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Research article

Ulam stability of linear differential equations using Fourier transform

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Abstract: The purpose of this paper is to study the Hyers-Ulam stability and generalized Hyers-Ulam stability of general linear differential equations of nth order with constant coefficients by using the Fourier transform method. Moreover, the Hyers-Ulam stability constants are obtained for these differential equations.

Keywords: Hyers-Ulam stability; generalized Hyers-Ulam stability; nth order linear differential equation; Fourier transform method; Hyers-Ulam constant
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1. Introduction

A classical question in the theory of functional equation is the following: "When is it true that a function which approximately satisfies a functional equation g must be close to an exact solution of g?" If the problem accepts a solution, we say that the equation g is stable.

The Hyers-Ulam stability problem was introduced by S. M. Ulam [27] in 1940. He motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Theorem 1.1. [27] *Let G_1 be a group and G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)\, f(y)) < \delta$, for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ satisfying $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?*

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [6] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the case of approximately additive mappings, when G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated theorem.

Theorem 1.2. [6] *Assume that G_1 and G_2 are Banach spaces. If a mapping $f : G_1 \to G_2$ satisfies the inequality $\|f(x + y) - f(x) - f(y)\| \leq \epsilon$ for some $\epsilon > 0$ and for all $x, y \in G_1$, then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in G_1$ and $A : G_1 \to G_2$ is the unique additive mapping such that

$$\|f(x) - A(x)\| \leq \epsilon \tag{1.1}$$

for all $x \in G_1$.

Taking the above fact into account, the additive functional equation

$$f(x + y) = f(x) + f(y)$$

is said to have *Hyers-Ulam stability* on (G_1, G_2). In the above theorem, an additive mapping A satisfying the inequality (1.1) is constructed directly from the given mapping f and it is the most powerful tool to study the stability of several functional equations. In course of time, the theorem formulated by Hyers was generalized by Aoki [4] for additive mappings.

There is no reason for the Cauchy difference $f(x+y)-f(x)-f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [24] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam-stability for the additive Cauchy equation. This terminology is justified because the theorem of Rassias has strongly influenced Mathematicians studying stability problems of functional equation. In fact, Rassias proved the following theorem.

Theorem 1.3. [24] *Let X and Y be Banach spaces. Let $\theta \in (0, \infty)$ and $p \in [0, 1)$. If a mapping $f : X \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta \, (\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \, \|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is $\mathbb{R}$-linear.

See [9] for more information on functional equations and their stability.

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', \cdots, x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', \cdots, x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and

$$\lim_{\epsilon \to 0} K(\epsilon) = 0.$$

If the preceeding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [20, 21]. Thereafter, In 1998, Alsina and Ger [3] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved the following theorem.

Theorem 1.4. [3] *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of Alsina and Ger [3] has been generalized by Takahasi [26]. He proved in [26] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [7, 8, 10, 11, 17].

In 2006, Jung [12] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, Wang, Zhou and Sun [28] studied the Hyers-Ulam stability of a class of first-order linear differential equations. Rus [25] discussed four types of Ulam stability: Hyers-Ulam stability, generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability of the ordinary differential equation

$$u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b].$$

In 2014, Alqifiary and Jung [1] proved the Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \, x^{(k)}(t) \;=\; f(t)$$

by using the Laplace transform method, where α_k are scalars and x and f are n times continuously differentiable function and of the exponential order, respectively.

Recently, Liu *et al.* [16] established the Hyers-Ulam stability of linear Caputo-Fabrizio fractional differential equation using the Laplace transform method, derived a generalized Hyers-Ulam stability result via the Gronwall inequality and established the existence and uniqueness of solutions for nonlinear Caputo-Fabrizio fractional differential equations using the generalized Banach fixed point theorem and Schaefer's fixed point theorem.

In 2019, Onitsuka [22] studied the Hyers-Ulam stability of the linear differential equation of the form $x' = ax + f(t)$ without the assumption of continuity of $f(t)$. He also obtained the Best constant in Hyers-Ulam stability of the first-order homogeneous linear differential equations with a periodic coefficient in [5]. That is, he concerned with the Hyers-Ulam stability of the first-order homogeneous linear differential equation $x' - a(t)x = 0$ on $\mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function. However, sufficient conditions for Hyers-Ulam stability are presented in spite of $a(t)$ has infinitely many zeros and changes sign. Furthermore, the best constant in Hyers-Ulam stability is clarified.

These days, the Hyers-Ulam stability of differential equations has been investigated in a series of papers [2, 13–15, 18, 19, 23, 29–34] and the investigation is ongoing.

Motivated and connected by the above facts, in this paper, by applying Fourier transform method, we prove the Hyers-Ulam stability and the generalized Hyers-Ulam stability of the nth order homogeneous linear differential equation of the form

$$x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) = 0 \tag{1.2}$$

and the non-homogeneous linear differential equation

$$x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) = r(t), \tag{1.3}$$

where $x(t)$ is an n times continuously differentiable function, a_i's are scalars and $r(t)$ is a continuously differentiable function.

2. Preliminaries

In this section, we introduce some standard notations, definitions and theorems, which are very useful to prove our main results.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|f(t)| \le Ae^{tB}$ for all $t > 0$.

For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, let g denote the Fourier transform of f so that

$$g(s) = \int_{-\infty}^{\infty} f(t)\, e^{-its}\, ds.$$

Then, at points of continuity of f, we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{-its}\, ds,$$

which is called the inverse Fourier transform. The Fourier transform of f is denoted by $\mathcal{F}(\xi)$. We also introduce the convolution of two functions.

Definition 2.1. (Convolution). Given two functions f and g, both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(t) = \int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 2.2. *The Fourier transform of the convolution of $f(t)$ and $g(t)$ is the product of the Fourier transform of $f(t)$ and $g(t)$. That is,*

$$\mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{f(t)\}\,\mathcal{F}\{g(t)\} = F(s)\,G(s)$$

or

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)\,g(t-s)\,ds\right\} = F(s)\,G(s),$$

where $F(s)$ and $G(s)$ are the Fourier transforms of $f(t)$ and $g(t)$, respectively.

Now, we give the definitions of Hyers-Ulam stability and generalized Hyers-Ulam stability of the differential equations (1.2) and (1.3).

Definition 2.3. The homogeneous linear differential equation (1.2) is said to have the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ and an n times continuously differentiable function $x(t)$ satisfying the inequality

$$\left|x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t)\right| \le \epsilon,$$

there exists some $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t) - y(t)| \le K\epsilon$ for all $t > 0$. We call such K as the Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.4. We say that the non-homogeneous linear differential equation (1.3) has the Hyers-Ulam stability if there exists a positive constant K satisfying the following condition: For every $\epsilon > 0$ and an n times continuously differentiable function $x(t)$ satisfying the inequality

$$\left|x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t) - r(t)\right| \le \epsilon,$$

there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation (1.3) such that $|x(t)-y(t)| \le K\epsilon$ for all $t > 0$. We call such K as the Hyers-Ulam stability constant for the differential equation (1.3).

Definition 2.5. We say that the homogeneous linear differential equation (1.2) has the generalized Hyers-Ulam stability, if there exists a constant $K > 0$ satisfying the following property: For every $\epsilon > 0$, an n times continuously differentiable function $x(t)$ and a function $\phi : (0,\infty) \to (0,\infty)$ satisfying the inequality

$$\left|x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t)\right| \le \phi(t)\epsilon,$$

there exists some $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t)-y(t)| \le K\,\phi(t)\epsilon$ for all $t > 0$. We call such K as the generalized Hyers-Ulam stability constant for differential equation (1.2).

Definition 2.6. The linear differential equation (1.3) is said to have the generalized Hyers-Ulam stability if there exists a positive constant K satisfying the following condition: For every $\epsilon > 0$, an n times continuously differentiable function $x(t)$ and a function $\phi : (0,\infty) \to (0,\infty)$ satisfying the inequality

$$\left|x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t) - r(t)\right| \le \phi(t)\epsilon,$$

there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation (1.3) such that $|x(t) - y(t)| \le K\,\phi(t)\epsilon$ for all $t > 0$. We call such K as the generalized Hyers-Ulam stability constant for the differential equation (1.3).

3. Hyers-Ulam stability

In the following theorems, we prove the Hyers-Ulam stability of the homogeneous and non-homogeneous linear differential equations (1.2) and (1.3). First of all, we establish the Hyers-Ulam stability of the differential equation (1.2).

Theorem 3.1. *Let $a_i \in \mathbb{F}$. Assume that for every $\epsilon > 0$ there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ which is n times continuously differentiable function satisfying the inequality*

$$\left| x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n \, x(t) \right| \leq \epsilon \tag{3.1}$$

for all $t > 0$. Then there exists a solution $y : (0, \infty) \to \mathbb{F}$ of the differential equation (1.2) *such that $|x(t) - y(t)| \leq K\epsilon$ for all $t > 0$.*

Proof. Assume that $x(t)$ is n times continuously differentiable function satisfying the inequality (3.1). Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that

$$p(t) =: x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n \, x(t)$$

for each $t > 0$. In view of (3.1), we have $|p(t)| \leq \epsilon$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\left\{ x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n \, x(t) \right\}, \\ P(\xi) &= \left\{ (-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n \right\} X(\xi), \\ X(\xi) &= \frac{P(\xi)}{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}. \end{aligned}$$

Let $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$ be the distinct roots of the characteristic equation

$$\nu^n + a_1 \nu^{n-1} + \cdots + a_{n-1}\nu + a_n = 0.$$

Since $a_i's$ are constants in $\mathbb{F}$ such that there exist $\nu_1, \nu_2, \nu_3, \cdots, \nu_n \in \mathbb{F}$ with

$$\text{the sum of the roots} = \nu_1 + \nu_2 + \nu_3 + \cdots + \nu_n = \sum_{j=1}^{n} \nu_\ell = -a_1,$$

$$\text{the sum of the roots at 2 time} = \nu_1\nu_2 + \nu_1\nu_3 + \cdots + \nu_{n-1}\nu_n = \sum_{j<k}^{n} \nu_i \nu_j = a_2,$$

$$\text{the sum of the roots at 3 time} = \nu_1\nu_2\nu_3 + \cdots + \nu_{n-2}\nu_{n-1}\nu_n = \sum_{j<k<l}^{n} \nu_i \nu_j \nu_l = -a_3$$

$$\cdots$$

$$\text{the product of the roots} = \nu_1 \ \nu_2 \ \nu_3 \ \cdots \ \nu_n = \prod_{j=1}^{n} \nu_j = (-1)^n \, a_n,$$

we have

$$(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n = (i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3) \cdots (i\xi - \nu_n).$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}. \tag{3.2}$$

Taking $Q(\xi) = \dfrac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}$, we have

$$q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}\right\}.$$

Now, let us define

$$y(t) = \sum_{j=1}^{n} \nu_j e^{-\nu_j t}$$

and taking the Fourier transform to $y(t)$, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \nu_j e^{-\nu_j t}\, e^{ist}\, dt = 0. \tag{3.3}$$

Now, $\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1} y'(t) + a_n\, y(t)\}$

$$= \left\{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n\right\}\, Y(\xi).$$

Then by using (3.3), we have

$$\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1} y'(t) + a_n\, y(t)\} = 0.$$

Since $\mathcal{F}$ is one-to-one, $y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1} y'(t) + a_n\, y(t) = 0$. Hence $y(t)$ satisfies the differential equation (1.2) and thus $y(t)$ is a solution of (1.2). Then by using (3.2) and (3.3) we can obtain

$$\begin{aligned}\mathcal{F}\{x(t)\} - F\{y(t)\} = X(\xi) - Y(\xi) &= \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}\\ &= P(\xi)\, Q(\xi) = \mathcal{F}\{p(t)\}\,\mathcal{F}\{q(t)\},\end{aligned}$$

which implies

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(s)\, q(t-s)\, ds\right| \le |p(t)| \left|\int_{-\infty}^{\infty} q(t-s)\, ds\right| \le K\epsilon,$$

where $K = \left|\int_{-\infty}^{\infty} q(t-s)\, ds\right|$ and the integral exists for each value of t. Then by the virtue of Definition 2.3, the homogeneous linear differential equation (1.2) has the Hyers-Ulam stability. □

Now, we prove the Hyers-Ulam stability of the non-homogeneous linear differential equation (1.3) by using the Fourier transform method.

Theorem 3.2. *Let* $a_i \in \mathbb{F}$*. Assume that for every* $\epsilon > 0$ *there exists a positive constant* K *such that* $x : (0, \infty) \to \mathbb{F}$ *is an* n *times continuously differentiable function satisfying the inequality*

$$\left| x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t) \right| \leq \epsilon \tag{3.4}$$

for all $t > 0$*. Then there exists a solution* $y : (0, \infty) \to \mathbb{F}$ *of the differential equation* (1.3) *such that* $|x(t) - y(t)| \leq K\epsilon$ *for all* $t > 0$*.*

Proof. Assume that $x(t)$ is n times continuously differentiable function satisfying the inequality (3.1). Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that

$$p(t) =: x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t)$$

for each $t > 0$. In view of (3.4), we have $|p(t)| \leq \epsilon$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\left\{ x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t) \right\}, \\ P(\xi) &= \left\{ (-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n \right\} X(\xi) - R(\xi), \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}. \end{aligned}$$

Let $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$ be the distinct roots of the characteristic equation

$$\nu^n + a_1 \nu^{n-1} + \cdots + a_{n-1}\nu + a_n = 0.$$

Since $a_i' s$ are constants in $\mathbb{F}$ such that there exist $\nu_1, \nu_2, \nu_3, \cdots, \nu_n \in \mathbb{F}$ with

$$(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n = (i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3) \cdots (i\xi - \nu_n),$$

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3) \cdots (i\xi - \nu_n)}. \tag{3.5}$$

Taking $Q(\xi) = \dfrac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3) \cdots (i\xi - \nu_n)}$, we have

$$q(t) = \mathcal{F}^{-1}\left\{ \frac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3) \ldots (i\xi - \nu_n)} \right\}.$$

Now, let us define

$$y(t) = \sum_{j=1}^{n} \nu_j e^{-\nu_j t} + (r(t) * q(t))$$

and taking the Fourier transform to $y(t)$, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \nu_j e^{-\nu_j t}\, e^{ist}\, dt + R(\xi)\, Q(\xi)$$

$$= \frac{R(\xi)}{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}. \tag{3.6}$$

Now, $\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\}$

$$= \left\{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n\right\}\, Y(\xi) = R(\xi).$$

Then by using (3.6), we have

$$\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\} = \mathcal{F}\{r(t)\}.$$

Since $\mathcal{F}$ is one-to-one, $y^{(n)}(t)+a_1y^{(n-1)}+\cdots+a_{n-1}y'(t)+a_n\,y(t) = r(t)$. Hence $y(t)$ satisfies the differential equation (1.3) and thus $y(t)$ is a solution of (1.3). Then by using (3.5) and (3.6) we can have

$$\begin{aligned}\mathcal{F}\{x(t)\} - F\{y(t)\} = X(\xi) - Y(\xi) &= \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}\\ &= P(\xi)\,Q(\xi) = \mathcal{F}\{p(t)\}\,\mathcal{F}\{q(t)\} = \mathcal{F}\{p(t) * q(t)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(s)\,q(t-s)\,ds\right| \le |p(t)|\,\left|\int_{-\infty}^{\infty} q(t-s)\,ds\right| \le K\epsilon,$$

where $K = \left|\int_{-\infty}^{\infty} q(t-s)\,ds\right|$ and the integral exists for each value of t. Then by the virtue of Definition 2.4, the non-homogeneous linear differential equation (1.3) has the Hyers-Ulam stability. □

4. Generalized Hyers-Ulam stability

In the following theorems, we are going to study the generalized Hyers-Ulam stability of the differential equations (1.2) and (1.3). First, we investigate the generalized Hyers-Ulam stability of the differential equation (1.2).

Theorem 4.1. *Let $a_i \in \mathbb{F}$. Assume that for every $\epsilon > 0$ there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ is an n times continuously differentiable function and $\phi : (0,\infty) \to (0,\infty)$ is an integrable function satisfying the inequality*

$$\left|x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t)\right| \le \phi(t)\epsilon \tag{4.1}$$

for all $t > 0$. Then there exists a solution $y : (0,\infty) \to \mathbb{F}$ of the differential equation (1.2) such that $|x(t) - y(t)| \le K\phi(t)\epsilon$ for all $t > 0$.

Proof. Assume that $x(t)$ is n times continuously differentiable function and $\phi : (0,\infty) \to (0,\infty)$ is an integrable function satisfying the inequality (4.1). Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that

$$p(t) =: x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t)$$

for each $t > 0$. In view of (4.1), we have $|p(t)| \leq \phi(t)\epsilon$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned}
\mathcal{F}\{p(t)\} &= \mathcal{F}\left\{x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1}x'(t) + a_n\, x(t)\right\},\\
P(\xi) &= \left\{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n\right\} X(\xi),\\
X(\xi) &= \frac{P(\xi)}{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}.
\end{aligned}$$

Let $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$ be the distinct roots of the characteristic equation

$$\nu^n + a_1\nu^{n-1} + \cdots + a_{n-1}\nu + a_n = 0.$$

Since $a_i's$ are constants in $\mathbb{F}$ such that there exist $\nu_1, \nu_2, \nu_3, \cdots, \nu_n \in \mathbb{F}$ with

$$(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n = (i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n),$$

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}. \tag{4.2}$$

Taking $Q(\xi) = \dfrac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}$, we have

$$q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}\right\}.$$

Now, let us define

$$y(t) = \sum_{j=1}^{n} \nu_j e^{-\nu_j t}$$

and taking the Fourier transform to $y(t)$, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \nu_j e^{-\nu_j t}\, e^{ist}\, dt = 0. \tag{4.3}$$

Now, $\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\}$

$$= \left\{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n\right\}\, Y(\xi).$$

Then by using (4.3), we have

$$\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\} = 0.$$

Since $\mathcal{F}$ is one-to-one, $y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t) = 0$. Hence $y(t)$ satisfies the differential equation (1.2) and thus $y(t)$ is a solution of (1.2). Then by using (4.2) and (4.3) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = X(\xi) - Y(\xi) = \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}$$

$$= P(\xi)\, Q(\xi) = \mathcal{F}\{p(t)\}\, \mathcal{F}\{q(t)\} = \mathcal{F}\{p(t) * q(t)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(s)\, q(t-s)\, ds \right| \leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\, ds \right| \leq K\phi(t)\epsilon,$$

where $K = \left| \int_{-\infty}^{\infty} q(t-s)\, ds \right|$ and the integral exists for each value of t and $\phi(t)$ is an integrable function. Then by the virtue of Definition 2.5, the differential equation (1.2) has the generalized Hyers-Ulam stability. □

Finally, we prove the Hyers-Ulam stability of the non-homogeneous linear differential equation (1.3) by using the Fourier transform method.

Theorem 4.2. *Let $a_i \in \mathbb{F}$. Assume that for every $\epsilon > 0$ there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ which is n times continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ is an integrable function satisfying the inequality*

$$\left| x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t) \right| \leq \phi(t)\epsilon \tag{4.4}$$

for all $t > 0$. Then there exists a solution $y : (0, \infty) \to \mathbb{F}$ of the differential equation (1.3) *such that $|x(t) - y(t)| \leq K\phi(t)\epsilon$ for all $t > 0$.*

Proof. Assume that $x(t)$ is n times continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ is an integrable function satisfying the inequality (4.4). Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that

$$p(t) =: x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t)$$

for each $t > 0$. In view of (4.4), we have $|p(t)| \leq \phi(t)\epsilon$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} \mathcal{F}\{p(t)\} &= \mathcal{F}\left\{ x^{(n)}(t) + a_1 x^{(n-1)} + \cdots + a_{n-1} x'(t) + a_n\, x(t) - r(t) \right\}, \\ P(\xi) &= \left\{ (-i\xi)^n + a_1 (-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n \right\} X(\xi) - R(\xi), \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{(-i\xi)^n + a_1 (-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}. \end{aligned}$$

Let $v_1, v_2, v_3, \cdots, v_n$ be the distinct roots of the characteristic equation

$$v^n + a_1 v^{n-1} + \cdots + a_{n-1} v + a_n = 0.$$

Since $a_i' s$ are constants in $\mathbb{F}$ such that there exist $v_1, v_2, v_3, \cdots, v_n \in \mathbb{F}$ with

$$(-i\xi)^n + a_1 (-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n = (i\xi - v_1)\,(i\xi - v_2)\,(i\xi - v_3) \cdots (i\xi - v_n),$$

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}. \tag{4.5}$$

Taking $Q(\xi) = \dfrac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}$, we have

$$q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\,...\,(i\xi - \nu_n)}\right\}.$$

Now, let us define

$$y(t) = \sum_{j=1}^{n} \nu_j e^{-\nu_j t} + (r(t) * q(t))$$

and taking the Fourier transform to $y(t)$, we obtain

$$\begin{aligned}\mathcal{F}\{y(t)\} = Y(\xi) &= \int_{-\infty}^{\infty} \sum_{j=1}^{n} \nu_j e^{-\nu_j t}\, e^{ist}\, dt + R(\xi)\, Q(\xi)\\ &= \frac{R(\xi)}{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n}.\end{aligned} \tag{4.6}$$

Now, $\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\}$

$$= \left\{(-i\xi)^n + a_1(-i\xi)^{n-1} + \cdots + a_{n-1}(-i\xi) + a_n\right\}\, Y(\xi) = R(\xi).$$

Then by using (4.6), we have

$$\mathcal{F}\{y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t)\} = \mathcal{F}\{r(t)\}.$$

Since $\mathcal{F}$ is one-to-one, $y^{(n)}(t) + a_1 y^{(n-1)} + \cdots + a_{n-1}y'(t) + a_n\, y(t) = r(t)$. Hence $y(t)$ satisfies the differential equation (1.3) and thus $y(t)$ is a solution of (1.3). Then by using (4.5) and (4.6), we can have

$$\begin{aligned}\mathcal{F}\{x(t)\} - F\{y(t)\} = X(\xi) - Y(\xi) &= \frac{P(\xi)}{(i\xi - \nu_1)\,(i\xi - \nu_2)\,(i\xi - \nu_3)\cdots(i\xi - \nu_n)}\\ &= P(\xi)\, Q(\xi) = \mathcal{F}\{p(t)\}\,\mathcal{F}\{q(t)\} = \mathcal{F}\{p(t) * q(t)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, then we have $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(s)\, q(t-s)\, ds\right| \le |p(t)| \left|\int_{-\infty}^{\infty} q(t-s)\, ds\right| \le K\, \phi(t)\epsilon,$$

where $K = \left|\int_{-\infty}^{\infty} q(t-s)\, ds\right|$ and the integral exists for each value of t and $\phi(t)$ is an integrable function. Hence by the virtue of Definition 2.6, the differential equation (1.3) has the generalized Hyers-Ulam stability. □

5. Conclusion

In this paper, we have proved the Hyers-Ulam stability and the generalized Hyers-Ulam stability of the linear differential equations of nth order with constant coefficients using the Fourier transform method. That is, we have established the sufficient criteria for the Hyers-Ulam stability and the generalized Hyers-Ulam stability of the linear differential equation of nth order with constant coefficients by using the Fourier transform method. Additionally, this paper also provided another method to study the Hyers-Ulam stability of differential equations. Furthermore, we have showed that the Fourier transform method is more convenient to study the Hyers-Ulam stability and the generalized Hyers-Ulam stability of the linear differential equation with constant coefficients.

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Conflict of interest

The authors declare that they have no competing interests.

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