http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(3): 2196–2210.
DOI: 10.3934/math.2020145
Received: 04 October 2019
Accepted: 25 February 2020
Published: 27 February 2020

Research article

Bihomomorphisms and biderivations in Lie Banach algebras

Tae Hun Kim[1], **Ha Nuel Ju**[1], **Hong Nyeong Kim**[1], **Seong Yoon Jo**[1] **and Choonkil Park**[2,*]

[1] Mathematics Branch, Seoul Science High School, Seoul 03066, Korea
[2] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Email: baak@hanyang.ac.kr; Tel: +82222200892; Fax: +82222810019.

Abstract: In this paper, we solve the following bi-additive s-functional inequality

$$\begin{aligned} &\|f(x-y,y+z)+f(y+z,z-x)+f(z+x,x-z)-f(x-y,x+y)\| \\ &\le \|s(f(y-z,z+x)+f(z+x,x-y)+f(x+y,y-x)-f(y-z,y+z))\|, \end{aligned} \tag{0.1}$$

where s is a fixed nonzero complex number satisfying $|s| < 1$. Furthermore, we prove the Hyers-Ulam stability of bihomomorphisms and biderivations in Lie Banach algebras associated with the bi-additive s-functional inequality (0.1).

Keywords: Hyers-Ulam stability; bi-additive s-functional inequality; Lie Banach algebra; bihomomorphism; biderivation
Mathematics Subject Classification: 39B52, 47B47, 39B62, 17B40

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [6] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [7]. Fechner [8] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1.1). Park [10, 11] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [12–17]).

Bae and Park [18] proved the Hyers-Ulam stability of bihomomorphisms and biderivations in C^*-ternary algebras, Shokri, Park and Shin [19] proved the Hyers-Ulam stability of bihomomorphisms and biderivations in intuitionistic fuzzy ternary normed algebras, and Park [20] proved the Hyers-Ulam stability of biderivations and bihomomorphisms in Banach algebras.

Definition 1.1. Let A, B be Lie Banach algebras. A bi-additive mapping $H : A \times A \to B$ is called a *bihomomorphism* if H satisfies

$$\begin{aligned} H([x, y], [z, z]) &= [H(x, z), H(y, z)], \\ H([x, x], [y, z]) &= [H(x, y), H(x, z)] \end{aligned}$$

for all $x, y, z \in A$.

Definition 1.2. Let A be a Lie Banach algebra. A bi-additive mapping $\delta : A \times A \to A$ is called a *biderivation* if δ satisfies

$$\begin{aligned} \delta([x, y], z) &= [\delta(x, z), y] + [x, \delta(y, z)], \\ \delta(x, [y, z]) &= [\delta(x, y), z] + [y, \delta(x, z)] \end{aligned}$$

for all $x, y, z \in A$.

This paper is organized as follows: In Section 2, we solve the bi-additive s-functional inequality (0.1) and prove the Hyers-Ulam stability of the bi-additive s-functional inequality (0.1) in complex Banach spaces. In Section 3, we prove the Hyers-Ulam stability of bihomomorphisms and biderivations in Lie Banach algebras, associated with the bi-additive s-functional inequality (0.1).

Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Bi-additive s-functional inequality (0.1)

Throughout this section, let X be a complex normed space and Y a complex Banach space.

We solve and investigate the bi-additive s-functional inequality (0.1) in complex normed spaces.

Theorem 2.1. *If a mapping $f : X^2 \to Y$ satisfies* (0.1) *for all $x, y, z \in X$, then $f : X^2 \to Y$ is bi-additive.*

Proof. Assume that f satisfies (0.1). Replacing x by y, y by z and z by x in (0.1), we get

$$\begin{aligned} &\|f(y - z, z + x) + f(z + x, x - y) + f(x + y, y - x) - f(y - z, y + z)\| \\ &\quad \le \|s(f(z - x, x + y) + f(x + y, y - z) + f(y + z, z - y) - f(z - x, z + x))\| \qquad (2.1) \end{aligned}$$

for all $x, y, z \in X$. Replacing x by z, y by x, and z by y in (0.1), we get

$$\begin{aligned} &\|f(z-x, x+y) + f(x+y, y-z) + f(y+z, z-y) - f(z-x, z+x)\| \\ &\quad \le \|s(f(x-y, y+z) + f(y+z, z-x) + f(z+x, x-z) - f(x-y, x+y))\| \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$. By (0.1), (2.1), (2.2), we obtain

$$\begin{aligned} &\|f(x-y, y+z) + f(y+z, z-x) + f(z+x, x-z) - f(x-y, x+y)\| \\ &\quad \le \|s^3(f(x-y, y+z) + f(y+z, z-x) + f(z+x, x-z) - f(x-y, x+y))\| \\ &\quad \le \|f(x-y, y+z) + f(y+z, z-x) + f(z+x, x-z) - f(x-y, x+y)\| \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. From (2.3), we get the equality

$$f(x-y, y+z) + f(y+z, z-x) + f(z+x, x-z) - f(x-y, x+y) = 0 \tag{2.4}$$

for all $x, y, z \in X$. By putting $x = y = z = 0$ in (2.4), we get

$$f(0, 0) = 0.$$

Then let's put in (2.4) $x = z = 0$. We have

$$f(y, 0) = 0.$$

Next we take in (2.4) $y = z = 0$. Then

$$f(x, 0) + f(0, -x) = 0.$$

But we already know that $f(x, 0) = 0$. Therefore,

$$f(0, -x) = 0.$$

Replacing x and y by $\frac{x-y}{2}$ and z by $\frac{x+y}{2}$ in (2.4), we get

$$f(x, y) + f(x, -y) = 0 \tag{2.5}$$

for all $x, y \in X$. Replacing x by $\frac{x}{2}$, y by $-\frac{x}{2}$, z by $\frac{x}{2} + y$ in (2.4), we get

$$f(x, y) + f(y, y) + f(x+y, -y) = 0 \tag{2.6}$$

for all $x, y \in X$. Replacing x by $x + y$ in (2.5), we get

$$f(x+y, y) + f(x+y, -y) = 0 \tag{2.7}$$

for all $x, y \in X$. It follows from (2.6) and (2.7) that

$$f(x, y) + f(y, y) - f(x+y, y) = 0 \tag{2.8}$$

for all $x, y \in X$. Replacing x by $\frac{x+y}{2}$, y by $\frac{y-x}{2}$, and z by $-\frac{x+y}{2}$ in (2.4), we get

$$f(x, -x) + f(-x, -x-y) - f(x, y) = 0 \tag{2.9}$$

for all $x, y \in X$. Replacing y by x in (2.5), we get $f(x, x) = -f(x, -x)$ for all $x \in X$, and together with (2.9), we obtain

$$f(x, x) + f(x, y) - f(-x, -x - y) = 0 \tag{2.10}$$

for all $x, y \in X$. Replacing x by $\frac{x}{2} + y$, y by $-\frac{x}{2} + y$, and z by $\frac{x}{2}$ in (2.4), we get

$$f(x, y) + f(y, -y) + f(x + y, y) - f(x, 2y) = 0 \tag{2.11}$$

for all $x, y \in X$. Adding (2.8) and (2.11), we obtain

$$2f(x, y) - f(x, 2y) = 2f(x, y) + f(y, y) + f(y, -y) - f(x, 2y) = 0 \tag{2.12}$$

where the first equality comes from replacing y by x in (2.5). Replacing x by $x - y$ and y by $y + z$ in (2.8), we get

$$-f(x - y, y + z) - f(y + z, y + z) + f(x + z, y + z) = 0 \tag{2.13}$$

for all $x, y, z \in X$. Replacing x by $-y - z$ and y by $x + y$ in (2.10), we get

$$f(-y - z, -y - z) + f(-y - z, x + y) - f(y + z, -x + z) = 0 \tag{2.14}$$

for all $x, y, z \in X$. Replacing x by $y + z$ and letting $y = 0$ in (2.10), we get

$$f(y + z, y + z) - f(-y - z, -y - z) = 0 \tag{2.15}$$

for all $x, y, z \in X$. Adding (2.4), (2.13), (2.14) and (2.15), we obtain

$$f(x + z, x - z) - f(x - y, x + y) + f(x + z, y + z) + f(-y - z, x + y) = 0 \tag{2.16}$$

for all $x, y, z \in X$. Replacing x by $\frac{x+y}{2}$, y by $\frac{y+z-x}{2}$, z by $\frac{x-y}{2}$ in (2.16), we get

$$f(x, y) - f\left(x - \frac{z}{2}, y + \frac{z}{2}\right) + f\left(x, \frac{z}{2}\right) + f\left(-\frac{z}{2}, y + \frac{z}{2}\right) = 0 \tag{2.17}$$

for all $x, y, z \in X$. Replacing x by $\frac{x-y-z}{2}$, y by $-\frac{x+y}{2}$, z by $\frac{x+y+z}{2}$ in (2.16), we get

$$f(x, -y - z) - f\left(x - \frac{z}{2}, -y - \frac{z}{2}\right) + f\left(x, \frac{z}{2}\right) + f\left(-\frac{z}{2}, -y - \frac{z}{2}\right) = 0 \tag{2.18}$$

for all $x, y, z \in X$. Adding (2.17) and (2.18), and from (2.5) and (2.12), we obtain

$$\begin{aligned} f(x, y) - f\left(x - \frac{z}{2}, y + \frac{z}{2}\right) - f\left(x - \frac{z}{2}, -y - \frac{z}{2}\right) + 2f\left(x, \frac{z}{2}\right) \\ + f\left(-\frac{z}{2}, y + \frac{z}{2}\right) + f\left(-\frac{z}{2}, -y - \frac{z}{2}\right) + f(x, -y - z) = 0 \end{aligned}$$

and so

$$f(x, y) + f(x, z) - f(x, y + z) = 0 \tag{2.19}$$

for all $x, y, z \in X$. Thus $f : X^2 \to Y$ is additive in the second variable.

Replacing y by $y - x$ and z by x in (2.19), we get

$$-f(x, y - x) - f(x, x) + f(x, y) = 0 \tag{2.20}$$

for all $x, y \in X$ and replacing y by $y - x$ in (2.10), we get

$$f(x, x) + f(x, y - x) - f(-x, -y) = 0 \tag{2.21}$$

for all $x, y \in X$. Adding (2.20) and (2.21), we get

$$f(x, y) - f(-x - y) = 0 \tag{2.22}$$

for all $x, y \in X$. Replacing y by z and z by y in (2.14), we get

$$f(x - z, y + z) + f(y + z, y - x) + f(x + y, x - y) - f(x - z, x + z) = 0 \tag{2.23}$$

for all $x, y, z \in X$. Using (2.5), (2.22) and (2.23), we obtain

$$\begin{aligned} & f(y + z, x - y) + f(z - x, y + z) + f(x - z, x + z) - f(x + y, x - y) \\ &= -f(y + z, y - x) - f(x - z, y + z) + f(x - z, x + z) - f(x + y, x - y) \\ &= -(f(x - z, y + z) + f(y + z, y - x) + f(x + y, x - y) - f(x - z, x + z)) = 0 \end{aligned} \tag{2.24}$$

for all $x, y, z \in X$.

Define a mapping $g : X^2 \to Y$ by $g(x, y) = f(y, x)$ for all $x, y \in X$. Then, from (2.24), g also satisfies (2.4). Thus, in a similar way, we can prove that g is additive in the second variable, which yields that $f : X^2 \to Y$ is additive in the first variable.

Therefore, $f : X^2 \to Y$ is a bi-additive mapping. □

Now, we prove the Hyers-Ulam stability of the bi-additive s-functional inequality (0.1).

Theorem 2.2. *Let $0 < r < 2$ and θ be nonnegative real number. If a mapping $f : X^2 \to Y$ satisfies $f(0, x) = f(x, 0) = 0$ and*

$$\begin{aligned} & \|f(x - y, y + z) + f(y + z, z - x) + f(z + x, x - z) - f(x - y, x + y)\| \\ & \quad \leq \|s(f(y - z, z + x) + f(z + x, x - y) + f(x + y, y - x) - f(y - z, y + z))\| \\ & \quad +\theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{2.25}$$

for all $x, y, z \in X$, then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that

$$\|f(x, y) - B(x, y)\| \leq \frac{2^r\theta}{4 - 2^r} E(x, y) \tag{2.26}$$

for all $x, y \in X$, where the function $E : X^2 \to \mathbb{R}$ is defined as

$$E(x, y) \quad = \quad (\frac{14}{1 - |s|} + 76 + 12|s|)\left\|\frac{x}{4}\right\|^r + (\frac{9}{1 - |s|} + 26 + 4|s|)\left\|\frac{x}{2}\right\|^r$$

$$
\begin{aligned}
&+ \quad (\frac{1}{1-|s|}+1)\left\|\frac{3x}{4}\right\|^r + (56+16|s|)\left\|\frac{x}{8}\right\|^r + (16+8|s|)\left\|\frac{3x}{8}\right\|^r \\
&+ \quad (\frac{3}{1-|s|}+5)\left\|\frac{x-2y}{4}\right\|^r + (\frac{4}{1-|s|}+18+2|s|)\left\|\frac{x+2y}{4}\right\|^r \\
&+ \quad (\frac{7}{1-|s|}+10)\left\|\frac{y}{2}\right\|^r + (\frac{4}{1-|s|}+3)\left\|\frac{x+y}{2}\right\|^r + \frac{1}{1-|s|}\left\|\frac{x-y}{2}\right\|^r
\end{aligned}
\tag{2.27}
$$

for all $x, y \in X$.

Proof. Replacing x by y, y by z, z by x in (2.25), we get

$$
\begin{aligned}
&\|f(y-z,z+x)+f(z+x,x-y)+f(x+y,y-x)-f(y-z,y+z)\| \\
&\quad \leq \|s(f(z-x,x+y)+f(x+y,y-z)+f(y+z,z-y)-f(z-x,z+x))\| \\
&\quad +\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}
\tag{2.28}
$$

for all $x, y, z \in X$. Replacing x by z, y by x, z by y in (2.25), we get

$$
\begin{aligned}
&\|f(z-x,x+y)+f(x+y,y-z)+f(y+z,z-y)-f(z-x,z+x)\| \\
&\quad \leq \|s(f(x-y,y+z)+f(y+z,z-x)+f(z+x,x-z)-f(x-y,x+y))\| \\
&\quad +\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}
\tag{2.29}
$$

for all $x, y, z \in X$. By (2.25), (2.28) and (2.29), we obtain

$$
\begin{aligned}
&\|f(x-y,y+z)+f(y+z,z-x)+f(z+x,x-z)-f(x-y,x+y)\| \\
&\quad \leq |s|^3\|f(x-y,y+z)+f(y+z,z-x)+f(z+x,x-z)-f(x-y,x+y)\| \\
&\quad +(1+|s|+|s|^2)\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}
$$

and so

$$
\begin{aligned}
&\|f(x-y,y+z)+f(y+z,z-x)+f(z+x,x-z)-f(x-y,x+y)\| \\
&\quad \leq \frac{1}{1-|s|}\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}
\tag{2.30}
$$

for all $x, y, z \in X$. Replacing x, y by $\frac{x-y}{2}$ and z by $\frac{x+y}{2}$ in (2.25), we get

$$
\|f(x,y)+f(x,-y)\| \leq \theta(2\left\|\frac{x-y}{2}\right\|^r + \left\|\frac{x+y}{2}\right\|^r)
\tag{2.31}
$$

for all $x, y \in X$. Replacing x by $\frac{x}{2}$, y by $-\frac{x}{2}$, z by $\frac{x}{2}+y$ in (2.25), we get

$$
\begin{aligned}
&\|f(x,y)+f(y,y)+f(x+y,-y)\| \\
&\leq |s|\|f(-x-y,x+y)+f(x+y,x)-f(-x-y,y)\| + \theta(2\left\|\frac{x}{2}\right\|^r + \left\|\frac{x}{2}+y\right\|^r)
\end{aligned}
\tag{2.32}
$$

for all $x, y \in X$. Replacing x by $-\frac{x}{2}$, y by $\frac{x}{2}+y$, z by $\frac{x}{2}$ in (2.25), we get

$$
|s|\|f(-x-y,x+y)+f(x+y,x)-f(-x-y,y)\| \leq |s|\theta(2\left\|\frac{x}{2}\right\|^r + \left\|\frac{x}{2}+y\right\|^r)
\tag{2.33}
$$

for all $x, y \in X$. Replacing x by $x+y$ in (2.31), we get

$$\|-f(x+y,y)-f(x+y,-y)\| \le \theta(2\left\|\frac{x}{2}\right\|^r+\left\|\frac{x}{2}+y\right\|^r) \tag{2.34}$$

for all $x, y \in X$. Adding (2.32), (2.33) and (2.34), we obtain

$$\|f(x,y)+f(y,y)-f(x+y,y)\| \le (2+|s|)\theta(2\left\|\frac{x}{2}\right\|^r+\left\|\frac{x}{2}+y\right\|^r) \tag{2.35}$$

for all $x, y \in X$. Replacing x by $\frac{x+y}{2}$, y by $\frac{-x+y}{2}$, z by $-\frac{x+y}{2}$ in (2.25), we get

$$\|f(x,y)-f(x,-x)-f(-x,-x-y)\| \le \theta(2\left\|\frac{x+y}{2}\right\|^r+\left\|\frac{x-y}{2}\right\|^r) \tag{2.36}$$

for all $x, y \in X$. Setting $x=y=0$ and replacing z by x in (2.25), we get

$$\|f(x,x)+f(x,-x)\| \le \theta(\|x\|^r) \tag{2.37}$$

for all $x \in X$. Adding (2.36) and (2.37), we get

$$\|f(x,x)+f(x,y)-f(-x,-x-y)\| \le \theta(2\left\|\frac{x+y}{2}\right\|^r+\left\|\frac{x-y}{2}\right\|^r+\|x\|^r) \tag{2.38}$$

for all $x, y \in X$. Replacing x by $\frac{x}{2}+y$ and y by $-\frac{x}{2}+y$ and z by $\frac{x}{2}$ in (2.30), we get

$$\|f(x,y)+f(y,-y)+f(x+y,y)-f(x,2y)\| \le \frac{1}{1-|s|}\theta(\left\|\frac{x}{2}+y\right\|^r+\left\|-\frac{x}{2}+y\right\|^r+\left\|\frac{x}{2}\right\|^r) \tag{2.39}$$

for all $x, y \in X$. Adding (2.35), (2.39) and (2.37) (here, we replace x by y), we obtain

$$\|2f(x,y)-f(x,2y)\| \le \theta E_1(x,y) \tag{2.40}$$

for all $x, y, z \in X$, where the function $E_1 : X^2 \to \mathbb{R}$ is defined as

$$E_1(x,y) = (\frac{1}{1-|s|}+2+|s|)\left\|\frac{x}{2}+y\right\|^r+\frac{1}{1-|s|}\left\|-\frac{x}{2}+y\right\|^r+(\frac{1}{1-|s|}+4+2|s|)\left\|\frac{x}{2}\right\|^r+\|y\|^r$$

for all $x, y \in X$. Replacing x by $x-y$ and y by $y+z$ in (2.35), we get

$$\begin{aligned}&\|-f(x-y,y+z)-f(y+z,y+z)+f(x+z,y+z)\|\\ &\le (2+|s|)\theta(2\|\frac{x-y}{2}\|^r+\|\frac{x+y+2z}{2}\|^r)\end{aligned} \tag{2.41}$$

for all $x, y, z \in X$. Replacing x by $-y-z$ and y by $x+y$ in (2.38) gives

$$\begin{aligned}&\|f(-y-z,-y-z)+f(-y-z,x+y)-f(y+z,-x+z)\|\\ &\le \theta(2\|\frac{x-z}{2}\|^r+\|\frac{x+2y+z}{2}\|^r+\|y+z\|^r)\end{aligned} \tag{2.42}$$

for all $x, y, z \in X$. Replacing x by $y+z$ and setting $y=0$ in (2.38), we get

$$\|f(y+z,y+z)-f(-y-z,-y-z)\| \le \theta(3\|\frac{y+z}{2}\|^r+\|y+z\|^r) \tag{2.43}$$

for all $x, y, z \in X$. Adding (2.30), (2.41), (2.42) and (2.43), we obtain

$$\|f(x+z, x-z) - f(x-y, x+y) + f(x+z, y+z) + f(-y-z, x+y)\| \leq \theta E_2(x, y, z) \tag{2.44}$$

for all $x, y, z \in X$, where the function $E_2 : X^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} E_2(x,y,z) &= \frac{1}{1-|s|}(\|x\|^r + \|y\|^r + \|z\|^r) + (4+2|s|)\left\|\frac{x-y}{2}\right\|^r + (2+|s|)\left\|\frac{x+y+2z}{2}\right\|^r \\ &+ 2\left\|\frac{x-z}{2}\right\|^r + \left\|\frac{x+2y+z}{2}\right\|^r + 2\|y+z\|^r + 3\left\|\frac{y+z}{2}\right\|^r \end{aligned} \tag{2.45}$$

for all $x, y, z \in X$. Replacing x by $\frac{y}{2}$, y by $\frac{-x+y}{2}$, z by $x - \frac{y}{2}$ in (2.44), we get

$$\begin{aligned} &\left\|-f(x, -x+y) + f\left(\frac{x}{2}, -\frac{x}{2}+y\right) - f\left(x, \frac{x}{2}\right) - f\left(-\frac{x}{2}, -\frac{x}{2}+y\right)\right\| \\ &\leq E_2\left(\frac{y}{2}, \frac{-x+y}{2}, x-\frac{y}{2}\right) \end{aligned} \tag{2.46}$$

for all $x, y \in X$. Replacing x by $\frac{x-y}{2}$, y by $-\frac{y}{2}$, z by $\frac{x+y}{2}$ in (2.44), we get

$$\left\|-f(x,-y) + f\left(\frac{x}{2}, \frac{x}{2}-y\right) - f\left(x, \frac{x}{2}\right) - f\left(-\frac{x}{2}, \frac{x}{2}-y\right)\right\| \leq E_2\left(\frac{x-y}{2}, -\frac{y}{2}, \frac{x+y}{2}\right) \tag{2.47}$$

for all $x, y \in X$. Replacing x by $-\frac{x}{2}$ and y by $-\frac{x}{2} + y$ in (2.31), we get

$$\left\|f\left(-\frac{x}{2}, -\frac{x}{2}+y\right) + f\left(-\frac{x}{2}, \frac{x}{2}-y\right)\right\| \leq \theta\left(2\left\|\frac{y}{2}\right\|^r + \left\|\frac{x-y}{2}\right\|^r\right) \tag{2.48}$$

for all $x, y \in X$. Replacing x by $\frac{x}{2}$ and y by $\frac{x}{2} - y$ in (2.31), we get

$$\left\|-f\left(\frac{x}{2}, \frac{x}{2}-y\right) - f\left(\frac{x}{2}, -\frac{x}{2}+y\right)\right\| \leq \theta\left(2\left\|\frac{y}{2}\right\|^r + \left\|\frac{x-y}{2}\right\|^r\right) \tag{2.49}$$

for all $x, y \in X$. Replacing y by $\frac{x}{2}$ in (2.40), we get

$$\|2f(x, \frac{x}{2}) - f(x,x)\| \leq \theta E_1(x, \frac{x}{2}) \tag{2.50}$$

for all $x \in X$. Replacing y by $y - x$ in (2.38), we get

$$\|-f(x,x) - f(x, y-x) + f(-x,-y)\| \leq \theta\left(2\left\|\frac{y}{2}\right\|^r + \left\|x - \frac{y}{2}\right\|^r + \|x\|^r\right) \tag{2.51}$$

for all $x, y \in X$. Adding (2.31), (2.46), (2.47), (2.48), (2.49), (2.50) and (2.51), we obtain

$$\|f(x,y) - f(-x,-y)\| \leq \theta E_3(x,y) \tag{2.52}$$

for all $x, y \in X$, where the function $E_3 : X^2 \to \mathbb{R}$ is defined as

$$E_3(x,y) = \left(\frac{1}{1-|s|} + 3 + |s|\right)\|x\|^r + (4+2|s|)\left\|\frac{3x}{4}\right\|^r + \left(\frac{1}{1-|s|} + 9 + 2|s|\right)\left\|\frac{x}{2}\right\|^r$$

$$+ \quad (14+4|s|)\left\|\frac{x}{4}\right\|^r + (\frac{2}{1-|s|}+9)\left\|\frac{y}{2}\right\|^r + (\frac{2}{1-|s|}+7)\left\|\frac{x-y}{2}\right\|^r$$
$$+ \quad (\frac{1}{1-|s|}+1)\left\|\frac{x+y}{2}\right\|^r + (\frac{1}{1-|s|}+1)\left\|x-\frac{y}{2}\right\|^r$$

for all $x, y \in X$. Replacing y by z and z by y in (2.30), we get

$$\begin{aligned} &\| -f(x-z, y+z) - f(y+z, -x+y) - f(x+y, x-y) + f(x-z, x+z)\| \\ &\leq \frac{1}{1-|s|}\theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{2.53}$$

for all $x, y, z \in X$. Replacing x by $-x+z$ and y by $y+z$ in (2.52), we get

$$\|f(-x+z, y+z) - f(x-z, -y-z)\| \leq \theta E_3(-x+z, y+z) \tag{2.54}$$

for all $x, y, z \in X$. Replacing x by $x-z$ and y by $y+z$ in (2.31), we get

$$\|f(x-z, y+z) + f(x-z, -y-z)\| \leq \theta(2\left\|\frac{x-y-2z}{2}\right\|^r + \left\|\frac{x+y}{2}\right\|^r) \tag{2.55}$$

for all $x, y, z \in X$. Replacing x by $y+z$ and y by $x-y$ in (2.31), we get

$$\|f(y+z, x-y) + f(y+z, -x+y)\| \leq \theta(2\|\frac{-x+2y+z}{2}\|^r + \|\frac{x+z}{2}\|^r) \tag{2.56}$$

for all $x, y, z \in X$. Adding (2.53), (2.54), (2.55) and (2.56), we obtain

$$\| -f(x+y, x-y) + f(x-z, x+z) + f(-x+z, y+z) + f(y+z, x-y)\| \leq \theta E_4(x, y, z) \tag{2.57}$$

for all $x, y, z \in X$, where the function $E_4 : X^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} E_4(x, y, z) &= \frac{1}{1-|s|}(\|x\|^r + \|y\|^r + \|z\|^r) + (\frac{2}{1-|s|}+9)\left\|\frac{y+z}{2}\right\|^r \\ &+(\frac{2}{1-|s|}+8)\left\|\frac{x+y}{2}\right\|^r + (\frac{1}{1-|s|}+3)\left\|\frac{-x+y+2z}{2}\right\|^r + (\frac{1}{1-|s|}+1)\left\|\frac{2x+y-z}{2}\right\|^r \\ &+2\left\|\frac{-x+2y+z}{2}\right\|^r + \left\|\frac{x+z}{2}\right\|^r + (\frac{1}{1-|s|}+3+|s|)\|x-z\|^r \\ &+(4+2|s|)\left\|\frac{3(x-z)}{4}\right\|^r + (\frac{1}{1-|s|}+9+2|s|)\left\|\frac{x-z}{2}\right\|^r + (14+4|s|)\left\|\frac{x-z}{4}\right\|^r \end{aligned}$$

for all $x, y, z \in X$. Replacing x by $x+y$, y by $x-y$, and z by y in (2.57), we get

$$\| -f(2x, 2y) + f(x, x+2y) + f(-x, x) + f(x, 2y)\| \leq \theta E_4(x+y, x-y, y) \tag{2.58}$$

for all $x, y \in X$. Replacing x by y and z by $x+y$ (2.57), we get

$$\| -f(-x, x+2y) - f(x, x+2y)\| \leq \theta E_4(y, y, x+y) \tag{2.59}$$

for all $x, y \in X$. Replacing x by y, y by $-y$, and z by $x+y$ (2.57), we get

$$\|f(-x, x+2y) + f(x, x) + f(x, 2y)\| \leq \theta E_4(y, -y, x+y) \tag{2.60}$$

for all $x, y \in X$. Setting $x = y = 0$ and replacing z by x in (2.57), we get

$$\| - f(-x, x) - f(x, x)\| \leq \theta E_4(0, 0, x) \tag{2.61}$$

for all $x \in X$. Adding (2.58), (2.59), (2.60) and (2.61) and adding (2.40) twice, we obtain

$$\|f(2x, 2y) - 4f(x, y)\| \leq E_5(x, y) \tag{2.62}$$

for all $x, y \in X$, where the function $E_5 : X^2 \to \mathbb{R}$ is defined as

$$\begin{aligned} E_5(x, y) &= (\frac{14}{1-|s|} + 76 + 12|s|)\left\|\frac{x}{2}\right\|^r + (\frac{9}{1-|s|} + 26 + 4|s|)\|x\|^r + (\frac{1}{1-|s|} + 1)\left\|\frac{3x}{2}\right\|^r \\ &+ (56 + 16|s|)\left\|\frac{x}{4}\right\|^r + (16 + 8|s|)\left\|\frac{3x}{4}\right\|^r + (\frac{3}{1-|s|} + 5)\left\|\frac{x}{2} - y\right\|^r \\ &+ (\frac{4}{1-|s|} + 18 + 2|s|)\left\|\frac{x}{2} + y\right\|^r + (\frac{7}{1-|s|} + 10)\|y\|^r \\ &+ (\frac{4}{1-|s|} + 3)\|x + y\|^r + \frac{1}{1-|s|}\|x - y\|^r \end{aligned}$$

for all $x, y \in X$.

For any positive integer n, replacing x by $2^{n-1}x$ and y by $2^{n-1}y$ in (2.62), and dividing both sides by 4^n, we obtain

$$\left\|\frac{1}{4^n}f(2^n x, 2^n y) - \frac{1}{4^{n-1}}f(2^{n-1}x, 2^{n-1}y)\right\| \leq \theta\left(\frac{2^r}{4}\right)^n E(x, y) \tag{2.63}$$

for all $x, y \in X$. For any nonnegative integers u, v satisfying $u < v$, by (2.63), we obtain

$$\begin{aligned} \left\|\frac{1}{4^v}f(2^v x, 2^v y) - \frac{1}{4^u}f(2^u x, 2^u y)\right\| &\leq \sum_{n=u+1}^{n=v} \left\|\frac{1}{4^n}f(2^n x, 2^n y) - \frac{1}{4^{n-1}}f(2^{n-1}x, 2^{n-1}y)\right\| \\ &\leq \sum_{n=u+1}^{n=v} \theta\left(\frac{2^r}{4}\right)^n E(x, y) = \theta\frac{(\frac{2^r}{4})^{u+1} - (\frac{2^r}{4})^{v+1}}{1 - \frac{2^r}{4}}E(x, y) \end{aligned} \tag{2.64}$$

for all $x, y \in X$. It follows from (2.64) that the sequence $\{\frac{1}{4^n}f(2^n x, 2^n y)\}$ is Cauchy for all $x, y \in X$. Since Y is a Banach space, then this sequence converges. So we can define the mapping $B : X^2 \to Y$ by

$$B(x, y) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x, 2^n y)$$

for all $x, y \in X$. Setting $v = 0$ and passing the limit $u \to \infty$ in (2.64), we get (2.26). Also, from (2.25),

$$\begin{aligned} &\|B(x - y, y + z) + B(y + z, z - x) + B(z + x, x - z) - B(x - y, x + y)\| \\ &= \lim_{n\to\infty} \|\frac{1}{4^n}(f(2^n(x - y), 2^n(y + z)) + f(2^n(y + z), 2^n(z - x)) \\ &\quad + f(2^n(z + x), 2^n(x - z)) - f(2^n(x - y), 2^n(x + y)))\| \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} |s|\|\frac{1}{4^n}(f(2^n(y-z),2^n(z+x)) + f(2^n(z+x),2^n(x-y)) \\
&\quad + f(2^n(x+y),2^n(y-x)) - f(2^n(y-z),2^n(y+z)))\| \\
&+ \lim_{n\to\infty} \frac{2^{nr}}{4^n}\theta(\|x\|^r + \|y\|^r + \|z\|^r) \\
&= |s|\|B(y-z,z+x) + B(z+x,x-y) + B(x+y,y-x) - B(y-z,y+z)\|
\end{aligned}
$$

for all $x, y, z \in X$. So, by Theorem 2.1, the mapping $B : X^2 \to Y$ is bi-additive.

Now, let $A : X^2 \to Y$ be another bi-additive specified what the conditions are. Then, for any positive integer n, we have

$$
\begin{aligned}
\|B(x,y) - A(x,y)\| &= \|\frac{1}{4^n}B(2^n x, 2^n y) - \frac{1}{4^n}A(2^n x, 2^n y)\| \\
&\leq \|\frac{1}{4^n}B(2^n x, 2^n y) - \frac{1}{4^n}f(2^n x, 2^n y)\| + \|\frac{1}{4^n}f(2^n x, 2^n y) - \frac{1}{4^n}A(2^n x, 2^n y)\| \\
&\leq \frac{1}{4^n}\frac{2^{r+1}\theta}{4-2^r}E(2^n x, 2^n y) = (\frac{2^r}{4})^n\frac{2^{r+1}\theta}{4-2^r}E(x,y) \qquad (2.65)
\end{aligned}
$$

for all $x, y \in X$. When n tends to infinity in (2.65), we have $B(x, y) = A(x, y)$ for all $x, y \in X$. This proves the uniqueness of the bi-additive mapping B, as desired. □

Theorem 2.3. *Let $r > 2$ and θ be nonnegative real number. If a mapping $f : X^2 \to Y$ satisfies $f(0, x) = f(x, 0) = 0$ and* (2.25) *for all $x, y, z \in X$, then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that*

$$
\|f(x,y) - B(x,y)\| \leq \frac{2^r\theta}{2^r-4}E(x,y) \qquad (2.66)
$$

for all $x, y \in X$, where the function $E : X^2 \to \mathbb{R}$ is defined in (2.27).

Proof. Assume that f satisfies (0.1). By the same inappropriate as in the proof of Theorem 2.2, we obatin (2.62). For any positive integer n, replacing x by $\frac{x}{2^{n+1}}$ and y by $\frac{y}{2^{n+1}}$ in (2.62), and multiplying both sides by 4^n, we obtain

$$
\left\|4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right\| \leq \theta\left(\frac{4}{2^r}\right)^n E(x,y) \qquad (2.67)
$$

for all $x, y \in X$. For any nonnegative integer u, v satisfying $u < v$, by (2.67), we obtain

$$
\begin{aligned}
\left\|4^u f\left(\frac{x}{2^u}, \frac{y}{2^u}\right) - 4^v f\left(\frac{x}{2^v}, \frac{y}{2^v}\right)\right\| &\leq \sum_{n=u}^{n=v-1}\left\|4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right)\right\| \\
&\leq \sum_{n=u}^{n=v-1}\theta\left(\frac{4}{2^r}\right)^n E(x,y) = \theta\frac{(\frac{4}{2^r})^u - (\frac{4}{2^r})^v}{1-\frac{4}{2^r}}E(x,y) \qquad (2.68)
\end{aligned}
$$

for all $x, y \in X$. It follows from (2.68) that the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ is Cauchy for all $x, y \in X$. Since Y is a Banach space, this sequence converges. So we can define the mapping $B : X^2 \to Y$ by

$$
B(x,y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})
$$

for all $x, y \in X$. Setting $v = 0$ and passing the limit $u \to \infty$ in (2.68), we obtain (2.66). Also, from (2.25),

$$\begin{aligned}
&\|B(x-y, y+z) + B(y+z, z-x) + B(z+x, x-z) - B(x-y, x+y)\| \\
&= \lim_{n\to\infty} \|4^n(f(\frac{x-y}{2^n}, \frac{y+z}{2^n}) + f(\frac{y+z}{2^n}, \frac{z-x}{2^n}) + f(\frac{z+x}{2^n}, \frac{x-z}{2^n}) - f(\frac{x-y}{2^n}, \frac{x+y}{2^n}))\| \\
&\le \lim_{n\to\infty} |s|\|4^n(f(\frac{y-z}{2^n}, \frac{z+x}{2^n}) + f(\frac{z+x}{2^n}, \frac{x-y}{2^n}) + f(\frac{x+y}{2^n}, \frac{y-x}{2^n}) - f(\frac{y-z}{2^n}, \frac{y+z}{2^n}))\| \\
&\quad + \lim_{n\to\infty} \frac{4^n}{2^{nr}}\theta(\|x\|^r + \|y\|^r + \|z\|^r) \\
&= |s|\|B(y-z, z+x) + B(z+x, x-y) + B(x+y, y-x) - B(y-z, y+z)\|
\end{aligned}$$

for all $x, y, z \in X$. So, by Theorem 2.1, the mapping $B : X^2 \to Y$ is bi-additive.

Now, let $A : X^2 \to Y$ be another bi-additive specified what the conditions are. Then, for any positive integer n, we have

$$\begin{aligned}
\|B(x,y) - A(x,y)\| &= \|4^n B(\frac{x}{2^n}, \frac{y}{2^n}) - 4^n A(\frac{x}{2^n}, \frac{y}{2^n})\| \\
&\le \|4^n B(\frac{x}{2^n}, \frac{y}{2^n}) - 4^n f(\frac{x}{2^n}, \frac{y}{2^n})\| + \|4^n f(\frac{x}{2^n}, \frac{y}{2^n}) - 4^n A(\frac{x}{2^n}, \frac{y}{2^n})\| \\
&\le 4^n \frac{2^{r+1}\theta}{2^r - 4} E(\frac{x}{2^n}, \frac{y}{2^n}) = (\frac{4}{2^r})^n \frac{2^{r+1}\theta}{2^r - 4} E(x,y) \qquad (2.69)
\end{aligned}$$

for all $x, y \in X$. When n tends to infinity in (2.69), we have $B(x, y) = A(x, y)$ for all $x, y \in X$. This proves the uniqueness of the bi-additive mapping B, as desired. □

3. Hyers-Ulam stability of bihomomorphisms and biderivations in Lie Banach algebras

Throughout this section, let X and Y be complex Lie Banach algebras.

We prove the Hyers-Ulam stability of bihomomorphisms associated with the bi-additive s-functional inequality (0.1).

Theorem 3.1. *Let $r \neq 2$ and θ be nonnegative real number. If a mapping $f : X^2 \to Y$ satisfies $f(0, x) = f(x, 0) = 0$ and* (2.25)*, and*

$$\|f([x,y],[z,z]) - [f(x,z), f(y,z)]\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)^2, \qquad (3.1)$$

$$\|f([x,x],[y,z]) - [f(x,y), f(x,z)]\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)^2 \qquad (3.2)$$

for all $x, y, z \in X$, then there exists a unique bihomomorphism $H : X^2 \to Y$ such that

$$\|f(x,y) - H(x,y)\| \le \frac{2^r\theta}{|2^r - 4|} E(x,y) \qquad (3.3)$$

for all $x, y \in X$, where the function $E : X^2 \to \mathbb{R}$ is defined as (2.27)

Proof. First, we deal with the case $r < 2$.

By Theorem 2.2, $H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$ is a unique bi-additive mapping which satisfies (3.3).

Replacing x by $2^n x$, y by $2^n y$, and z by $2^n z$ in (3.1), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \|\frac{1}{16^n} f(4^n[x,y], 4^n[z,z]) - \frac{1}{16^n}[f(2^n x, 2^n z), f(2^n y, 2^n z)]\| \\
&\quad = \lim_{n\to\infty} \frac{1}{16^n}\|f([2^n x, 2^n y], [2^n z, 2^n z]) - [f(2^n x, 2^n z), f(2^n y, 2^n z)]\| \\
&\quad \le \lim_{n\to\infty} (\frac{4^r}{16})^n \theta(\|x\|^r + \|y\|^r + \|z\|^r)^2 = 0 \qquad (3.4)
\end{aligned}$$

for all $x, y, z \in X$.

Adding (3.4), we get

$$\begin{aligned}
&\|H([x,y],[z,z]) - [H(x,z), H(y,z)]\| \\
&\quad = \lim_{n\to\infty} \|\frac{1}{4^{2n}} f(2^{2n}[x,y], 2^{2n}[z,z]) - [\frac{1}{4^n} f(2^n x, 2^n z), \frac{1}{4^n} f(2^n y, 2^n z)]\| \\
&\quad \le \lim_{n\to\infty} \|\frac{1}{16^n} f(4^n[x,y], 4^n[z,z]) - \frac{1}{16^n}[f(2^n x, 2^n z), f(2^n y, 2^n z)]\| \le 0
\end{aligned}$$

for all $x, y, z \in X$. Thus we have $H([x,y],[z,z]) = [H(x,z), H(y,z)]$ for all $x, y, z \in X$. By a similar method, we can also prove that $H([x,x],[y,z]) = [H(x,y), H(x,z)]$, and thus $H : X^2 \to Y$ is a bihomomorphism.

Now, assume that $r > 2$.

By Theorem 2.3, $H(x,y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ is a unique bi-additive mapping which satisfies (3.3).

Replacing x by $\frac{x}{2^n}$, y by $\frac{y}{2^n}$, and z by $\frac{z}{2^n}$ in (3.1), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \|16^n f(\frac{[x,y]}{4^n}, \frac{[z,z]}{4^n}) - 16^n[f(\frac{x}{2^n}, \frac{z}{2^n}), f(\frac{y}{2^n}, \frac{z}{2^n})]\| \\
&\quad = \lim_{n\to\infty} 16^n \|f([\frac{x}{2^n}, \frac{y}{2^n}], [\frac{z}{2^n}, \frac{z}{2^n}]) - [f(\frac{x}{2^n}, \frac{z}{2^n}), f(\frac{y}{2^n}, \frac{z}{2^n})]\| \\
&\quad \le \lim_{n\to\infty} (\frac{16}{4^r})^n \theta(\|x\|^r + \|y\|^r + \|z\|^r)^2 = 0 \qquad (3.5)
\end{aligned}$$

for all $x, y, z \in X$.

Adding (3.5), we get

$$\begin{aligned}
&\|H([x,y],[z,z]) - [H(x,z), H(y,z)]\| \\
&\quad = \lim_{n\to\infty} \|4^{2n} f(\frac{[x,y]}{2^{2n}}, \frac{[z,z]}{2^{2n}}) - [4^n f(\frac{x}{2^n}, \frac{z}{2^n}), 4^n f(\frac{y}{2^n}, \frac{z}{2^n})]\| \\
&\quad \le \lim_{n\to\infty} \|16^n f(\frac{[x,y]}{4^n}, \frac{[z,z]}{4^n}) - 16^n[f(\frac{x}{2^n}, \frac{z}{2^n}), f(\frac{y}{2^n}, \frac{z}{2^n})]\| \le 0
\end{aligned}$$

for all $x, y, z \in X$. Thus we have $H([x,y],[z,z]) = [H(x,z), H(y,z)]$ for all $x, y, z \in X$. By a similar method, we can also prove that $H([x,x],[y,z]) = [H(x,y), H(x,z)]$ and thus $H : X^2 \to Y$ is a bihomomorphism. □

Remark 3.2. We have defined the new useful bi-additive functional inequality (0.1), which was not appeared in any papers or any books, and solved the bi-additive functional inequality (0.1). Furthermore, we have proved the Hyers-Ulam-Rassais stability of the bi-additive functional inequality (0.1) by the direct method.

Many authors have only tried to investigate bihomomorphisms and biderivations in Banach algebras, C^*-ternary algebras and C^*-algebras. But in this paper, we have proved the Hyers-Ulam-Rassias stability of bihomomorphisms and biderivations in **Lie** Banach algebras associated with the bi-additive functional inequality (0.1).

4. Conclusions

In this paper, we have introduced and solved the bi-additive s-functional inequality (0.1) and we have proved the Hyers-Ulam stability of bihomomorphisms and biderivations in Lie Banach algebras associated with the bi-additive s-functional inequality (0.1).

Acknowledgments

This work was supported by the Seoul Science High School R&E program in 2018. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF -2017R1D1A1B04032937).

Conflict of interest

The authors declare that they have no competing interests.

References

1. S. M. Ulam, *Problems in Modern Mathematics,* Chapter VI, Science Ed. Wiley, New York, 1940.
2. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.
3. T. Aoki, *On the stability of the linear transformationin Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.
4. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Am. Math. Soc., **72** (1978), 297–300.
5. P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436. Doi.org/10.1006/jmaa.1994.1211.
6. A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math., **62** (2001), 303–309.
7. J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math., **66** (2003), 191–200.
8. W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math., **71** (2006), 149–161.

9. A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl., **5** (2002), 707–710.

10. C. Park, *Additive ρ-functional inequalities and equations,* J. Math. Inequal., **9** (2015), 17–26.

11. C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal., **9** (2015), 397–407.

12. G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math., **50** (1995), 143–190.

13. V. Govindan, C. Park, S. Pinelas, et al., *Solution of a 3-D cubic functional equation and its stability*, AIMS Math., **5** (2020), 1693–1705.

14. Pl. Kannappan, P. K. Sahoo, *Cauchy difference-a generalization of Hosszú functional equation*, Proc. Nat. Acad. Sci. India, **63** (1993), 541–550.

15. A. Najati, *Homomorphisms in quasi-Banach algebras associated with a Pexiderized Cauchy-Jensen functional equation*, Acta Math. Sin. (Engl. Ser.), **25** (2009), 1529–1542.

16. C. Park, Y. Cho, M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl., **2007**, Art. ID 41820 (2007).

17. M. Ramdoss, P. Selvan-Arumugam, C. Park, *Ulam stability of linear differential equations using Fourier transform*, AIMS Math., **5** (2020), 766–780.

18. J. Bae, W. Park, *Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras*, Bull. Korean Math. Soc., **47** (2010), 195–209.

19. J. Shokri, C. Park, D. Shin, *Approximate bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras*, J. Comput. Anal. Appl., **23** (2017), 713–722.

20. C. Park, *Biderivations and bihomomorphisms in Banach algebras,* Filomat, **33** (2019), 2317–2328.

AIMS Press