Two multi-cubic functional equations and some results on the stability in modular spaces

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Abstract

In this article, we study n-variable mappings which are cubic in each variable. We also show that such mappings can be described by an equation, say, multi-cubic functional equation. Furthermore, we study the stability of such functional equations in the modular space X_ρ by applying Δ_2-condition and the Fatou property (in some cases) on the modular function ρ. Finally, we show that, under some mild conditions, one of these new multi-cubic functional equations can be hyperstable.

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1 Introduction

The main motivation for the investigation of the stability of functional equations was given by Ulam in 1940 in his talk at the University of Wisconsin [47], where he presented a number of unsolved problems. Among these was the famous Ulam stability question concerning the stability of group homomorphisms as follows: Let G be a group and let H be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a homomorphism $h : G \longrightarrow H$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $g : G \longrightarrow H$ with $d(h(x), g(x)) < \epsilon$ for all $x \in G$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the equation: *Under what conditions does there exist a true solution near an approximate mapping differing slightly from a functional equation*? If the answer is affirmative, we say that the functional equation is *stable*. Hyers [16] gave a first affirmative answer to the question of Ulam for Banach spaces as follows: Let X and Y be Banach spaces. Suppose that $f : X \longrightarrow Y$ satisfies

$$\left\| f(x + y) - f(x) - f(y) \right\| \leq \epsilon$$

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for all $x, y \in X$ and for some $\epsilon \geq 0$. Then, there exists a unique additive mapping $T : X \longrightarrow Y$ such that

$$\left\| f(x) - T(x) \right\| \leq \epsilon$$

for all $x \in X$, and if $f(tx)$ is continuous in t for each fixed x, then f is a linear mapping. Later on, various generalizations and extensions of Hyers' result were established by Aoki [1], Th.M. Rassias [44], Găvruța [15], and J.M. Rassias [43] in different versions for the Cauchy (additive) functional equation $A(x + y) = A(x) + A(y)$ in Banach spaces. Since then, the stability problems of various functional equations on miscellaneous normed spaces have been extensively investigated by a number of authors; for instance, see [12–14, 28, 30, 31], and [33]. Moreover, for some results on positive linear operators, we refer to [32] and [34].

Let V be a commutative group, W a linear space, and $n \geq 2$ an integer. Recall from [10] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies the Cauchy functional equation) in each variable. Some facts on such mappings can be found in [23] and many other sources. Besides, f is said to be *multi-quadratic* if it is quadratic (satisfies the quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$) in each variable [11]. In [50], Zhao et al. proved that the mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if the following relation holds:

$$\sum_{t \in \{-1,1\}^n} f(x_1 + t x_2) = 2^n \sum_{j_1, j_2, \dots, j_n \in \{1,2\}} f(x_{1j_1}, x_{2j_2}, \dots, x_{nj_n}) \tag{1.1}$$

where $x_j = (x_{1j}, x_{2j}, \dots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. In [10] and [11], Ciepliński studied the generalized Hyers–Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [50]). The Jensen-type multi-quadratic mappings and their characterization can be found in [46].

The cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) = 6f(y)$$

was introduced by J.M. Rassias in [43] for the first time. He investigated the Ulam–Hyers stability problem for this functional equation. After that, the next alternative cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.2}$$

has been introduced by Jun and Kim in [17]. They also presented the cubic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y), \tag{1.3}$$

found its general solution, and studied the Hyers–Ulam stability problem for it [18]. For other forms of the (generalized) cubic functional equations and their stability on the various Banach spaces, we refer to [3–7, 29, 35, 36], and [49]. Recently, the stability of multi-cubic and multi-quartic mappings in Banach spaces via the fixed point method were investigated in [9] and [8], respectively.

The concept of modular spaces was first introduced by Nakano [39], and then Luxemburg [24] and Mazur, Musielak and Orlicz [27, 37, 38] developed it extensively. Since then, the theory of modulars and modular spaces is widely applied in the study of interpolation theory [22, 26] and various Orlicz spaces [40]. A modular yields less properties than a norm, but it makes more sense in many special situations. When we work in a modular space, it is frequently assumed that the modular satisfies extra additional properties like some relaxed continuity or some Δ_2-condition. As for the mentioned condition, Khamsi [19] studied the stability of quasicontraction mappings in modular spaces without Δ_2-condition by using the fixed point theorem. Concerning the stability theory in modular spaces, Sadeghi [45] has established generalized Hyers–Ulam stability via the Khamsi fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2-condition with $0 < \kappa \leq 2$. In addition, the stability of quadratic functional equations in modular spaces satisfying the Fatou property without using the Δ_2-condition was investigated in [48]. Park et al. [41] investigated the stability of additive and Jensen-additive functional equations without using the Δ_2-condition by a fixed point method. An alternative generalized Hyers–Ulam stability theorem of a modified quadratic functional equation in a modular spaces using Δ_3-condition without the Fatou property on a modular function was presented in [20]. Furthermore, a refined stability result and alternative stability results for additive and quadratic functional equations using the direct method in modular spaces are given in [21]. For the stability of mixed additive-quadratic-cubic mappings in modular spaces which were recently studied, we refer to [25].

In this paper we define multi-cubic mappings and include two characterizations of such mappings. In other words, we prove that every multi-cubic mapping can be shown to satisfy a single functional equation, and vice versa. Moreover, we investigate the generalized Hyers–Ulam stability and hyperstability for multi-cubic mappings in modular spaces by applying the direct method. As a direct consequence of our main results, we show that under some mild conditions these new multi-cubic mappings are hyperstable.

2 Preliminary notations

In this section, we recall some basic definitions and remarks concerning modular spaces with modular functionals, which are primitive notions corresponding to norms and metrics, as in [20, 27, 37]:

Definition 2.1 Let X be a linear space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized function $\rho : X \longrightarrow [0, \infty]$ is called a *modular* if it satisfies the following three conditions for elements $\alpha, \beta \in \mathbb{K}$, $x, y \in X$:

   (i) $\rho(x) = 0$ if and only if $x = 0$;
   (ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
   (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all scalar $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If condition (iii) is replaced by $\rho(\alpha x + \beta y) \leq \alpha^t \rho(x) + \beta^t \rho(y)$ when $\alpha^t + \beta^t = 1$ and $\alpha, \beta \geq 0$ with an $t \in (0, 1]$, then ρ is called *t-convex modular*. 1-convex modular are called *convex modular*. For a modular ρ, there corresponds a linear subspace X_ρ of X, given by $X_\rho := \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$. In this case X_ρ is called a *modular space*.

Let ρ be a convex modular. Then, the modular space X_ρ can be equipped with a norm called the *Luxemburg norm*, defined by $\|x\|_\rho = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$.

Here, we have some observations:

(1) If ρ is a modular on X, then $\rho(tx)$ is an increasing function of $t \geq 0$ for each fixed $x \in X$, that is, $\rho(ax) \leq \rho(bx)$ whenever $0 \leq a < b$;

(2) If ρ is a convex modular on X and $|\alpha| \leq 1$, then $\rho(ax) \leq |\alpha|\rho(x)$ for all $x \in X$. In particular, if $\alpha_j \geq 0$ ($j = 1, 2, \ldots, n$) with $0 < \sum_{j=1}^{n} \alpha_j \leq 1$, then $\rho(\sum_{j=1}^{n} \alpha_j x_j) \leq \sum_{j=1}^{n} \alpha_j \rho(x_j)$ for all $x_j \in X$.

Definition 2.2 Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. Then

(i) $\{x_n\}$ is ρ-convergent to a point $x_* \in X_\rho$, and we write $x_n \overset{\rho}{\to} x_*$, if $\rho(x_n - x_*) \to 0$ as $n \to \infty$;

(ii) $\{x_n\}$ is a ρ-Cauchy sequence if for any $\epsilon > 0$ one has $\rho(x_n - x_m) < \epsilon$ for all sufficiently large $m, n \in \mathbb{N}$;

(iii) A subset $Y \subseteq X_\rho$ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent to a point in Y.

Example 2.3 ([42]) Let $\psi : [0, \infty) \longrightarrow \mathbb{R}$ be a function defined by $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and $\lim_{t \to \infty} \psi(t) = \infty$. If, moreover, ψ is convex, continuous and nondecreasing, then ψ is called an *Orlicz function*. For a measure space $(X, \sum, \mu)$, suppose that $L^0(\mu)$ is the set of all measurable functions on X. For each $f \in L^0(\mu)$, define $\rho_\psi(f) = \int_X \psi(|f|) \, d\mu$. Then, ρ_ψ is a modular and the corresponding modular space is called an *Orlicz space* and denoted by

$$L_\psi = \{ f \in L^0(\mu) | \rho_\psi(\lambda f) \to 0 \text{ as } \lambda \to 0 \}.$$

One can check that L_ψ is ρ_ψ-complete.

A modular function ρ is said to satisfy the Δ_s-*condition* if there exists $\kappa > 0$ such that $\rho(sx) \leq \kappa\rho(x)$ for all $x \in X_\rho$. Throughout this paper, we say that the constant κ is a Δ_s-constant related to Δ_s-condition. Suppose that ρ is convex and satisfies Δ_s-condition with Δ_s-constant κ. If $\kappa < s$, then $\rho(x) \leq \kappa\rho(\frac{x}{s}) \leq \frac{\kappa}{s}\rho(x)$, which implies $\rho = 0$. Hence, we must have the Δ_s-constant $\kappa \geq s$ if ρ is convex modular. It is said that a modular ρ has the *Fatou property* if and only if $\rho(x) \leq \liminf_{n \to \infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x in the modular space X_ρ.

3 Characterization of multi-cubic mappings

Motivated by equation (1.3), we consider the functional equation

$$8f\left(\frac{x + 2y}{2}\right) + 8f\left(\frac{x - 2y}{2}\right) = 4f(x + y) + 4f(x - y) - 6f(x). \tag{3.1}$$

It is easy to check that $f(x) = ax^3$ is a solution of (3.1). Hence, it said to be a *cubic* functional equation. The above equation leads us to define n-multi-cubic mappings and characterize them in this section.

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $n \in \mathbb{N}$. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $q = (q_1, \ldots, q_n) \in \{-2, -1, 1, 2\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$, we write $lx := (lx_1, \ldots, lx_n)$ and $qx := (q_1 x_1, \ldots, q_n x_n)$, where lx stands, as usual, for the scalar product of an element l on x of the vector space V.

In the sequel, let V and W be vector spaces over the rationals, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. Let $x_1, x_2 \in V^n$ and $T \in \mathbb{N}_0$ with $0 \le T \le n$. Put $\mathcal{M} = \{\mathfrak{N}_n = (N_1, N_2, \ldots, N_n) | N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, where $j \in \{1, \ldots, n\}$. Consider

$$\mathcal{M}_T^n := \{\mathfrak{N}_n = (N_1, N_2, \ldots, N_n) \in \mathcal{M} \,|\, \mathrm{Card}\{N_j : N_j = x_{1j}\} = T\}.$$

We say that a mapping $f : V^n \longrightarrow W$ is *n-multi-cubic* or *multi-cubic* if f is cubic in each variable (see equation (3.1)). From now on, for such mappings, we use the following notations:

$$f(\mathcal{M}_T^n) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n), \tag{3.2}$$

$$f(\mathcal{M}_T^n, z) := \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} f(\mathfrak{N}_n, z) \quad (z \in V).$$

We say that a mapping $f : V^n \longrightarrow W$ satisfies *the r-power condition* in the jth variable if

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $z_1, \ldots, z_n \in V^n$

Remark 3.1 It is easily verified that if f is a multi-cubic mapping, then it satisfies the 3-power condition in all variables. But the converse is not true. Here, by means of an example we show that the 3-power condition in all variables for a mapping f does not imply that it is multi-cubic. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Fix a vector a_0 in $\mathcal{A}$ (not necessarily unit). Define the mapping $h : \mathcal{A}^n \longrightarrow \mathcal{A}$ by $h(a_1, \ldots, a_n) = \prod_{j=1}^n \|a_j\|^3 a_0$ for $(a_1, \ldots, a_n) \in \mathcal{A}^n$. It is easily verified that the mapping h satisfies the 3-power condition in all variables but is not multi-cubic even for $n = 1$, that is, h does not satisfy equation (3.1).

In what follows, $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \ge k$ by $n!/(k!(n-k)!)$. In the upcoming result, we show that a multi-cubic mapping $f : V^n \longrightarrow W$ has the form

$$8^n \sum_{q \in \{-2,2\}^n} f\left(\frac{x_1 + qx_2}{2}\right) = \sum_{k=0}^n 4^{n-k}(-6)^k f(\mathcal{M}_k^n) \tag{3.3}$$

for all $x_1, x_2 \in V^n$, where $f(\mathcal{M}_k^n)$ is defined in (3.2).

Theorem 3.2 *Consider the mapping $f : V^n \longrightarrow W$. Then, the following assertions are equivalent*:

  (i) *f is multi-cubic*;
 (ii) *f satisfies equation* (3.3) *with the 3-power condition in all variables.*

Proof (i) $\Rightarrow$ (ii) We first note that it is not hard to show that f satisfies the 3-power condition in all variables. Assume that f is multi-cubic. We prove that f satisfies equation (3.3)

by induction on n. For $n = 1$, it is trivial that f satisfies equation (3.1). If (3.3) is valid for some positive integer $n > 1$, then

$$8^{n+1} \sum_{q \in \{-2,2\}^{n+1}} f\left(\frac{x_1^{n+1} + qx_2^{n+1}}{2}\right) = 4 \times 8^n \sum_{q \in \{-2,2\}^n} f\left(\frac{x_1^n + qx_2^n}{2}, x_{1n+1} + x_{2n+1}\right)$$

$$+ 4 \times 8^n \sum_{q \in \{-2,2\}^n} f\left(\frac{x_1^n + qx_2^n}{2}, x_{1n+1} - x_{2n+1}\right)$$

$$- 6 \times 8^n \sum_{q \in \{-2,2\}^n} f\left(\frac{x_1^n + qx_2^n}{2}, x_{1n+1}\right)$$

$$= 4 \sum_{k=0}^{n} \sum_{q \in \{-2,2\}} 4^{n-k}(-6)^k f\left(\mathcal{M}_k^n, x_{1n+1} + qx_{2n+1}\right)$$

$$- 6 \sum_{k=0}^{n} 4^{n-k}(-6)^k f\left(\mathcal{M}_k^n, x_{1n+1}\right)$$

$$= \sum_{k=0}^{n+1} 4^{n+1-k}(-6)^k f\left(\mathcal{M}_k^{n+1}\right).$$

This means that (3.3) holds for $n + 1$.

(ii) $\Rightarrow$ (i) Fix $j \in \{1, \ldots, n\}$ and put

$$f^*(x_{1j}, x_{2j}) := f(x_{11}, \ldots, x_{1j-1}, x_{1j} + x_{2j}, x_{1j+1}, \ldots, x_{1n})$$

$$+ f(x_{11}, \ldots, x_{1j-1}, x_{1j} - x_{2j}, x_{1j+1}, \ldots, x_{1n}). \tag{3.4}$$

Putting $x_{2k} = 0$ for all $k \in \{1, \ldots, n\} \setminus \{j\}$ in (3.3) and using the assumption, we get

$$\frac{2^{n-1}}{8^{n-1}} \times 8^n f\left(x_{11}, \ldots, x_{1j-1}, \frac{x_{1j} + 2x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n}\right)$$

$$+ \frac{2^{n-1}}{8^{n-1}} \times 8^n f\left(x_{11}, \ldots, x_{1j-1}, \frac{x_{1j} - 2x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n}\right)$$

$$= 2^{n-1} \times 8^n f\left(\frac{x_{11}}{2}, \ldots, \frac{x_{1j-1}}{2}, \frac{x_{1j} + 2x_{2j}}{2}, \frac{x_{1j+1}}{2}, \ldots, \frac{x_{1n}}{2}\right)$$

$$+ 2^{n-1} \times 8^n f\left(\frac{x_{11}}{2}, \ldots, \frac{x_{1j-1}}{2}, \frac{x_{1j} - 2x_{2j}}{2}, \frac{x_{1j+1}}{2}, \ldots, \frac{x_{1n}}{2}\right)$$

$$= 2^{n-1} \times 4^n f^*(x_{1j}, x_{2j})$$

$$+ \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} 4^{n-k} \times 2^{n-k} \times (-6)^k\right] f(x_{11}, \ldots, x_{1n})$$

$$+ \sum_{k=1}^{n-1} \left[\binom{n-1}{k} 4^{n-k} \times 2^{n-k-1} \times (-6)^k\right] f^*(x_{1j}, x_{2j})$$

$$+ (-6)^n f(x_{11}, \ldots, x_{1n})$$

$$= \left[2^{n-1} \times 4^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} 4^{n-k} \times 2^{n-k-1} \times (-6)^k \right] \right] f^*(x_{1j}, x_{2j})$$

$$+ \left[(-6)^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} 8^{n-k} \times (-6)^k \right] \right] f(x_{11}, \ldots, x_{1n}). \tag{3.5}$$

On the other hand, we have

$$2^{n-1} \times 4^n + \sum_{k=1}^{n-1} \binom{n-1}{k} 4^{n-k} \times 2^{n-k-1} \times (-6)^k$$

$$= 2^{n-1} \times 4^n + 4 \sum_{k=1}^{n-1} \binom{n-1}{k} 4^{n-k-1} \times (-3)^k$$

$$= 4 \times 2^{n-1} (4-3)^{n-1} = 4 \times 2^{n-1}. \tag{3.6}$$

In addition,

$$(-6)^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 8^{n-k} \times (-6)^k = (-6)^n + \sum_{k=0}^{n-2} \binom{n-1}{k} 8^{n-k-1} \times (-6)^{k+1}$$

$$= (-6)^n + (-6) \sum_{k=0}^{n-2} \binom{n-1}{k} 8^{n-k-1} (-6)^k$$

$$= (-6)^n - 6 \left(2^{n-1} - (-6)^{n-1} \right) = -6 \times 2^{n-1}. \tag{3.7}$$

Relations (3.5), (3.7), and (3.6) imply that

$$8f \left(x_{11}, \ldots, x_{1j-1}, \frac{x_{1j} + 2x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n} \right)$$

$$+ 8f \left(x_{11}, \ldots, x_{1j-1}, \frac{x_{1j} - 2x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n} \right)$$

$$= 4f^*(x_{1j}, x_{2j}) - 6f(x_{11}, \ldots, x_{1n}).$$

This means that f is cubic in the jth variable. Since j is arbitrary, we obtain the desired result.   □

4 An alternative characterization of multi-cubic mappings

It is easy to check that equation (1.2) implies that $f(\frac{x}{2}) = \frac{1}{8} f(x)$. On the other hand, the equation

$$8f \left(\frac{2x+y}{2} \right) + 8f \left(\frac{2x-y}{2} \right) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{4.1}$$

shows that $f(2x) = 8f(x)$. Summing up, we have

Proposition 4.1 *A mapping $f : V \longrightarrow W$ satisfies equation* (1.2) *if and only if it satisfies equation* (4.1).

Here, we introduce a different multi-cubic mapping by using equation (4.1). In other words, a mapping $f : V^n \longrightarrow W$ is said to be *n-multi-cubic* or *multi-cubic* if f is cubic in each variable (see equation (4.1)).

Analogous to Theorem 3.2, we show that multi-cubic mappings in the sense above can be unified using an equation without any extra condition on the components as follows:

Theorem 4.2 *A mapping $f : V^n \longrightarrow W$ is multi-cubic (*in the sense above*) if and only if f satisfies the equation*

$$8^n \sum_{q\in\{-1,1\}^n} f\left(\frac{2x_1 + qx_2}{2}\right) = \sum_{k=0}^n 2^{n-k} 12^k f\left(\mathcal{M}_k^n\right). \tag{4.2}$$

for all $x_1, x_2 \in V^n$, where $f(\mathcal{M}_k^n)$ is defined in (3.2).

Proof (Necessity) Assume that f is multi-cubic. We prove that f satisfies equation (4.2) by induction on n. For $n = 1$, it is trivial that f satisfies equation (4.1). If (4.2) is valid for some positive integer $n > 1$, then

$$8^{n+1} \sum_{q\in\{-1,1\}^{n+1}} f\left(\frac{2x_1^{n+1} + qx_2^{n+1}}{2}\right) = 2 \times 8^n \sum_{q\in\{-1,1\}^n} f\left(\frac{2x_1^n + qx_2^n}{2}, x_{1n+1} + x_{2n+1}\right)$$

$$+ 2 \times 8^n \sum_{q\in\{-1,1\}^n} f\left(\frac{2x_1^n + qx_2^n}{2}, x_{1n+1} - x_{2n+1}\right)$$

$$+ 12 \times 8^n \sum_{q\in\{-1,1\}^n} f\left(\frac{2x_1^n + qx_2^n}{2}, x_{1n+1}\right)$$

$$= 2 \sum_{k=0}^n \sum_{q\in\{-1,1\}} 2^{n-k} 12^k f\left(\mathcal{M}_k^n, x_{1n+1} + qx_{2n+1}\right)$$

$$+ 12 \sum_{k=0}^n 2^{n-k} 12^k f\left(\mathcal{M}_k^n, x_{1n+1}\right)$$

$$= \sum_{k=0}^{n+1} 2^{n+1-k} 12^k f\left(\mathcal{M}_k^{n+1}\right).$$

This means that (4.2) holds for $n + 1$.

(Sufficiency) Fix $j \in \{1,\dots,n\}$ and $f^*(x_{1j}, x_{2j})$ as in (3.4). Putting $x_{2k} = 0$ for all $k \in \{1,\dots,n\}\setminus\{j\}$ in (4.2), we get

$$2^{n-1} \times 8^n \left[f\left(x_{11},\dots,x_{1j-1}, \frac{2x_{1j} + x_{2j}}{2}, x_{1j+1},\dots,x_{1n}\right) \right.$$

$$\left. + f\left(x_{11},\dots,x_{1j-1}, \frac{2x_{1j} - x_{2j}}{2}, x_{1j+1},\dots,x_{1n}\right) \right]$$

$$= 2^{n-1} \times 2^n f^*(x_{1j}, x_{2j})$$

$$+ \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} 2^{2(n-k)} \times 12^k \right] f(x_{11},\dots,x_{1n})$$

$$+ \sum_{k=1}^{n-1} \left[\binom{n-1}{k} 2^{2(n-k)-1} \times 12^k \right] f^*(x_{1j}, x_{2j})$$

$$+ 12^n f(x_{11}, \ldots, x_{1n})$$

$$= \left[2^{2n-1} \times 2^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} 2^{2(n-k)-1} \times 12^k \right] \right] f^*(x_{1j}, x_{2j})$$

$$+ \left[12^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} 2^{2(n-k)} \times 12^k \right] \right] f(x_{11}, \ldots, x_{1n}). \tag{4.3}$$

On the other hand, we have

$$2^{2n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k} 2^{2(n-k)-1} \times 12^k = 2^{2n-1} \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} 3^k \right)$$

$$= 2^{2n-1}(1+3)^{n-1} = 2^{4n-3}. \tag{4.4}$$

Furthermore,

$$12^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{2(n-k)} \times 12^k = 12^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{2(n-k)} \times 2^{2k} \times 3^k$$

$$= 12^n + 3 \times 2^{2n} \sum_{k=0}^{n-2} \binom{n-1}{k-1} 3^k$$

$$= 12^n + 3 \times 2^{2n} \left(\sum_{k=0}^{n-1} \left[\binom{n-1}{k-1} 3^k \right] - 3^{n-1} \right)$$

$$= 12^n + 3 \times 2^{2n} \left((1+3)^{n-1} - 3^{n-1} \right)$$

$$= 12^n + 3 \times 2^{2n} \left(2^{2(n-1)} - 3^{n-1} \right)$$

$$= 12 \times 2^{4(n-1)}. \tag{4.5}$$

Relations (4.3), (4.4), and (4.5) imply that

$$8f \left(x_{11}, \ldots, x_{1j-1}, \frac{2x_{1j} + x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n} \right)$$

$$+ 8f \left(x_{11}, \ldots, x_{1j-1}, \frac{2x_{1j} - x_{2j}}{2}, x_{1j+1}, \ldots, x_{1n} \right)$$

$$= 2f^*(x_{1j}, x_{2j}) + 12f(x_{11}, \ldots, x_{1n}).$$

This means that f is cubic in the jth variable, and so it is a multi-cubic mapping.　□

5 Stability results for (3.3)

In this section, we prove the generalized Hyers–Ulam stability of multi-cubic functional equation (3.3) in the modular space X_ρ by applying the Δ_2-condition and Fatou property on the modular function ρ.

Here and subsequently, given a mapping $f : V^n \longrightarrow W$, we define the difference operator $\Gamma f : V^n \times V^n \longrightarrow W$ through

$$\Gamma f(x_1, x_2) := 8^n \sum_{q \in \{-2,2\}^n} f\left(\frac{x_1 + qx_2}{2}\right) - \sum_{k=0}^{n} 4^{n-k}(-6)^k f\left(\mathcal{M}_k^n\right)$$

for $x_1, x_2 \in V^n$, where $f(\mathcal{M}_k^n)$ is defined in (3.2).

From now on we assume that V is a real vector space and X_ρ is a complete modular space satisfying the Δ_2-condition, which has the Fatou property unless otherwise stated explicitly. In the next theorem, we establish the stability of functional equation (3.3).

Theorem 5.1 *Let $s \in \{1, -1\}$, and let $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ be a function such that*

$$\sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3n|s-1|j}}{2^{3nj}} \phi\left(2^{sj+1}x_1, 2^{sj+1}x_2\right) < \infty \tag{5.1}$$

for all $x_1, x_2 \in V^n$. Suppose that $f : V^n \longrightarrow X_\rho$ is a mapping satisfying the inequality

$$\rho\left(\Gamma f(x_1, x_2)\right) \leq \phi(x_1, x_2) \tag{5.2}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow X_\rho$ such that

$$\rho\left(f(x) - \mathcal{Q}(x)\right) \leq \frac{1}{2^{n+\frac{3}{2}|s+1|n}} \sum_{j=\frac{|s-1|}{2}}^{\infty} \frac{\kappa^{3n|s-1|j}\phi(2^{js+1}x, 0)}{2^{3nj}} \tag{5.3}$$

for all $x \in V^n$.

Proof We first consider the case $s = 1$. Replacing (x_1, x_2) by $(2x_1, 0)$ in (5.2), we have

$$\rho\left(2^n \times 8^n f(x) - \sum_{k=0}^{n} \binom{n}{k} 4^{n-k} 2^{n-k} (-6)^k f(2x)\right) \leq \phi(2x, 0)$$

for all $x = x_1 \in V^n$. Hence,

$$\rho\left(16^n f(x) - 2^n f(2x)\right) \leq \phi(2x, 0) \tag{5.4}$$

for all $x \in V^n$. Inequality (5.4) implies that

$$\rho\left(f(x) - \frac{f(2x)}{2^{3n}}\right) \leq \frac{1}{2^{4n}} \phi(2x, 0) \tag{5.5}$$

for all $x \in V^n$. Once more, by induction on k, one can prove the following functional inequality:

$$\rho\left(f(x) - \frac{f(2^k x)}{2^{3nk}}\right) \leq \frac{1}{2^{4n}} \sum_{j=0}^{k-1} \frac{\phi(2^{j+1}x, 0)}{2^{3nj}} \tag{5.6}$$

for all $x \in V^n$. Now, interchanging x by $2^l x$ in (5.6), we have

$$\rho\left(\frac{f(2^l x)}{2^{3nl}} - \frac{f(2^{k+l}x)}{2^{3n(k+l)}}\right) \le \frac{1}{2^{4n}} \sum_{j=l}^{k+l-1} \frac{\phi(2^{j+1}x, 0)}{2^{3nj}}$$

for all $x \in V^n$. Since the right-hand side of the above inequality tends to zero as l goes to infinity, the sequence $\{\frac{f(2^k x)}{2^{3nk}}\}$ is a ρ-Cauchy sequence in X_ρ and so the mentioned sequence is ρ-convergent on X_ρ. Thus, we may define the mapping $\mathcal{C} : V \longrightarrow X_\rho$ via $\mathcal{C}(x) = \rho - \lim_{k\to\infty} \frac{f(2^k x)}{2^{3nk}}$ for all $x \in V^n$. In other words, $\lim_{k\to\infty} \rho(\frac{f(2^k x)}{2^{3nk}} - \mathcal{C}(x)) = 0$. Replacing (x_1, x_2) by $(2^k x_1, 2^k x_2)$ in (5.2), and dividing the resulting inequality by 2^{3nk}, we get

$$\rho\left(\frac{1}{2^{3nk}}\Gamma f(2^k x_1, 2^k x_2)\right) \le \frac{1}{2^{3nk}}\rho\left(\Gamma f(2^k x_1, 2^k x_2)\right) \le \frac{1}{2^{3nk}}\phi(2^k x_1, 2^k x_2) \tag{5.7}$$

for all $x_1, x_2 \in V^n$. Consider a fixed real number M such that $M \ge 8^n + 2^n$. By the property $\rho(\alpha t) \le \alpha \rho(t)$ when $0 < \alpha \le 1$, we have

$$\begin{aligned}
\rho\left(\frac{1}{M}\Gamma\mathcal{C}(x_1, x_2)\right) &= \rho\left(\frac{1}{M}\left[\Gamma\mathcal{C}(x_1, x_2) - \frac{\Gamma f(2^k x_1, 2^k x_2)}{2^{3nk}} + \frac{\Gamma f(2^k x_1, 2^k x_2)}{2^{3nk}}\right]\right) \\
&\le \frac{8^n}{M} \sum_{q\in\{-2,2\}^n} \rho\left(\mathcal{C}\left(\frac{x_1 + q x_2}{2}\right) - \frac{1}{2^{3nk}}f\left(\frac{2^k(x_1 + q x_2)}{2}\right)\right) \\
&\quad + \frac{1}{M}\sum_{k=0}^n 4^{n-k}(-6)^k \sum_{\mathfrak{N}_n \in \mathcal{M}_T^n} \rho\left(\mathcal{C}(\mathfrak{N}_n) - \frac{f(2^k \mathfrak{N}_n)}{2^{3nk}}\right) \\
&\quad + \frac{1}{M}\rho\left(\frac{\Gamma f(2^k x_1, 2^k x_2)}{2^{3nk}}\right)
\end{aligned}$$

for all $x_1, x_2 \in V^n$ and all positive integers k. Taking the limit as $k \to \infty$ and using (5.7), we see that $\rho(\frac{1}{M}\Gamma\mathcal{C}(x_1, x_2)) = 0$, and hence $\Gamma\mathcal{C}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that $\mathcal{C}$ is a multi-cubic mapping. Now, the Fatou property of modular ρ implies that

$$\rho\left(f(x) - \mathcal{C}(x)\right) \le \liminf_{\rho\to\infty}\left(f(x) - \frac{f(2^k x)}{2^{3nk}}\right) \le \frac{1}{2^{4n}} \sum_{j=0}^{k-1} \frac{\phi(2^{j+1}x, 0)}{2^{3nj}}$$

for all $x \in V^n$, which shows that the relation (5.3) holds. For the uniqueness of $\mathcal{C}$, we assume that there exists another multi-cubic mapping $\mathcal{C}_0 : V^n \longrightarrow X_\rho$ such that

$$\rho\left(f(x) - \mathcal{C}_0(x)\right) \le \frac{1}{2^{4n}} \sum_{j=0}^{\infty} \frac{\phi(2^{j+1}x, 0)}{2^{3nj}} \tag{5.8}$$

for all $x \in V^n$ such that $\mathcal{C}_0(x_*) \ne \mathcal{C}(x_*)$ for some $x_* \in V^n$. In other words, there is a positive constant $\delta > 0$ such that $\rho(\mathcal{C}_0(x_*) - \mathcal{C}(x_*)) > \delta$. On the other hand, there is a positive integer $p_0 \in \mathbb{N}$ such that $\frac{1}{2^{4n}} \sum_{j=p_0}^{\infty} \frac{\phi(2^{j+1}x, 0)}{2^{3nj}} < \delta$. Since $\mathcal{C}$ and $\mathcal{C}_0$ are multi-cubic mappings, we have

$C_0(2^{p_0}x) = 2^{3np_0}C_0(x)$ and $C(2^{p_0}x) = 2^{3np_0}C(x)$. Thus,

$$\delta < \rho\big(C_0(x) - C(x)\big) \le \frac{1}{2^{3np_0}}\rho\big(C_0\big(2^{p_0}x\big) - f\big(2^{p_0}x\big)\big) + \frac{1}{2^{3np_0}}\rho\big(f\big(2^{p_0}x\big) - C\big(2^{p_0}x\big)\big)$$

$$\le \frac{1}{2^{4n}}\sum_{j=0}^{\infty}\frac{\phi(2^{j+1+p_0}x,0)}{2^{3n(j+1+p_0)}} = \frac{1}{2^{4n}}\sum_{j=p_0}^{\infty}\frac{\phi(2^{j+1}x,0)}{2^{3nj}} < \delta,$$

which is a contradiction. Now, assume that $s = -1$. It follows (5.4) that

$$\rho\left(f(x) - 2^{3n}f\left(\frac{x}{2}\right)\right) \le \frac{1}{2^n}\phi(x,0) \tag{5.9}$$

for all $x \in V^n$. By the convexity of the modular ρ, Δ_2-condition and (5.9), we have

$$\rho\left(f(x) - 2^{6n}f\left(\frac{x}{2^2}\right)\right)$$

$$\le \frac{1}{2^{3n}}\rho\left(2^{3n}f(x) - 2^{6n}f\left(\frac{x}{2}\right)\right) + \frac{1}{2^{3n}}\rho\left(2^{6n}f\left(\frac{x}{2}\right) - 2^{9n}f\left(\frac{x}{2^2}\right)\right)$$

$$\le \frac{1}{2^n}\left[\frac{\kappa^{3n}}{2^{3n}}\phi(x,0) + \frac{\kappa^{6n}}{2^{3n}}\phi\left(\frac{x}{2},0\right)\right] \tag{5.10}$$

for all $x \in V^n$. It is routine to show by induction on $k > 1$ that

$$\rho\left(f(x) - 2^{3nk}f\left(\frac{x}{2^k}\right)\right) \le \frac{1}{2^n}\left[\sum_{j=1}^{k-1}\frac{\kappa^{3n(2j-1)}}{2^{3nj}}\phi\left(\frac{x}{2^{j-1}},0\right) + \frac{\kappa^{6n(k-1)}}{2^{3n(k-1)}}\phi\left(\frac{x}{2^{k-1}},0\right)\right] \tag{5.11}$$

for all $x \in V^n$. Replacing x by $\frac{x}{2^l}$ in (5.11), we get

$$\rho\left(2^{3nl}f\left(\frac{x}{2^l}\right) - 2^{3n(k+l)}f\left(\frac{x}{2^{k+l}}\right)\right)$$

$$\le \kappa^{3nl}\rho\left(f\left(\frac{x}{2^l}\right) - 2^{3nk}f\left(\frac{x}{2^{k+l}}\right)\right)$$

$$\le \frac{\kappa^{3nl}}{2^n}\left[\sum_{j=1}^{k-1}\frac{\kappa^{3n(2j-1)}}{2^{3nj}}\phi\left(\frac{x}{2^{j-1+l}},0\right) + \frac{\kappa^{6n(k-1)}}{2^{3n(k-1)}}\phi\left(\frac{x}{2^{k-1+l}},0\right)\right]$$

$$\le \frac{1}{2^n}\frac{2^{3nl}}{\kappa^{3nl}}\left[\sum_{j=1+l}^{k+l-1}\frac{\kappa^{3n(2j-1)}}{2^{3nj}}\phi\left(\frac{x}{2^{j-1}},0\right) + \frac{\kappa^{3n(k+l-1)}}{2^{3n(k+l-1)}}\phi\left(\frac{x}{2^{k-1+l}},0\right)\right] \tag{5.12}$$

for all $x \in V^n$. It follows from (5.1) and (5.12) that the sequence $\{2^{3nk}f(\frac{x}{2^k})\}$ is a ρ-Cauchy sequence in X_ρ and so there exists the mapping $C : V^n \longrightarrow X_\rho$ such that $C(x) = \rho - \lim_{k\to\infty} 2^{3nk}f(\frac{x}{2^k})$. This means that the mentioned sequence is a ρ-convergent to $C(x)$. Using the Δ_2-condition without applying the Fatou property, we obtain

$$\rho\big(f(x) - C(x)\big) \le \frac{1}{2^{3n}}\rho\left(2^{3n}f(x) - 2^{3(k+1)n}f\left(\frac{x}{2^k}\right)\right) + \frac{1}{2^{3n}}\rho\left(2^{3(k+1)n}f\left(\frac{x}{2^k}\right) - 2^{3n}C(x)\right)$$

$$\le \frac{\kappa^{3n}}{2^{3n}}\rho\left(f(x) - 2^{3nk}f\left(\frac{x}{2^k}\right)\right) + \frac{\kappa^{3n}}{2^{3n}}\rho\left(2^{3nk}f\left(\frac{x}{2^k}\right) - C(x)\right)$$

$$\leq \frac{1}{2^n} \frac{\kappa^{3n}}{2^{3n}} \left[\sum_{j=1}^{k-1} \frac{\kappa^{3n(2j-1)}}{2^{3nj}} \phi\left(\frac{x}{2^{j-1}},0\right) + \frac{\kappa^{6n(k-1)}}{2^{3n(k-1)}} \phi\left(\frac{x}{2^{k-1}},0\right) \right]$$
$$+ \frac{\kappa^{3n}}{2^{3n}} \rho\left(2^{3nk} f\left(\frac{x}{2^k}\right) - \mathcal{C}(x) \right)$$

for all $x \in V^n$. Letting $k \to \infty$, we see that (5.3) holds. The rest of the proof is similar to the previous case. This completes the proof. $\qquad\square$

The following corollaries are the direct consequences of Theorem 5.1 concerning the stability of (3.3).

Corollary 5.2 *Suppose $\theta > 0$ and let $r > 0$ be such that $r \neq 3n, \log_2^{\frac{\kappa^{6n}}{2^{3n}}}$. Let V be a normed space and let X_ρ be a ρ-complete convex modular space. If $f : V^n \longrightarrow X_\rho$ is a mapping satisfying*

$$\rho\big(\Gamma f(x_1,x_2)\big) \leq \theta \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^r$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow X_\rho$ such that

$$\rho\big(f(x) - \mathcal{C}(x)\big) \leq \begin{cases} \frac{2^r \theta}{2^n(2^{3n}-2^r)} \sum_{j=1}^{n} \|x_{1j}\|^r & r \in (0,3n), \\ \frac{2^r \times \kappa^{6n}\theta}{2^n(2^{3n+r}-\kappa^{6n})} \sum_{j=1}^{n} \|x_{1j}\|^r & r \in (\log_2^{\frac{\kappa^{6n}}{2^{3n}}},\infty), \end{cases}$$

for all $x = x_1 \in V^n$.

Proof Putting $\phi(x_1,x_2) = \theta \sum_{k=1}^{2} \sum_{j=1}^{n} \|x_{kj}\|^r$ in Theorem 5.1, one can obtain the first and second inequalities for $s = 1$ and $s = -1$, respectively. $\qquad\square$

Here, we have the following observations:
- In the case $s = 1$ of Theorem 5.1, we have used the Fatou property, while the Δ_2-condition was not applied, and vice versa for the case $s = -1$.
- In Corollary 5.2, if the Δ_2-constant is $\kappa = 2$, then $\log_2^{\frac{\kappa^{6n}}{2^{3n}}} = 3n$. Hence, in this case the second condition converts to $r \in (3n,\infty)$.

Corollary 5.3 *Consider a $\delta > 0$, let V be a normed space, and let X_ρ a ρ-complete convex modular space. If $f : V^n \longrightarrow X_\rho$ is a mapping satisfying*

$$\rho\big(\Gamma f(x_1,x_2)\big) \leq \delta$$

for all $x_1, x_2 \in V^n$, then there exists a unique multi-cubic mapping $\mathcal{C} : V^n \longrightarrow X_\rho$ such that

$$\rho\big(f(x) - \mathcal{C}(x)\big) \leq \frac{\delta}{2^n(2^{3n}-1)}$$

for all $x \in V^n$.

Proof Letting $\phi(x_1, x_2) = \delta$ in Theorem 5.1 for the case $s = 1$, we establish the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Recall that a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$. Under some conditions, the functional equation (3.3) can be hyperstable as follows.

Corollary 5.4 *Suppose $\theta > 0$, let V be a normed space, and let X_ρ be a ρ-complete convex modular space. Suppose that $\alpha_{kj} > 0$ for $k \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$ fulfill $\sum_{k=1}^{2} \sum_{j=1}^{n} \alpha_{kj} \neq 3n, \log_2^{\frac{\kappa^{6n}}{2^{3n}}}$. If $f : V^n \longrightarrow X_\rho$ is a mapping satisfying*

$$\rho\big(\Gamma f(x_1, x_2)\big) \leq \theta \prod_{k=1}^{2} \prod_{j=1}^{n} \|x_{kj}\|^{\alpha_{kj}}$$

for all $x_1, x_2 \in V^n$, then it is multi-cubic.

6 Conclusion

We studied n-variable mappings which are cubic in each variable and showed that such mappings can be characterized by an equation. We also established the stability and hyperstability of such functional equation by using the direct method in the modular spaces.

7 Conventions

1 *Multi-cubic mapping*: An n-variable mapping f is multi-cubic if it is cubic in each variable.
2 *Stability*: A functional equation $\mathcal{F}$ is stable when a mapping that approximately satisfies $\mathcal{F}$ is close to an exact solution of $\mathcal{F}$.
3 *Hyperstability*: A functional equation $\mathcal{F}$ is hyperstable if any mapping f satisfying equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$.

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