi(t_1)+\int_{t_0}^{t}A_*(s)\,\sigma(s)\,B(s)\,ds<\infty$$

which contradicts (21). Therefore, every solution of (1) is oscillatory. □

In the next theorem, we establish new oscillation results for Equation (1) by using the comparison technique with the first-order differential inequality:

Theorem 2. *Let $n\geq 2$ be even and $r'(t)>0$. Assume that for some constant $\lambda\in(0,1)$, the differential equation*

$$\varphi'(t)+\frac{q(t)}{r(\tau(t))}\left(\frac{\lambda\tau^{n-1}(t)}{(n-1)!}\right)^{p-1}\varphi(\tau(t))=0 \tag{24}$$

is oscillatory. Then every solution of (1) is oscillatory.

Proof. Let (1) have a nonoscillatory solution y. Without loss of generality, we can assume that $y(t) > 0$ for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large. Since $r'(t) > 0$, we have

$$y'(t) > 0,\ y^{(n-1)}(t) > 0 \text{ and } y^{(n)}(t) < 0. \tag{25}$$

From Lemma 3, we get

$$y(t) \geq \frac{\lambda t^{n-1}}{(n-1)! r^{1/p-1}(t)} r^{1/p-1}(t)\, y^{(n-1)}(t), \tag{26}$$

for every $\lambda \in (0,1)$. Thus, if we set

$$\varphi(t) = r(t)\left[y^{(n-1)}(t)\right]^{p-1} > 0,$$

then we see that φ is a positive solution of the inequality

$$\varphi'(t) + \frac{q(t)}{r(\tau(t))}\left(\frac{\lambda \tau^{n-1}(t)}{(n-1)!}\right)^{p-1} \varphi(\tau(t)) \leq 0. \tag{27}$$

From [22] (Theorem 1), we conclude that the corresponding Equation (24) also has a positive solution, which is a contradiction.

Theorem 2 is proved. □

Corollary 1. *Assume that (2) holds and let $n \geq 2$ be even. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{q(s)}{r(\tau(s))}\left(\tau^{n-1}(s)\right)^{p-1} ds > \frac{((n-1)!)^{p-1}}{e}, \tag{28}$$

then every solution of (1) is oscillatory.

Next, we give the following example to illustrate our main results.

Example 1. *Consider the equation*

$$y^{(4)}(t) + \frac{\gamma}{t^4} y\left(\frac{9}{10}t\right) = 0,\ t \geq 1, \tag{29}$$

where $\gamma > 0$ is a constant. We note that $n = 4,\ r(t) = 1,\ p = 2,\ \tau(t) = 9t/10$ and $q(t) = \gamma/t^4$. If we set $H(t,s) = H_*(t,s) = (t-s)^2$, $A(s) = A_*(s) = 1$, $\delta(s) = t^3$, $\sigma(s) = t$, $h(t,s) = (t-s)\left(5 - 3ts^{-1}\right)$ and $h_*(t,s) = (t-s)\left(3 - ts^{-1}\right)$ then we get

$$\eta(s) = \int_{t_0}^{\infty} \frac{1}{r^{1/(p-1)}(s)} ds = \infty$$

and

$$\begin{aligned} B(t) &= \frac{1}{(n-4)!}\int_t^{\infty} (\theta - t)^{n-4} \left(\frac{\int_\theta^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{p-1} ds}{r(\theta)}\right)^{1/(p-1)} d\theta \\ &= 3\gamma/\left(20t^2\right). \end{aligned}$$

Hence conditions (20) and (21) become

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)\,A(s)\,\delta(s)\,q(s)\left(\frac{\tau^{n-1}(s)}{s^{n-1}}\right)^{p-1}-D(s)\right)ds$$
$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{729\gamma}{1000}t^2s^{-1}+\frac{729\gamma}{1000}s-\frac{729\gamma}{500}t-\frac{s}{2\mu}\left(25+9t^2s^{-2}-30ts^{-1}\right)\right]ds$$
$$=\infty\quad(\text{if }\gamma>500/81)$$

and

$$\limsup_{t\to\infty}\frac{1}{H_*(t,t_0)}\int_{t_0}^{t}\left(H_*(t,s)\,A_*(s)\,\sigma(s)\,B(s)-\frac{\sigma(s)\,|h_*(t,s)|^2}{4H_*(t,s)\,A_*(s)}\right)ds$$
$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{3\gamma}{20}t^2s^{-1}+\frac{3\gamma}{20}s-\frac{3\gamma}{10}t-\frac{s}{4}\left(9-630ts^{-1}+t^2s^{-2}\right)\right]ds$$
$$=\infty\quad(\text{if }\gamma>5/3).$$

Thus, by Theorem 1, every solution of Equation (29) is oscillatory if $\gamma > 500/81$.

3. Conclusions

In this work, we have discussed the oscillation of the higher-order differential equation with a p-Laplacian-like operator and we proved that Equation (1) is oscillatory by using the following methods:

1. The Riccati transformation technique.
2. Comparison principles.
3. The Integral averaging technique.

Additionally, in future work we could try to get some oscillation criteria of Equation (1) under the condition $\int_{t_0}^{\infty}\frac{1}{r^{1/(p-1)}(t)}dt<\infty$. Thus, we would discuss the following two cases:

$$(C_1)\quad y(t)>0,\ y^{(n-1)}(t)>0,\ y^{(n)}(t)<0,$$
$$(C_2)\quad y(t)>0,\ y^{(n-2)}(t)>0,\ y^{(n-1)}(t)<0.$$

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