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Research article

Fixed point results for dominated mappings in rectangular b-metric spaces with applications

Abdullah Shoaib[1], Tahair Rasham[2], Giuseppe Marino[3], Jung Rye Lee[4,*] and Choonkil Park[5,*]

[1] Department of Mathematics and Statistics, Riphah International University, Islamabad 44000, Pakistan

[2] Department of Mathematics, International Islamic University, H-10, Islamabad 44000, Pakistan

[3] Dipartimento di Matematica e Informatica, Universita della Calabria, 87036, Arcavacata di Rende (CS), Italy

[4] Department of Mathematics, Daejin University, Kyunggi 11159, Korea

[5] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Email: jrlee@daejin.ac.kr; baak@hanyang.ac.kr.

Abstract: In this paper, we establish some fixed point results for α-dominated mappings fulfilling new generalized locally Ćirić type rational contraction conditions in complete rectangular b-metric space. As an application, we establish the existence of fixed point of $\preceq$-dominated mappings in an ordered complete rectangular b-metric space. The notion of graph dominated mappings is introduced. Fixed point results with graphic contractions for such mappings are established.

Keywords: fixed point; complete rectangular b-metric space; α-dominated mapping; Ćirić type rational contraction condition; partial order; $\preceq$-dominated mapping; graph dominated mapping
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1. Introduction and preliminaries

Let W be a set and $H : W \longrightarrow W$ be a mapping. A point $w \in W$ is called a fixed point of H if $w = Hw$. Fixed point theory plays a fundamental role in functional analysis (see [15]). Shoaib [17] introduced the concept of α-dominated mapping and obtained some fixed point results (see also [1,2]). George *et al.* [11] introduced a new space and called it rectangular b-metric space ($r.b.m.$ space). The triangle inequality in the b-metric space was replaced by rectangle inequality. Useful results on $r.b.m.$ spaces can be seen in ([5, 6, 8–10]). Ćirić introduced new types of contraction and proved some metrical fixed point results (see [4]). In this article, we introduce Ćirić type rational contractions for

α-dominated mappings in *r.b.m.* spaces and proved some metrical fixed point results. New interesting results in metric spaces, rectangular metric spaces and b-metric spaces can be obtained as applications of our results.

Definition 1.1. [11] Let U be a nonempty set. A function $d_{lb} : U \times U \to [0, \infty)$ is said to be a rectangular b-metric if there exists $b \geq 1$ such that
(i) $d_{lb}(\theta, \nu) = d_{lb}(\nu, \theta)$;
(ii) $d_{lb}(\theta, \nu) = 0$ if and only if $\theta = \nu$;
(iii) $d_{lb}(\theta, \nu) \leq b[d_{lb}(\theta, q) + d_{lb}(q, l) + d_{lb}(l, \nu)]$ for all $\theta, \nu \in U$ and all distinct points $q, l \in U \setminus \{\theta, \nu\}$.
The pair (U, d_{lb}) is said a rectangular b-metric space (in short, *r.b.m.* space) with coefficient b.

Definition 1.2. [11] Let (U, d_{lb}) be an *r.b.m.* space with coefficient b.
(i) A sequence $\{\theta_n\}$ in (U, d_{lb}) is said to be Cauchy sequence if for each $\varepsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d_{lb}(\theta_m, \theta_n) < \varepsilon$ or $\lim_{n,m\to+\infty} d_{lb}(\theta_n, \theta_m) = 0$.
(ii) A sequence $\{\theta_n\}$ is rectangular b-convergent (for short, (d_{lb})-converges) to θ if $\lim_{n\to+\infty} d_{lb}(\theta_n, \theta) = 0$. In this case θ is called a (d_{lb})-limit of $\{\theta_n\}$.
(iii) (U, d_{lb}) is complete if every Cauchy sequence in U d_{lb}-converges to a point $\theta \in U$.

Let ϖ_b, where $b \geq 1$, denote the family of all nondecreasing functions $\delta_b : [0, +\infty) \to [0, +\infty)$ such that $\sum_{k=1}^{+\infty} b^k \delta_b^k(t) < +\infty$ and $b\delta_b(t) < t$ for all $t > 0$, where δ_b^k is the k^{th} iterate of δ_b. Also $b^{n+1}\delta_b^{n+1}(t) = b^n b\delta_b(\delta_b^n(t)) < b^n\delta_b^n(t)$.

Example 1.3. [11] Let $U = \mathbb{N}$. Define $d_{lb} : U \times U \to \mathbb{R}^+ \cup \{0\}$ such that $d_{lb}(u, v) = d_{lb}(v, u)$ for all $u, v \in U$ and $\alpha > 0$

$$d_{lb}(u, v) = \begin{cases} 0, \text{ if } u = v; \\ 10\alpha, \text{ if } u = 1,\ v = 2; \\ \alpha, \text{ if } u \in \{1, 2\} \text{ and } v \in \{3\}; \\ 2\alpha, \text{ if } u \in \{1, 2, 3\} \text{ and } v \in \{4\}; \\ 3\alpha, \text{ if } u \text{ or } v \notin \{1, 2, 3, 4\} \text{ and } u \neq v. \end{cases}$$

Then (U, d_{lb}) is an *r.b.m.* space with $b = 2 > 1$. Note that

$$d(1, 4) + d(4, 3) + d(3, 2) = 5\alpha < 10\alpha = d(1, 2).$$

Thus d_{lb} is not a rectangular metric.

Definition 1.4. [17] Let (U, d_{lb}) be an *r.b.m.* space with coefficient b. Let $S : U \to U$ be a mapping and $\alpha : U \times U \to [0, +\infty)$. If $A \subseteq U$, we say that the S is α-dominated on A, whenever $\alpha(i, Si) \geq 1$ for all $i \in A$. If $A = U$, we say that S is α-dominated.

For $\theta, \nu \in U$, $a > 0$, we define $D_{lb}(\theta, \nu)$ as

$$D_{lb}(\theta, \nu) = \max\{d_{lb}(\theta, \nu), \frac{d_{lb}(\theta, S\theta) . d_{lb}(\nu, S\nu)}{a + d_{lb}(\theta, \nu)}, d_{lb}(\theta, S\theta), d_{lb}(\nu, S\nu)\}.$$

2. Main result

Now, we present our main result.

Theorem 2.1. *Let (U, d_{lb}) be a complete r.b.m. space with coefficient b, $\alpha : U \times U \to [0, \infty)$, $S : U \to U$, $\{\theta_n\}$ be a Picard sequence and S be a α-dominated mapping on $\{\theta_n\}$. Suppose that, for some $\delta_b \in \varpi_b$, we have*

$$d_{lb}(S\theta, S\nu) \leq \delta_b(D_{lb}(\theta, \nu)), \tag{2.1}$$

for all $\theta, \nu \in \{\theta_n\}$ with $\alpha(\theta, \nu) \geq 1$. Then $\{\theta_n\}$ converges to $\theta^* \in U$. Also, if (2.1) holds for θ^* and $\alpha(\theta_n, \theta^*) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S has a fixed point θ^* in U.

Proof. Let $\theta_0 \in U$ be arbitrary. Define the sequence $\{\theta_n\}$ by $\theta_{n+1} = S\theta_n$ for all $n \in \mathbb{N} \cup \{0\}$. We shall show that $\{\theta_n\}$ is a Cauchy sequence. If $\theta_n = \theta_{n+1}$, for some $n \in \mathbb{N}$, then θ_n is a fixed point of S. So, suppose that any two consecutive terms of the sequence are not equal. Since $S : U \to U$ be an α-dominated mapping on $\{\theta_n\}$, $\alpha(\theta_n, S\theta_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and then $\alpha(\theta_n, \theta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now by using inequality (2.1), we obtain

$$\begin{aligned} d_{lb}(\theta_{n+1}, \theta_{n+2}) &= d_{lb}(S\theta_n, S\theta_{n+1}) \leq \delta_b(D_{lb}(\theta_n, \theta_{n+1})) \\ &\leq \delta_b(\max\{d_{lb}(\theta_n, \theta_{n+1}), \frac{d_{lb}(\theta_n, \theta_{n+1}) . d_{lb}(\theta_{n+1}, \theta_{n+2})}{a + d_{lb}(\theta_n, \theta_{n+1})}, \\ &\quad d_{lb}(\theta_n, \theta_{n+1}), d_{lb}(\theta_{n+1}, \theta_{n+2})\}) \\ &\leq \delta_b(\max\{d_{lb}(\theta_n, \theta_{n+1}), d_{lb}(\theta_{n+1}, \theta_{n+2})\}). \end{aligned}$$

If $\max\{d_{lb}(\theta_n, \theta_{n+1}), d_{lb}(\theta_{n+1}, \theta_{n+2})\} = d_{lb}(\theta_{n+1}, \theta_{n+2})$, then

$$\begin{aligned} d_{lb}(\theta_{n+1}, \theta_{n+2}) &\leq \delta_b(d_{lb}(\theta_{n+1}, \theta_{n+2})) \\ &\leq b\delta_b(d_{lb}(\theta_{n+1}, \theta_{n+2})). \end{aligned}$$

This is the contradiction to the fact that $b\delta_b(t) < t$ for all $t > 0$. So

$$\max\{d_{lb}(\theta_n, \theta_{n+1}), d_{lb}(\theta_{n+1}, \theta_{n+2})\} = d_{lb}(\theta_n, \theta_{n+1}).$$

Hence, we obtain

$$d_{lb}(\theta_{n+1}, \theta_{n+2}) \leq \delta_b(d_{lb}(\theta_n, \theta_{n+1})) \leq \delta_b^2(d_{lb}(\theta_{n-1}, \theta_n))$$

Continuing in this way, we obtain

$$d_{lb}(\theta_{n+1}, \theta_{n+2}) \leq \delta_b^{n+1}(d_{lb}(\theta_0, \theta_1)). \tag{2.2}$$

Suppose for some $n, m \in \mathbb{N}$ with $m > n$, we have $\theta_n = \theta_m$. Then by (2.2)

$$\begin{aligned} d_{lb}(\theta_n, \theta_{n+1}) &= d_{lb}(\theta_n, S\theta_n) = d_{lb}(\theta_m, S\theta_m) = d_{lb}(\theta_m, \theta_{m+1}) \\ &\leq \delta_b^{m-n}(d_{lb}(\theta_n, \theta_{n+1})) < b\delta_b(d_{lb}(\theta_n, \theta_{n+1})) \end{aligned}$$

As $d_{lb}(\theta_n, \theta_{n+1}) > 0$, so this is not true, because $b\delta_b(t) < t$ for all $t > 0$. Therefore, $\theta_n \neq \theta_m$ for any $n, m \in \mathbb{N}$. Since $\sum_{k=1}^{+\infty} b^k \delta_b^k(t) < +\infty$, for some $\nu \in \mathbb{N}$, the series $\sum_{k=1}^{+\infty} b^k \delta_b^k(\delta_b^{\nu-1}(d_{lb}(\theta_0, \theta_1)))$ converges. As $b\delta_b(t) < t$, so

$$b^{n+1}\delta_b^{n+1}(\delta_b^{\nu-1}(d_{lb}(\theta_0, \theta_1))) < b^n\delta_b^n(\delta_b^{\nu-1}(d_{lb}(\theta_0, \theta_1))), \text{ for all } n \in \mathbb{N}.$$

Fix $\varepsilon > 0$. Then $\frac{\varepsilon}{2} = \varepsilon' > 0$. For ε', there exists $\nu(\varepsilon') \in \mathbb{N}$ such that

$$b\delta_b(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + b^2\delta_b^2(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + \cdots < \varepsilon' \tag{2.3}$$

Now, we suppose that any two terms of the sequence $\{\theta_n\}$ are not equal. Let $n, m \in \mathbb{N}$ with $m > n > \nu(\varepsilon')$. Now, if $m > n + 2$,

$$\begin{aligned}
d_{lb}(\theta_n, \theta_m) &\leq b[d_{lb}(\theta_n, \theta_{n+1}) + d_{lb}(\theta_{n+1}, \theta_{n+2}) + d_{lb}(\theta_{n+2}, \theta_m)] \\
&\leq b[d_{lb}(\theta_n, \theta_{n+1}) + d_{lb}(\theta_{n+1}, \theta_{n+2})] + b^2[d_{lb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + d_{lb}(\theta_{n+3}, \theta_{n+4}) + d_{lb}(\theta_{n+4}, \theta_m)] \\
&\leq b[\delta_b^n (d_{lb}(\theta_0, \theta_1)) + \delta_b^{n+1} (d_{lb}(\theta_0, \theta_1))] + b^2[\delta_b^{n+2} (d_{lb}(\theta_0, \theta_1)) \\
&\quad + \delta_b^{n+3} (d_{lb}(\theta_0, \theta_1))] + b^3[\delta_b^{n+4} (d_{lb}(\theta_0, \theta_1)) + \delta_b^{n+5} (d_{lb}(\theta_0, \theta_1))] + \cdots \\
&\leq b\delta_b^n(d_{lb}(\theta_0, \theta_1)) + b^2\delta_b^{n+1}(d_{lb}(\theta_0, \theta_1)) + b^3\delta_b^{n+2} (d_{lb}(\theta_0, \theta_1)) + \cdots \\
&= b\delta_b(\delta_b^{n-1}(d_{lb}(\theta_0, \theta_1))) + b^2\delta_b^2(\delta_b^{n-1}(d_{lb}(\theta_0, \theta_1))) + \cdots .
\end{aligned}$$

By using (2.3), we have

$$\begin{aligned}
&d_{lb}(\theta_n, \theta_m) \\
&< b\delta_b(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + b^2\delta_b^2(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + \cdots < \varepsilon' < \varepsilon.
\end{aligned}$$

Now, if $m = n + 2$, then we obtain

$$\begin{aligned}
&d_{lb}(\theta_n, \theta_{n+2}) \\
&\leq b[d_{lb}(\theta_n, \theta_{n+1}) + d_{lb}(\theta_{n+1}, \theta_{n+3}) + d_{lb}(\theta_{n+3}, \theta_{n+2})] \\
&\leq b[d_{lb}(\theta_n, \theta_{n+1}) + b[d_{lb}(\theta_{n+1}, \theta_{n+2}) + d_{lb}(\theta_{n+2}, \theta_{n+4}) + d_{lb}(\theta_{n+4}, \theta_{n+3})] \\
&\quad + d_{lb}(\theta_{n+3}, \theta_{n+2})] \\
&\leq bd_{lb}(\theta_n, \theta_{n+1}) + b^2d_{lb}(\theta_{n+1}, \theta_{n+2}) + bd_{lb}(\theta_{n+2}, \theta_{n+3}) + b^2d_{lb}(\theta_{n+3}, \theta_{n+4}) \\
&\quad + b^3[d_{lb}(\theta_{n+2}, \theta_{n+3}) + d_{lb}(\theta_{n+3}, \theta_{n+5}) + d_{lb}(\theta_{n+5}, \theta_{n+4})] \\
&\leq bd_{lb}(\theta_n, \theta_{n+1}) + b^2d_{lb}(\theta_{n+1}, \theta_{n+2}) + \left(b + b^3\right) d_{lb}(\theta_{n+2}, \theta_{n+3}) + b^2d_{lb}(\theta_{n+3}, \theta_{n+4}) \\
&\quad + b^3d_{lb}(\theta_{n+5}, \theta_{n+4}) + b^4[d_{lb}(\theta_{n+3}, \theta_{n+4}) + d_{lb}(\theta_{n+4}, \theta_{n+6}) + d_{lb}(\theta_{n+6}, \theta_{n+5})] \\
&\leq bd_{lb}(\theta_n, \theta_{n+1}) + b^2d_{lb}(\theta_{n+1}, \theta_{n+2}) + \left(b + b^3\right) d_{lb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \left(b^2 + b^4\right) d_{lb}(\theta_{n+3}, \theta_{n+4}) + b^3d_{lb}(\theta_{n+5}, \theta_{n+4}) + b^4d_{lb}(\theta_{n+6}, \theta_{n+5}) \\
&\quad + b^5[d_{lb}(\theta_{n+4}, \theta_{n+5}) + d_{lb}(\theta_{n+5}, \theta_{n+7}) + d_{lb}(\theta_{n+7}, \theta_{n+6})] \\
&\leq bd_{lb}(\theta_n, \theta_{n+1}) + b^2d_{lb}(\theta_{n+1}, \theta_{n+2}) + \left(b + b^3\right) d_{lb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + \left(b^2 + b^4\right) d_{lb}(\theta_{n+3}, \theta_{n+4}) + \left(b^3 + b^5\right) d_{lb}(\theta_{n+4}, \theta_{n+5}) + \cdots \\
&< 2[bd_{lb}(\theta_n, \theta_{n+1}) + b^2d_{lb}(\theta_{n+1}, \theta_{n+2}) + b^3d_{lb}(\theta_{n+2}, \theta_{n+3}) \\
&\quad + b^4d_{lb}(\theta_{n+3}, \theta_{n+4}) + b^5d_{lb}(\theta_{n+4}, \theta_{n+5}) + \cdots] \\
&\leq 2[b\delta_b^n(d_{lb}(\theta_0, \theta_1)) + b^2\delta_b^{n+1}(d_{lb}(\theta_0, \theta_1)) + b^3\delta_b^{n+2} (d_{lb}(\theta_0, \theta_1)) + \cdots] \\
&< 2[b\delta_b(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + b^2\delta_b^2(\delta_b^{\nu(\varepsilon')-1}(d_{lb}(\theta_0, \theta_1))) + \cdots] < 2\varepsilon' = \varepsilon.
\end{aligned}$$

It follows that
$$\lim_{n,m\to+\infty} d_{lb}(\theta_n,\theta_m)=0. \tag{2.4}$$
Thus $\{\theta_n\}$ is a Cauchy sequence in (U,d_{lb}). As (U,d_{lb}) is complete, so there exists θ^* in U such that $\{\theta_n\}$ converges to θ^*, that is,
$$\lim_{n\to+\infty} d_{lb}(\theta_n,\theta^*)=0. \tag{2.5}$$
Now, suppose that $d_{lb}(\theta^*,S\theta^*)>0$. Then
$$\begin{aligned} d_{lb}(\theta^*,S\theta^*) &\leq b[d_{lb}(\theta^*,\theta_n)+d_{lb}(\theta_n,\theta_{n+1})+d_{lb}(\theta_{n+1},S\theta^*) \\ &\leq b[d_{lb}(\theta^*,\theta_{n+1})+d_{lb}(\theta_n,\theta_{n+1})+d_{lb}(S\theta_n,S\theta^*). \end{aligned}$$
Since $\alpha(\theta_n,\theta^*)\geq 1$, we obtain
$$\begin{aligned} d_{lb}(\theta^*,S\theta^*) \leq\; & bd_{lb}(\theta^*,\theta_{n+1})+bd_{lb}(\theta_n,\theta_{n+1})+b\delta_b(\max\{d_{lb}(\theta_n,\theta^*), \\ & \frac{d_{lb}\left(\theta^*,S\theta^*\right).d_{lb}\left(\theta_n,\theta_{n+1}\right)}{a+d_{lb}\left(\theta_n,\theta^*\right)},\ d_{lb}(\theta_n,\theta_{n+1})\, d_{lb}(\theta^*,S\theta^*)\}). \end{aligned}$$
Letting $n\to+\infty$, and using the inequalities (2.4) and (2.5), we obtain $d_{lb}(\theta^*,S\theta^*)\leq b\delta_b(d_{lb}(\theta^*,S\theta^*))$. This is not true, because $b\delta_b(t)<t$ for all $t>0$ and hence $d_{lb}(\theta^*,S\theta^*)=0$ or $\theta^*=S\theta^*$. Hence S has a fixed point θ^* in U. □

Remark 2.2. By taking fourteen different proper subsets of $D_{lb}(\theta,\nu)$, we can obtainvnew results as corollaries of our result in a complete *r.b.m.* space with coefficient b.

We have the following result without using α-dominated mapping.

Theorem 2.3. *Let (U,d_{lb}) be a complete r.b.m. space with coefficient b, $S:U\to U$, $\{\theta_n\}$ be a Picard sequence. Suppose that, for some $\delta_b\in\varpi_b$, we have*
$$d_{lb}(S\theta,S\nu)\leq\delta_b(D_{lb}(\theta,\nu)) \tag{2.6}$$
for all $\theta,\nu\in\{\theta_n\}$. Then $\{\theta_n\}$ converges to $\theta^*\in U$. Also, if (2.6) holds for θ^*, then S has a fixed point θ^* in U.

We have the following result by taking $\delta_b\,(t)=ct$, $t\in\mathbb{R}^+$ with $0<c<\frac{1}{b}$ without using α-dominated mapping.

Theorem 2.4. *Let (U,d_{lb}) be a complete r.b.m. space with coefficient b, $S:U\to U$, $\{\theta_n\}$ be a Picard sequence. Suppose that, for some $0<c<\frac{1}{b}$, we have*
$$d_{lb}(S\theta,S\nu)\leq c(D_{lb}(\theta,\nu)) \tag{2.7}$$
for all $\theta,\nu\in\{\theta_n\}$. Then $\{\theta_n\}$ converges to $\theta^*\in U$. Also, if (2.7) holds for θ^*, then S has a fixed point θ^* in U.

Ran and Reurings [16] gave an extension to the results in fixed point theory and obtained results in partially ordered metric spaces. Arshad *et al.* [3] introduced $\preceq$-dominated mappings and established some results in an ordered complete dislocated metric space. We apply our result to obtain results in ordered complete *r.b.m.* space.

Definition 2.5. $(U, \preceq, d_{lb})$ is said to be an ordered complete *r.b.m.* space with coefficient b if
(i) $(U, \preceq)$ is a partially ordered set.
(ii) (U, d_{lb}) is an *r.b.m.* space.

Definition 2.6. [3] Let U be a nonempty set, $\preceq$ is a partial order on θ. A mapping $S : U \to U$ is said to be $\preceq$-dominated on A if $a \preceq Sa$ for each $a \in A \subseteq \theta$. If $A = U$, then $S : U \to U$ is said to be $\preceq$-dominated.

We have the following result for $\preceq$-dominated mappings in an ordered complete *r.b.m.* space with coefficient b.

Theorem 2.7. *Let $(U, \preceq, d_{lb})$ be an ordered complete r.b.m. space with coefficient b, $S : U \to U$, $\{\theta_n\}$ be a Picard sequence and S be a $\preceq$-dominated mapping on $\{\theta_n\}$. Suppose that, for some $\delta_b \in \varpi_b$, we have*

$$d_{lb}(S\theta, S\nu) \leq \delta_b(D_{lb}(\theta, \nu)), \tag{2.8}$$

for all $\theta, \nu \in \{\theta_n\}$ with $\theta \preceq \nu$. Then $\{\theta_n\}$ converges to $\theta^* \in U$. Also, if (2.8) holds for θ^* and $\theta_n \preceq \theta^*$ for all $n \in \mathbb{N} \cup \{0\}$. Then S has a fixed point θ^* in U.

Proof. Let $\alpha : U \times U \to [0, +\infty)$ be a mapping defined by $\alpha(\theta, \nu) = 1$ for all $\theta, \nu \in U$ with $\theta \preceq \nu$ and $\alpha(\theta, \nu) = \frac{4}{11}$ for all other elements $\theta, \nu \in U$. As S is the dominated mappings on $\{\theta_n\}$, so $\theta \preceq S\theta$ for all $\theta \in \{\theta_n\}$. This implies that $\alpha(\theta, S\theta) = 1$ for all $\theta \in \{\theta_n\}$. So $S : U \to U$ is the α-dominated mapping on $\{\theta_n\}$. Moreover, inequality (2.8) can be written as

$$d_{lb}(S\theta, S\nu) \leq \delta_b(D_{lb}(\theta, \nu))$$

for all elements θ, ν in $\{\theta_n\}$ with $\alpha(\theta, \nu) \geq 1$. Then, as in Theorem 2.1, $\{\theta_n\}$ converges to $\theta^* \in U$. Now, $\theta_n \preceq \theta^*$ implies $\alpha(\theta_n, \theta^*) \geq 1$. So all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, S has a fixed point θ^* in U. □

Now, we present an example of our main result. Note that the results of George *et al.* [11] and all other results in rectangular b-metric space are not applicable to ensure the existence of the fixed point of the mapping given in the following example.

Example 2.8. Let $U = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5\}\}$ and $B = [1, \infty]$. Define $d_l : U \times U \to [0, \infty)$ such that $d_l(\theta, \nu) = d_l(\nu, \theta)$ for $\theta, \nu \in U$ and

$$\begin{cases} d_l(\frac{1}{2}, \frac{1}{3}) = d_l(\frac{1}{4}, \frac{1}{5}) = 0.03 \\ d_l(\frac{1}{2}, \frac{1}{5}) = d_l(\frac{1}{3}, \frac{1}{4}) = 0.02 \\ d_l(\frac{1}{2}, \frac{1}{4}) = d_l(\frac{1}{5}, \frac{1}{3}) = 0.6 \\ d_l(\theta, \nu) = |\theta - \nu|^2 \quad \text{otherwise} \end{cases}$$

be a complete *r.b.m.* space with coefficient $b = 4 > 1$ but (U, d_l) is neither a metric space nor a rectangular metric space. Take $\delta_b(t) = \frac{t}{10}$, then $b\delta_b(t) < t$. Let $S : U \to U$ be defined as

$$S\theta = \begin{cases} \frac{1}{5} & \text{if } \theta \in A \\ \frac{1}{3} & \text{if} \theta = 1 \\ 9\theta^{100} + 85 & \text{otherwise.} \end{cases}$$

Let $\theta_0 = 1$. Then the Picard sequence $\{\theta_n\}$ is $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \cdots\}$. Define

$$\alpha(\theta, \nu) = \begin{cases} \frac{8}{5} & \text{if} \theta, \nu \in \{\theta_n\} \\ \frac{4}{7} & \text{otherwise.} \end{cases}$$

Then S is an α-dominated mapping on $\{\theta_n\}$. Now, S satisfies all the conditions of Theorem 2.1. Here $\frac{1}{5}$ is the fixed point in U.

3. Fixed point results for graphic contractions

Jachymski [13] proved the contraction principle for mappings on a metric space with a graph. Let (U, d) be a metric space and $\triangle$ represents the diagonal of the cartesian product $U \times U$. Suppose that G be a directed graph having the vertices set $V(G)$ along with U, and the set $E(G)$ denoted the edges of U included all loops, i.e., $E(G) \supseteq \triangle$. If G has no parallel edges, then we can unify G with pair $(V(G), E(G))$. If l and m are the vertices in a graph G, then a path in G from l to m of length N ($N \in \mathbb{N}$) is a sequence $\{\theta_i\}_{i=o}^{N}$ of $N + 1$ vertices such that $l_o = l$, $l_N = m$ and $(l_{n-1}, l_n) \in E(G)$ where $i = 1, 2, \cdots N$ (see for detail [7, 8, 12, 14, 18, 19]). Recently, Younis *et al.* [20] introduced the notion of graphical rectangular b-metric spaces (see also [5, 6, 21]). Now, we present our result in this direction.

Definition 3.1. Let θ be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = U$ and $A \subseteq U$. A mapping $S : U \to U$ is said to be graph dominated on A if $(\theta, S\theta) \in E(G)$ for all $\theta \in A$.

Theorem 3.2. *Let (U, d_{lb}) be a complete rectangular b-metric space endowed with a graph G, $\{\theta_n\}$ be a Picard sequence and $S : U \to U$ be a graph dominated mapping on $\{\theta_n\}$. Suppose that the following hold:*
(i) there exists $\delta_b \in \varpi_b$ such that

$$d_{lb}(S\theta, S\nu) \leq \delta_b(D_{lb}(\theta, \nu)), \tag{3.1}$$

for all $\theta, \nu \in \{\theta_n\}$ and $(\theta_n, \nu) \in E(G)$. Then $(\theta_n, \theta_{n+1}) \in E(G)$ and $\{\theta_n\}$ converges to θ^*. Also, if (3.1) holds for θ^* and $(\theta_n, \theta^*) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$, then S has a fixed point θ^* in U.

Proof. Define $\alpha : U \times U \to [0, +\infty)$ by

$$\alpha(\theta, \nu) = \begin{cases} 1, & \text{if } \theta, \nu \in U, \ (\theta, \nu) \in E(G) \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Since S is a graph dominated on $\{\theta_n\}$, for $\theta \in \{\theta_n\}$, $(\theta, S\theta) \in E(G)$. This implies that $\alpha(\theta, S\theta) = 1$ for all $\theta \in \{\theta_n\}$. So $S : U \to U$ is an α-dominated mapping on $\{\theta_n\}$. Moreover, inequality (3.1) can be written as

$$d_{lb}(S\theta, S\nu) \leq \delta_b(D_{lb}(\theta, \nu)),$$

for all elements θ, ν in $\{\theta_n\}$ with $\alpha(\theta, \nu) \geq 1$. Then, by Theorem 2.1, $\{\theta_n\}$ converges to $\theta^* \in U$. Now, $(\theta_n, \theta^*) \in E(G)$ implies that $\alpha(\theta_n, \theta^*) \geq 1$. So all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, S has a fixed point θ^* in U. □

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Conflict of interest

The authors declare that they have no competing interests.

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