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On New Extensions of Darbo's Fixed Point Theorem with Applications

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Abstract: In this paper, we extend Darbo's fixed point theorem via weak JS-contractions in a Banach space. Our results generalize and extend several well-known comparable results in the literature. The technique of *measure of non-compactness* is the main tool in carrying out our proof. As an application, we study the existence of solutions for a system of integral equations. Finally, we present a concrete example to support the effectiveness of our results.

Keywords: fixed point; complete metric space; measure of non-compactness; integral equation

1. Introduction and Preliminaries

The notion of a measure of non-compactness (shortly, MNC) was introduced by Kuratowski [1] in 1930. This concept is a very useful tool in functional analysis, such as in metric fixed point theory and operator equation theory in Banach spaces. This notion is also applied in studies of the existence of solutions of ODE and PDE problems, integral and integro-differential equations, etc. For more details, we refer the reader to [2–6]. In [7], the authors generalized the Darbo's fixed point theorem via the concept of the class of operators $O(f;.)$.

The aim of this paper is to generalize the Darbo's fixed point theorem via weak JS-contractions in a Banach space and to study the existence of solutions for the following system of integral equations:

$$\begin{cases} \kappa_1(\eta) = f\left(\eta, \kappa_1(\alpha(\eta)), \kappa_2(\alpha(\eta)), \int_0^{\beta(\eta)} g\left(\eta, \varrho, \kappa_1(\alpha(\varrho)), \kappa_2(\alpha(\varrho))\right) d\varrho\right) \\ \kappa_2(\eta) = f\left(\eta, \kappa_2(\alpha(\eta)), \kappa_1(\alpha(\eta)), \int_0^{\beta(\eta)} g\left(\eta, \varrho, \kappa_2(\alpha(\varrho)), \kappa_1(\alpha(\varrho))\right) d\varrho\right) \end{cases} \tag{1}$$

where $\eta \in [0, T]$.

We introduce some notations and definitions which are used throughout this paper. Let $\mathbb{R}$ denotes the set of real numbers and let $\mathbb{R}_+ = [0, +\infty)$. Let $(\mathcal{E}, \|\cdot\|)$ be a real Banach space. Moreover, $\overline{B}(\eta, r)$

denotes the closed ball centered at η with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For Λ, a nonempty subset of $\mathcal{E}$, we denote by $\overline{\Lambda}$ and $Conv\Lambda$ the closure and the closed convex hull of Λ, respectively. Furthermore, let us denote by $\mathfrak{M}_{\mathcal{E}}$ the family of nonempty bounded subsets of $\mathcal{E}$, and by $\mathfrak{N}_{\mathcal{E}}$, its subfamily consisting of all relatively compact subsets of $\mathcal{E}$.

Definition 1 ([8]). *A mapping $\mu : \mathfrak{M}_{\mathcal{E}} \longrightarrow \mathbb{R}_+$ is said to be a measure of non-compactness in $\mathcal{E}$ if it satisfies the following conditions:*

1° *The family $ker\mu = \{\Lambda \in \mathfrak{M}_{\mathcal{E}} : \mu(\Lambda) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_{\mathcal{E}}$;*
2° $\Lambda \subset \Sigma \Longrightarrow \mu(\Lambda) \leq \mu(\Sigma)$;
3° $\mu(\overline{\Lambda}) = \mu(\Lambda)$;
4° $\mu(Conv\Lambda) = \mu(\Lambda)$;
5° $\mu(\lambda\Lambda + (1-\lambda)\Sigma) \leq \lambda\mu(\Lambda) + (1-\lambda)\mu(\Sigma)$ *for all* $\lambda \in [0,1]$;
6° *if $\{\Lambda_n\}$ is a sequence of closed subsets from $\mathfrak{M}_{\mathcal{E}}$ such that $\Lambda_{n+1} \subset \Lambda_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(\Lambda_n) = 0$, then $\Lambda_\infty = \cap_{n=1}^{\infty} \Lambda_n \neq \emptyset$.*

We denote by Θ the set of all functions $\theta : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

θ_1. θ is a continuous strictly increasing function;
θ_2. for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$.

Let Φ be the class of all functions $\phi : [1,\infty) \to [0,\infty)$ satisfying the following properties:

ϕ_1. ϕ is lower semi-continuous;
ϕ_2. $\phi(1) = 0$.
ϕ_3. for each sequence $\{t_n\} \subseteq (1,\infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 1$.

Remark 1 ([9]). *It is clear that $f(t) = t - \sqrt[n]{t}$ $(n \geq 1)$ belongs to Φ. Other examples are $f(t) = e^{t-1} - 1$ and $f(t) = \ln t$.*

Definition 2 ([9]). *Let (Λ, d) be a metric space and Υ be a self-mapping on Λ. We say that Υ is a weak JS-contraction if, for all $\eta, \varrho \in \Lambda$ with $d(\Upsilon\eta, \Upsilon\varrho) > 0$, we have*

$$\theta\big(d(\Upsilon\eta, \Upsilon\varrho)\big) \leq \theta\big(d(\eta,\varrho)\big) - \phi\big(\theta(d(\eta,\varrho))\big) \tag{2}$$

where $\phi \in \Phi$ and $\theta \in \Theta$.

Theorem 1 ([9]). *Let (Λ, d) be a complete metric space. Let $\Upsilon : \Lambda \to \Lambda$ be a self-mapping satisfying the following assertions:*

(i) Υ *is a weak JS-contraction,*
(ii) Υ *is continuous.*

Then, Υ has a unique fixed point.

Now we recall two important theorems playing a key role in the fixed point theory.

Theorem 2 ([10]). *Let $\mathcal{C}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{E}$. Then, each continuous and compact map $\Upsilon : \mathcal{C} \to \mathcal{C}$ has at least one fixed point in the set $\mathcal{C}$.*

Obviously, the above theorem is the well-known Schauder fixed point principle. Its generalization, called Darbo's fixed point theorem, is arranged as follows.

Theorem 3 ([11]). *Let $\mathcal{C}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon : \mathcal{C} \to \mathcal{C}$ be a continuous mapping. Assume that there exists a constant $K \in [0,1)$ such that $\mu(\Upsilon\Lambda) \leq K\mu(\Lambda)$ for any nonempty subset Λ of $\mathcal{C}$, where μ is an MNC defined in $\mathcal{E}$. Then, Υ has at least one fixed point in $\mathcal{C}$.*

2. Main Results

Now, we state one of the main results in this article, which extends and generalizes Darbo's fixed point theorem by using the concept of weak JS-contractions.

Theorem 4. *Let $\mathcal{C}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon : \mathcal{C} \to \mathcal{C}$ be a continuous operator such that*

$$\theta\big(\mu(\Upsilon(\Lambda))\big) \leq \theta\big(\mu(\Lambda)\big) - \phi\big(\theta(\mu(\Lambda))\big) \tag{3}$$

for all $\Lambda \subseteq \mathcal{C}$, where $\phi \in \Phi$, $\theta \in \Theta$, and μ is an arbitrary MNC. Then, Υ has at least one fixed point in $\mathcal{C}$.

Proof. Define a sequence $\{\mathcal{C}_n\}$ such that $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_{n+1} = \overline{Conv}(\Upsilon(\mathcal{C}_n))$ for all $n \in \mathbb{N}$.

If there exists an integer $N \in \mathbb{N}$ such that $\mu(\mathcal{C}_N) = 0$, then $\mathcal{C}_N$ is relatively compact, and the Schauder fixed point theorem implies that Υ has a fixed point. So, we assume that $\mu(\mathcal{C}_N) > 0$ for each $n \in \mathbb{N}$.

It is clear that $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed and convex sets such that

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_n \supseteq \mathcal{C}_{n+1}.$$

We know that $\{\mu(\mathcal{C}_n)\}_{n\in\mathbb{N}}$ is a positive decreasing and bounded-below sequence of real numbers. Thus, $\{\mu(\mathcal{C}_n)\}_{n\in\mathbb{N}}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \mu(\mathcal{C}_n) = r$. Also, we have

$$\mu(\mathcal{C}_{n+1}) = \mu(\overline{Conv}(\Upsilon(\mathcal{C}_n))) = \mu(\Upsilon(\mathcal{C}_n)).$$

In view of condition (3), we have

$$\begin{aligned} \theta(\mu(\mathcal{C}_{n+1})) &= \theta(\mu(\Upsilon(\mathcal{C}_n))) \\ &\leq \theta(\mu(\mathcal{C}_n)) - \phi(\theta(\mu(\mathcal{C}_n))). \end{aligned} \tag{4}$$

Taking the limsup in the above inequality, we have

$$\limsup_{n\to\infty} \theta(\mu(\mathcal{C}_{n+1})) \leq \limsup_{n\to\infty} \theta(\mu(\mathcal{C}_n)) - \liminf_{n\to\infty} \phi(\theta(\mu(\mathcal{C}_n))).$$

Therefore,

$$\theta(r) \leq \theta(r) - \phi(\theta(r)).$$

Hence, $\phi(\theta(r))$ must be 0, which means that $\theta(r) = 1$. Consequently, $r = 0$. Therefore, $\lim_{n\to\infty} \mu(\mathcal{C}_n) = 0$. According to axiom (6°) of Definition 1, we derive that the set $\mathcal{C}_\infty = \bigcap_{n=1}^{\infty} \mathcal{C}_n$ is a nonempty, closed and convex set, that it is invariant under the operator Υ, and that it belongs to $Ker\mu$. Then, in view of the Schauder theorem, Υ has a fixed point. □

Remark 2. *We obtain the Darbo's fixed point theorem if we take $\theta(t) = e^t$, $\phi(t) = t - t^\lambda$ in Theorem 4, where $\lambda \in (0,1)$.*

In [12], Bhaskar and Lakshmikantham introduced the notion of coupled fixed point and proved some coupled fixed point theorems for some mappings, and discussed the existence and uniqueness of solutions for periodic boundary value problems.

Definition 3 ([12]). *An element $(\eta,\varrho)\in\mathcal{E}\times\mathcal{E}$ is called a coupled fixed point of the mapping $\Upsilon:\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ if $\Upsilon(\eta,\varrho)=\eta$ and $\Upsilon(\varrho,\eta)=\varrho$.*

Theorem 5 ([6]). *Suppose that $\mu_1,\mu_2,\ldots,\mu_n$ are measures of non-compactness in Banach spaces $\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n$, respectively. Moreover, assume that the function $\Gamma:[0,\infty)^n\longrightarrow[0,\infty)$ is convex and $\Gamma(\eta_1,\ldots,\eta_n)=0$ if and only if $\eta_i=0$ for $i=1,2,\ldots,n$. Then*

$$\tilde{\mu}(\Lambda)=\Gamma(\mu_1(\Lambda_1),\mu_2(\Lambda_2),\ldots,\mu_n(\Lambda_n))$$

defines a measure of non-compactness in $\mathcal{E}_1\times\mathcal{E}_2\times\ldots\times\mathcal{E}_n$, where Λ_i denotes the natural projection of Λ into $\mathcal{E}_i$, for $i=1,2,\ldots,n$.

From now on, we assume that θ is a subadditive mapping. For instance, a concave function $f:[0,\infty)\to[0,\infty)$ with $f(0)\geq 0$ is a subadditive function. $\theta(t)=\sqrt{t+1}$, $\theta(t)=\sqrt{t}+1$ and $\theta(t)=1+\ln(t+1)$ are some subadditive functions which belong to Θ.

Theorem 6. *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ be a continuous function such that*

$$\theta\big[\mu(\Upsilon(\Lambda_1\times\Lambda_2))\big]\leq\frac{1}{2}\big[\theta(\mu(\Lambda_1)+\mu(\Lambda_2))-\phi(\theta(\mu(\Lambda_1)+\mu(\Lambda_2)))\big] \tag{5}$$

for all subsets Λ_1,Λ_2 of $\mathcal{C}$, where μ is an arbitrary MNC and θ,ϕ are as in Theorem 4. In addition, we assume that θ is a subadditive mapping. Then, Υ has at least one coupled fixed point.

Proof. We define the mapping $\widetilde{\Upsilon}:\mathcal{C}^2\to\mathcal{C}^2$ by

$$\widetilde{\Upsilon}(\eta,\varrho)=(\Upsilon(\eta,\varrho),\Upsilon(\varrho,\eta)),$$

where $\mathcal{C}^2=\mathcal{C}\times\mathcal{C}$. It is clear that $\widetilde{\Upsilon}$ is continuous. We show that $\widetilde{\Upsilon}$ satisfies all of the conditions of Theorem 4. Let $\Lambda\subset\mathcal{C}^2$ be a nonempty subset. We know that $\widetilde{\mu}(\Lambda)=\mu(\Lambda_1)+\mu(\Lambda_2)$ is an MNC (see, [8]), where Λ_1 and Λ_2 denote the natural projections of Λ into $\mathcal{E}$. From (5), we have

$$\begin{aligned}\theta[\widetilde{\mu}(\widetilde{\Upsilon}(\Lambda))]&\leq\theta(\widetilde{\mu}(\Upsilon(\Lambda_1\times\Lambda_2)\times\Upsilon(\Lambda_2\times\Lambda_1)))\\&=\theta[\mu(\Upsilon(\Lambda_1\times\Lambda_2))+\mu(\Upsilon(\Lambda_2\times\Lambda_1))]\\&\leq\frac{1}{2}[\theta(\mu(\Lambda_1)+\mu(\Lambda_2))-\phi(\theta(\mu(\Lambda_1)+\mu(\Lambda_2)))]\\&\quad+\frac{1}{2}[\theta(\mu(\Lambda_2)+\mu(\Lambda_1))-\phi(\theta(\mu(\Lambda_2)+\mu(\Lambda_1)))]\\&=\theta(\widetilde{\mu}(\Lambda))-\phi(\theta(\widetilde{\mu}(\Lambda))).\end{aligned}$$

Now, from Theorem 4, we deduce that $\widetilde{\Upsilon}$ has at least one fixed point which implies that Υ has at least one coupled fixed point. □

Taking $\phi(t)=t-t^{\lambda}$ ($\lambda\in(0,1)$) in Theorem 6, we have the following result.

Corollary 1. *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ be a continuous function such that*

$$\theta[\mu(\Upsilon(\Lambda_1\times\Lambda_2))]\leq\frac{1}{2}(\theta[\mu(\Lambda_1)+\mu(\Lambda_2)])^{\lambda} \tag{6}$$

for all subsets Λ_1,Λ_2 of $\mathcal{C}$, where μ is an arbitrary MNC, $\lambda\in(0,1)$ and the subadditive mapping θ is as in Theorem 4. Then, Υ has at least one coupled fixed point.

Corollary 2. *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a continuous function such that*

$$1 + \ln[1 + \mu(\Upsilon(\Lambda_1 \times \Lambda_2))] \leq \frac{[1 + \ln(1 + \mu(\Lambda_1) + \mu(\Lambda_2))]^{\lambda}}{2} \tag{7}$$

for all subsets Λ_1, Λ_2 of $\mathcal{C}$, where μ is an arbitrary MNC and $\lambda \in (0,1)$. Then, Υ has at least one coupled fixed point.

Theorem 7. *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a continuous function such that*

$$\theta[\mu(\Upsilon(\Lambda_1 \times \Lambda_2))] \leq \theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\}) - \phi(\theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\})) \tag{8}$$

for all subsets Λ_1, Λ_2 of $\mathcal{C}$, where μ is an arbitrary MNC and θ, ϕ are as in Theorem 4. In addition, we assume that θ is a subadditive mapping. Then, Υ has at least one coupled fixed point.

Proof. We define the mapping $\widetilde{\Upsilon} : \mathcal{C}^2 \to \mathcal{C}^2$ by

$$\widetilde{\Upsilon}(\eta, \varrho) = (\Upsilon(\eta, \varrho), \Upsilon(\varrho, \eta)).$$

It is clear that $\widetilde{\Upsilon}$ is continuous. We show that $\widetilde{\Upsilon}$ satisfies all of the conditions of Theorem 4. We know that $\widetilde{\mu}(\Lambda) = \max\{\mu(\Lambda_1), \mu(\Lambda_2)\}$ is an MNC (see, [8]), where Λ_1 and Λ_2 denote the natural projections of Λ into $\mathcal{E}$. Let $\Lambda \subset \mathcal{C}^2$ be a nonempty subset. From (8), we have

$$\begin{aligned}
\theta[\widetilde{\mu}(\widetilde{\Upsilon}(\Lambda))] &\leq \theta(\widetilde{\mu}(\Upsilon(\Lambda_1 \times \Lambda_2) \times \Upsilon(\Lambda_2 \times \Lambda_1))) \\
&= \theta[\max\{\mu(\Upsilon(\Lambda_1 \times \Lambda_2)), \mu(\Upsilon(\Lambda_2 \times \Lambda_1))\}] \\
&\leq \max\{\theta[\mu(\Upsilon(\Lambda_1 \times \Lambda_2))], \theta[\mu(\Upsilon(\Lambda_2 \times \Lambda_1))]\} \\
&\leq \max\{\theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\}) - \phi(\theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\})), \\
&\quad \theta(\max\{\mu(\Lambda_2), \mu(\Lambda_1)\}) - \phi(\theta(\max\{\mu(\Lambda_2), \mu(\Lambda_1)\}))\} \\
&= \theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\}) - \phi(\theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\})) \\
&= \theta(\widetilde{\mu}(\Lambda)) - \phi(\theta(\widetilde{\mu}(\Lambda))).
\end{aligned}$$

Now, from Theorem 4, we deduce that $\widetilde{\Upsilon}$ has at least one fixed point which implies that Υ has at least one coupled fixed point. □

Corollary 3. *Let $\mathcal{C}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{E}$ and let $\Upsilon : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a continuous function such that*

$$\theta[\mu(\Upsilon(\Lambda_1 \times \Lambda_2))] \leq (\theta[\max\{\mu(\Lambda_1), \mu(\Lambda_2)\}])^{\lambda} \tag{9}$$

for all subsets Λ_1, Λ_2 of $\mathcal{C}$, where μ is an arbitrary MNC, $\lambda \in (0,1)$ and the subadditive mapping θ is as in Theorem 4. Then, Υ has at least one coupled fixed point.

3. Application

In this section, as an application of Corollary 3, we study the existence of solutions for the system of integral Equation (1).

Let the space $\Xi := \mathcal{C}(\mathcal{J}, \mathbb{R})(\mathcal{J} = [0, T])$ consists of all real valued functions which are bounded and continuous on $\mathcal{J}$ equipped with the standard norm

$$\|\eta\| = \sup\{|\eta(t)| : t \in \mathcal{J}\}.$$

Recall that the modulus of continuity of a function $\eta \in \Xi$ is defined by

$$\omega(\eta, \epsilon) = \sup\{|\eta(t) - \eta(s)| : t, s \in \mathcal{J},\ |t - s| \leq \epsilon\}.$$

Since η is uniformly continuous on $\mathcal{J}$, then $\omega(\eta, \epsilon) \to 0$ as $\epsilon \to 0$, and the Hausdorff measure of non-compactness for all bounded sets Ω of Ξ is

$$\chi(\Omega) = \lim_{\epsilon \to 0} \Big\{ \sup_{\eta \in \Omega} \omega(\eta, \epsilon) \Big\}.$$

(See [6], for more details).

Theorem 8. *Suppose that the following assumptions are satisfied:*

(i) $\alpha, \beta : \mathcal{J} \longrightarrow \mathcal{J}$ *are continuous functions;*
(ii) *the function* $f : \mathcal{J} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is continuous and*

$$\theta(|f(\eta, \kappa_1, \kappa_2, \varrho) - f(\eta, \nu_1, \nu_2, \zeta)|) \leq (\theta(\max\{|\kappa_1 - \nu_1|, |\kappa_2 - \nu_2|\} + |\varrho - \zeta|))^{\lambda},$$

where $\lambda \in (0, 1)$ *and* θ *is a subadditive mapping such that* $\theta \in \Theta$*;*
(iii) $N := \sup\{|f(\eta, 0, 0, 0)| : \eta \in \mathcal{J}\}$*;*
(iv) $g : \mathcal{J} \times \mathcal{J} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous and*

$$G := \sup\Big\{ \left| \int_0^{\beta(\eta)} g(\eta, \varrho, \kappa_1(\alpha(\varrho)), \kappa_2(\alpha(\varrho))) d\varrho \right| : \eta, \varrho \in \mathcal{J},\ \kappa_1, \kappa_2 \in \Xi \Big\};$$

(iv) *there exists a positive solution* r_0 *to the inequality*

$$\theta(r + G)^{\lambda} + \theta(N) \leq r.$$

Then, the system of the integral Equation (1) *has at least one solution in the space* Ξ^2*.*

Proof. Let us consider the operator

$$\Upsilon : \Xi \times \Xi \longrightarrow \Xi$$

with the formula

$$\begin{aligned} \Upsilon(\kappa_1, \kappa_2)(\eta) = f(\eta, \kappa_1(\alpha(\eta)), \kappa_2(\alpha(\eta)), \\ \int_0^{\beta(\eta)} g(\eta, \varrho, \kappa_1(\alpha(\varrho)), \kappa_2(\alpha(\varrho))) d\varrho). \end{aligned} \tag{10}$$

We observe that for any $\eta \in \mathcal{J}$, the function Υ is continuous. For arbitrary fixed $\eta \in \mathcal{J}$, by applying the assumptions (i)–(iv), we have

$$\begin{aligned}\theta(|\Upsilon(\kappa_1,\kappa_2)(\eta)|) &\leq \theta\Big(|f(\eta,\kappa_1(\alpha(\eta)),\kappa_2(\alpha(\eta)),\\ &\qquad \int_0^{\beta(\eta)} g(\eta,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))d\varrho\\ &\qquad - f(\eta,0,0,0)| + |f(\eta,0,0,0)|\Big)\\ &\leq \theta\Big(\max\{|\kappa_1(\alpha(\eta))|,|\kappa_2(\alpha(\eta))|\}\\ &\qquad + |\int_0^{\beta(\eta)} g(\eta,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))d\varrho|\Big)^{\lambda}\\ &\qquad + \theta(|f(\eta,0,0,0)|)\\ &\leq \theta\left(\max\{\|\kappa_1\|,\|\kappa_2\|\}+G\right)^{\lambda}+\theta(N).\end{aligned}$$

Therefore,

$$\theta(\|\Upsilon(\kappa_1,\kappa_2)\|) \leq \theta\left(\max\{\|\kappa_1\|,\|\kappa_2\|\}+G\right)^{\lambda}+\theta(N). \tag{11}$$

Due to the inequality (11) and using (iv), the function Υ maps $(\bar{B}_{r_0})^2$ into $(\bar{B}_{r_0})$. Now, we prove that the operator Υ is a continuous operator on $(\bar{B}_{r_0})^2$. Let us arbitrarily fix $\varepsilon > 0$ and take $(\kappa_1,\kappa_2),(\nu_1,\nu_2) \in (\bar{B}_{r_0})^2$ such that $\max\{\|\kappa_1-\nu_1\|,\|\kappa_2-\nu_2\|\} < \varepsilon$. Then, for all $\eta \in \mathcal{J}$, we have

$$\begin{aligned}&\theta(|\Upsilon(\kappa_1,\kappa_2)(\eta)-\Upsilon(\nu_1,\nu_2)(\eta)|)\\ &\quad\leq \theta\Big(|f(\eta,\kappa_1(\alpha(\eta)),\kappa_2(\alpha(\eta)),\int_0^{\beta(\eta)} g(\eta,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))d\varrho)\\ &\qquad - f(\eta,\nu_1(\alpha(\eta)),\nu_2(\alpha(\eta)),\int_0^{\beta(\eta)} g(\eta,\varrho,\nu_1(\alpha(\varrho)),\nu_2(\alpha(\varrho))d\varrho)|\Big)\\ &\quad\leq \theta\Big(\max\{|\kappa_1(\alpha(\eta))-\nu_1(\alpha(\eta))|,|\kappa_2(\alpha(\eta))-\nu_2(\alpha(\eta))|\}\\ &\qquad + |\int_0^{\beta(\eta)}[g(\eta,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))-g(\eta,\varrho,\nu_1(\alpha(\varrho)),\nu_2(\alpha(\varrho)))]d\varrho|\Big)^{\lambda}\\ &\quad\leq \theta(\varepsilon+\beta(T)\omega^T(g,\varepsilon))^{\lambda},\end{aligned}$$

where

$$\begin{aligned}\omega^T(g,\varepsilon) = \sup\{&|g(\eta,\varrho,\kappa_1,\kappa_2)-g(\eta,\varrho,\nu_1,\nu_2)| : \eta\in\mathcal{J},\varrho\in[0,\beta(T)],\\ &\kappa_1,\kappa_2,\nu_1,\nu_2\in[-r_0,r_0],\max\{\|\kappa_1-\nu_1\|,\|\kappa_2-\nu_2\|\}<\varepsilon\}\end{aligned}$$

and

$$\beta(T) = \sup\{\beta(\eta) : \eta \in \mathcal{J}\}.$$

Applying the continuity of g on $\mathcal{J}\times[0,\beta(T)]\times[-r_0,r_0]^2$, we have $\omega^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$, hence $\theta(\varepsilon+\beta(T)\omega^T(g,\varepsilon))^{\lambda} \to 1$ as $\varepsilon \to 0$, which implies that Υ is a continuous function on $(\bar{B}_{r_0})^2$.

Now, we prove that Υ satisfies all conditions of Corollary 3. To do this, let Λ_1 and Λ_2 be nonempty and bounded subsets of $\bar{B}_{r_0}$, and assume that $T > 0$ and $\varepsilon > 0$ are arbitrary constants. Let $\eta_1,\eta_2 \in \mathcal{J}$, with $|\eta_2-\eta_1| \leq \varepsilon$ and $(\kappa_1,\kappa_2) \in \Lambda_1\times\Lambda_2$. Then, we have

$$\begin{aligned}
&\theta(|\Upsilon(\kappa_1,\kappa_2)(\eta_2)-\Upsilon(\kappa_1,\kappa_2)(\eta_1)|)\\
&\leq \theta\Big(|f(\eta_2,\kappa_1(\alpha(\eta_2)),\kappa_2(\alpha(\eta_2)),\int_0^{\beta(\eta_2)} g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)\\
&\quad - f(\eta_2,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)|\\
&\quad + |f(\eta_2,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)\\
&\quad - f(\eta_1,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)|\\
&\quad + |f(\eta_1,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)\\
&\quad - f(\eta_1,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)|\\
&\quad + |f(\eta_1,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_2)} g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)\\
&\quad - f(\eta_1,\kappa_1(\alpha(\eta_1)),\kappa_2(\alpha(\eta_1)),\int_0^{\beta(\eta_1)} g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho))d\varrho)|\Big)\\
&\leq \Big(\theta\Big(\max\{|\kappa_1(\alpha(\eta_2))-\kappa_1(\alpha(\eta_1))|,|\kappa_2(\alpha(\eta_2))-\kappa_2(\alpha(\eta_1))|\}+\omega_{r_0,G}(f,\varepsilon)\\
&\quad + \int_0^{\beta(\eta_2)} |g(\eta_2,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))-g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))|d\varrho\\
&\quad + |\int_0^{\beta(\eta_2)} g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))d\varrho-\int_0^{\beta(\eta_1)} g(\eta_1,\varrho,\kappa_1(\alpha(\varrho)),\kappa_2(\alpha(\varrho)))d\varrho|\Big)\Big)^{\lambda}\\
&\leq \Big(\theta\Big(\max\{\omega(\kappa_1,\omega(\alpha,\varepsilon)),\omega(\kappa_2,\omega(\alpha,\varepsilon))\}+\omega_{r_0,G}(f,\varepsilon)+\beta(T)\omega_{r_0}(g,\varepsilon)+U_{r_0}\omega(\beta,\varepsilon)\Big)\Big)^{\lambda} \qquad (12)
\end{aligned}$$

where

$$\omega(\alpha,\varepsilon)=\sup\{|\alpha(\eta_2)-\alpha(\eta_1)|:\eta_1,\eta_2\in\mathcal{J},\ |\eta_2-\eta_1|\leq\varepsilon\},$$

$$\omega(\kappa_1,\omega(\alpha,\varepsilon))=\sup\{|\kappa_1(\eta_2)-\kappa_1(\eta_1)|:\eta_1,\eta_2\in\mathcal{J},\ |\eta_2-\eta_1|\leq\omega(\alpha,\varepsilon)\},$$

$$\beta(T)=\sup\{\beta(\eta):\eta\in\mathcal{J}\},$$

$$U_{r_0}=\sup\{|g(\eta,\varrho,\kappa_1,\kappa_2)|:\eta\in\mathcal{J},\ \varrho\in[0,\beta(T)],\ \kappa_1,\kappa_2\in[-r_0,r_0]\},$$

$$G=\beta(T)\sup\{|g(\eta,\varrho,\kappa_1,\kappa_2)|:\eta\in\mathcal{J},\ \varrho\in[0,\beta_\Lambda],\ \kappa_1,\kappa_2\in[-r_0,r_0]\},$$

$$\begin{aligned}\omega_{r_0,G}(f,\varepsilon)=\sup\{&|f(\eta_2,\kappa_1,\kappa_2,z)-f(\eta_1,\kappa_1,\kappa_2,z)|:\eta_1,\eta_2\in\mathcal{J},\\ &|\eta_2-\eta_1|\leq\varepsilon,\ \kappa_1,\kappa_2\in[-r_0,r_0],\ z\in[-G,G]\},\end{aligned}$$

$$\begin{aligned}\omega_{r_0}(g,\varepsilon)=\sup\{&|g(\eta_2,\varrho,\kappa_1,\kappa_2)-g(\eta_1,\varrho,\kappa_1,\kappa_2)|:\eta_1,\eta_2\in\mathcal{J},\\ &|\eta_2-\eta_1|\leq\varepsilon,\ \kappa_1,\kappa_2\in[-r_0,r_0],\ \varrho\in[0,\beta(T)]\}.\end{aligned}$$

Since (κ_1, κ_2) is an arbitrary element of $\Lambda_1 \times \Lambda_2$ in (12), we have

$$\theta(\omega(\Upsilon(\Lambda_1 \times \Lambda_2), \varepsilon)) \leq \Big(\theta\Big(\omega_{r_0,G}(f,\varepsilon) + \max\{\omega(\Lambda_1, \omega(\alpha,\varepsilon)), \omega(\Lambda_2, \omega(\alpha,\varepsilon))\} \\ + \beta(T)\omega_{r_0}(g,\varepsilon) + U_{r_0}\omega(\beta,\varepsilon)\Big)\Big)^{\lambda},$$

and by the uniform continuity of f and g on the compact sets

$$[0,T] \times [-r_0, r_0] \times [-r_0, r_0] \times [-G, G]$$

and

$$[0,T] \times [0, \beta(T)] \times [-r_0, r_0] \times [-r_0, r_0]$$

respectively, we infer that $\omega_{r_0,G}(f,\varepsilon) \longrightarrow 0$, $\omega_{r_0}(g,\varepsilon) \longrightarrow 0$ and $\omega(\beta,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Therefore, we obtain

$$\theta(\mu[\Upsilon(\Lambda_1 \times \Lambda_2)]) \leq (\theta(\max\{\mu(\Lambda_1), \mu(\Lambda_2)\}))^{\lambda}.$$

Thus, from Corollary 3, we deduce that the operator Υ has a coupled fixed point. Therefore, the system of the functional integral Equation (1) has at least one solution in Ξ^2. $\square$

Example 1. *Consider the following system of integral equations:*

$$\begin{cases} \eta(t) &= \frac{1}{2}e^{-t^2} + \frac{\arctan \eta(t) + \sinh^{-1} \varrho(t)}{8\pi + t^8} \\ & \quad + \dfrac{1}{8}\displaystyle\int_0^t \frac{s^2|\sin \eta(s)| + \sqrt{e^s(1+\eta^2(s))(1+\sin^2 \varrho(s))}}{e^t(1+\eta^2(s))(1+\sin^2 \varrho(s))} ds, \\ \varrho(t) &= \frac{1}{2}e^{-t^2} + \frac{\arctan \varrho(t) + \sinh^{-1} \eta(t)}{8\pi + t^8} \\ & \quad + \dfrac{1}{8}\displaystyle\int_0^t \frac{s^2|\sin \varrho(s)| + \sqrt{e^s(1+y^2(s))(1+\sin^2 \eta(s))}}{e^t(1+\varrho^2(s))(1+\sin^2 \eta(s))} ds. \end{cases} \tag{13}$$

We observe that the system of integral Equation (13) is a special case of (1) with $\Xi = C([0,1], \mathbb{R})$ *and*

$$\alpha(t) = \beta(t) = t, \qquad t \in [0,1],$$

$$f(t, \eta, \varrho, p) = \frac{1}{2}e^{-t^2} + \frac{\arctan \eta + \sinh^{-1} \varrho}{8\pi + t^8} + \frac{p}{8},$$

$$g(t, s, \eta, \varrho) = \frac{s^2|\sin \eta| + \sqrt{e^s(1+\eta^2)(1+\sin^2 \varrho)}}{e^t(1+\eta^2)(1+\sin^2 \varrho)}.$$

To solve this system, we need to verify the conditions (i)–(iv) of Theorem 8.
Let $\theta(t) = \sqrt{t+1}$ *and* $\lambda = \frac{1}{2}$. *Condition (i) is clearly evident. On the other hand,*

$$
\begin{aligned}
&\theta(|f(t,\eta,\varrho,m)-f(t,\kappa,\nu,n)|)\\
&\quad\leq\sqrt{\frac{|\arctan\eta-\arctan\kappa|+|\sinh^{-1}\varrho-\sinh^{-1}\nu|}{8\pi+t^8}+\frac{|m-n|}{8}+1}\\
&\quad\leq\sqrt{\frac{\arctan|\eta-\kappa|}{8\pi}+\frac{\sinh^{-1}|\varrho-\nu|}{8\pi}+\frac{|m-n|}{8}+1}\\
&\quad\leq\sqrt{\frac{|\eta-\kappa|}{8\pi}+\frac{|\varrho-\nu|}{8\pi}+\frac{|m-n|}{8}+1}\\
&\quad\leq\sqrt{\theta(\max\{|\eta-\kappa|,|\varrho-\nu|\}+|m-n|)}.
\end{aligned}
$$

So, we find that f satisfies condition (ii) of Theorem 8. In addition,

$$
N=\sup\{|f(t,0,0,0)|:t\in[0,1]\}=\sup\{\frac{1}{2}e^{-t^2}:t\in[0,1]\}=0.5.
$$

Hence, condition (iii) of Theorem 8 is valid. Moreover, g is continuous on $[0,1]\times[0,1]\times\mathbb{R}^2$ and

$$
\begin{aligned}
G=\sup\Big\{\Big|\int_0^t&\frac{t^2|\sin\eta(s)|+\sqrt{e^s(1+\eta^2(s))(1+\sin^2\varrho(s))}}{e^t(1+\eta^2(s))(1+\sin^2\varrho(s))}ds\Big|\\
&:t,s\in[0,1],\ \eta,\varrho\in\Xi\Big\}\\
<\sup\frac{t^2}{e^t}&\simeq 0.4.
\end{aligned}
$$

Furthermore, it is easy to see that $r\geq 2.65$ satisfies the inequality in condition (iv), i.e.,

$$
\theta(r+G)^{\frac{1}{2}}+\theta(N)=\sqrt{\sqrt{r+0.4+1}}+\sqrt{1.5}\leq r.
$$

Consequently, all of the conditions of Theorem 8 are satisfied. Hence, the system of integral Equation (13) has at least one solution which belongs to the space $(\mathcal{C}([0,1],\mathbb{R}))^2$.

4. Conclusions

There are many generalizations of Darbo's fixed point theorem. Some authors have made generalizations via certain contraction conditions. On the other hand, many authors have generalized Darbo's fixed point theorem by changing the domain of mappings which possess a fixed point. In this paper, we used the notion of weak JS-contractions to verify that a mapping defined on a nonempty, bounded, closed, and convex subset of a given Banach space has at least one fixed point. We applied our results to prove the existence of solutions for a system of functional integral equations.

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References

1. Kuratowski, K. Sur les espaces. *Fund. Math.* **1930**, *15*, 301–309. [CrossRef]
2. Aghajani, A.; Mursaleen, M.; Shole Haghighi, A. A generalization of Darbo's theorem with application to the solvability of systems of integral equations. *J. Comp. Appl. Math.* **2014**, *260*, 68–77. [CrossRef]
3. Banaei, S.; Ghaemi, M.B. A generalization of the Meir-Keeler condensing operators and its application to solvability of a system of nonlinear functional integral equations of Volterra type. *Sahand Commun. Math. Anal.* **2019**, *15*, 19–35.
4. Banaei, S. An extension of Darbo's theorem and its application to existence of solution for a system of integral equations. *Cogent Math. Stat.* **2019**, *6*, 1614319. [CrossRef]
5. Banaei, S. Solvability of a system of integral equations of Volterra type in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$ via measure of noncompactness. *Filomat* **2018**, *32*, 5255–5263. [CrossRef]
6. Banás, J.; Jleli, M.; Mursaleen, M.; Samet, B. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*; Springer: Singapore, 2017.
7. Banaei, S.; Ghaemi, M.B.; Saadati, R. An extension of Darbo's theorem and its application to system of neutral diferential equations with deviating argument. *Miskolc Math. Notes* **2017**, *18*, 83–94. [CrossRef]
8. Banás, J.; Goebel, K. *Measures of Noncompactness in Banach Spaces*; Lecture Notes in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 1980.
9. Parvaneh, V.; Husssain, N.; Mukheimer, A.; Aydi, H. On fixed point results for modified JS-contractions with applications. *Axioms* **2019**, *8*, 84. [CrossRef]
10. Agarwal, R.P.; Meehan, M.; O'Regan, D. *Fixed Point Theory and Applications*; Cambridge University Press: Cambridge, UK, 2004.
11. Darbo, G. Punti uniti in trasformazioni a codominio non compatto. *Rend. Sem. Mat. Univ. Padova* **1955**, *24*, 84–92.
12. Gnana Bhaskar, T.; Lakshmikantham, V. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.* **2006**, *65*, 1379–1393. [CrossRef]