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Research article

Hyers-Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces

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Abstract: In this paper, we introduce a mixed type finite variable functional equation deriving from quadratic and additive functions and obtain the general solution of the functional equation and investigate the Hyers-Ulam stability for the functional equation in quasi-Banach spaces.

Keywords: additive functional equation; quadratic functional equation; Hyers-Ulam stability; quasi-Banach space; p-Banach space
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1. Introduction

The stability problem of functional equations originated from a question of Ulam [27] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x, y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [12] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Rassias [23] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Quadratic functional equation was used to characterize inner product spaces [1, 2, 13]. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [1,16]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic Eq (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [1, 16]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}\left(f(x + y) - f(x - y)\right). \tag{1.2}$$

A Hyers-Ulam stability problem for the quadratic functional Eq (1.1) was proved by Skof [25] for mappings $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space ([16]). Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In [5], Czerwik proved the Hyers-Ulam stability of the quadratic functional Eq (1.1). Grabiec [11] generalized these results mentioned above.

Elqorachi and M. Th. Rassias [6] have been extensively studied the Hyers-Ulam stability of the generalized trigonometric functional equations

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) + 2h(y), \quad x, y \in S, \tag{1.3}$$

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(y)g(x) + 2h(x), \quad x, y \in S, \tag{1.4}$$

where S is a semigroup, $\sigma : S \to S$ is an involutive morphism, and $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Jung [19] proved the stability theorems for n-dimensional quartic-cubic-quadratic-additive type functional equations of the form $\sum_{i=1}^{l} c_i f(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) = 0$ by applying the direct method. These stability theorems can save us the trouble of proving the stability of relevant solutions repeatedly appearing in the stability problems for various functional equations. Lee [18] introduced general quintic functional equation and general sextic functional equations such as the additive functional equation and the quadratic functional equation. He investigated the Hyers-Ulam stability results. Kayal et al. [24] established the Hyers-Ulam stability results belonging to two different set valued functional equations in several variables, namely, additive and cubic. The results were obtained in the contexts of Banach spaces. See [10, 15, 20] for more information on functional equations and their stability.

Jun and Kim [14] obtained the Hyers-Ulam stability for a mixed type of cubic and additive functional equations. In addition theHyers-Ulam for a mixed type of quadratic and additive functional equations

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 2f(2x) - 2f(x) \tag{1.5}$$

in quasi-Banach spaces have been investigated by Najati and Moghimi [21]. Najati and Eskandani [22] introduced the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2f(2x\_4f(x). \tag{1.6}$$

It is easy to see that the function $f(x) = ax^3 + bx$ is a solution of the functional Eq (1.6). They established the general solution and the Hyers-Ulam stability for the functional Eq (1.6) in quasi-Banach spaces. In 2009, Eshaghi Gordji et al. [7] introduced the following mixed type cubic, quadratic and additive functional equations for a fixed integer k with $k \neq 0, \pm 1$:

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2) f(x) \tag{1.7}$$

and proved the function $f(x) = ax^3 + bx^2 + cx$ is a solution of the functional Eq (1.7). They investigated the general solution of (1.7) in vector spaces, and established the Hyers-Ulam stability of the functional Eq (1.7) in quasi-Banach spaces.

In this paper, we introduce the following mixed type finite variable functional equation deriving from quadratic and additive functions

$$\phi\left(\sum_{i=1}^{l} t_i\right) = \sum_{1 \leq i < j \leq l} \phi\left(t_i + t_j\right) - (l + 2) \sum_{i=1}^{l} \left[\frac{\phi(t_i) + \phi(-t_i)}{2}\right] - l \sum_{i=1}^{l} \left[\frac{\phi(t_i) - \phi(-t_i)}{2}\right] + \sum_{j=1}^{l} \phi(2t_j) \tag{1.8}$$

where $\phi(0) = 0$ and $l \geq 4$ is a fixed positive integer, which generalizes a quadratic-additive functional equation given in [17,21]. It is easy to see that the function $\phi(t) = at^2 + bt$ is a solution of the functional Eq (1.8). The primary goal of this paper is to obtain the general solution of the functional Eq (1.8) and investigate the Hyers-Ulam stability for the functional Eq (1.8) in quasi-Banach spaces. Our results generalize the results given by Najati and Moghimi [21].

Definition 1.1. ([3]) Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K (\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from condition (iii) that

$$\left\|\sum_{i=1}^{2n} x_i\right\| \leq K^n \sum_{i=1}^{2n} \|x_i\| \quad \Rightarrow \quad \left\|\sum_{i=1}^{2n+1} x_i\right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \cdots, x_{2n+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem (see [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, we restrict our attention mainly to p-norms. Moreover in [26], Tabor investiagted a version of Hyers-Ulam theorem in quasi-Banach spaces (see [8, 9]).

2. Solution of the functional Eq (1.8)

Throughout this section, P and Q will be real vector spaces.

Lemma 2.1. *If an odd mapping $\phi : P \to Q$ satisfies (1.8) for all $t_1, t_2, \cdots, t_l \in P$, then ϕ is additive.*

Proof. In the view of the oddness of ϕ, we have $\phi(-t) = -\phi(t)$ for all $t \in P$. Now, (1.8) becomes

$$\phi\left(\sum_{i=1}^{l} t_i\right) = \sum_{1\leq i<j\leq l} \phi\left(t_i + t_j\right) - l\sum_{i=1}^{l} \phi(t_i) + \sum_{j=1}^{l} \phi(2t_j). \tag{2.1}$$

Setting $(t_1, t_2, \cdots, t_l) = (0, 0, \cdots, 0)$ in (2.1), we get $\phi(0) = 0$. Now, letting $(t_1, t_2, \cdots, t_l) = (t, 0, \cdots, 0)$ in (2.1), we obtain

$$\phi(2t) = 2\phi(t) \tag{2.2}$$

for all $t \in P$. Replacing t by $2t$ in (2.2), we get

$$\phi(2^2 t) = 2^2\phi(t) \tag{2.3}$$

for all $t \in P$. Again replacing t by $2t$ in (2.3), we have

$$\phi(2^3 t) = 2^3\phi(t)$$

for all $t \in P$. In general, for any positive integer l, we obtain

$$\phi(2^l t) = 2^l\phi(t)$$

for all $t \in P$. Therefore, (2.1) now becomes

$$\phi\left(\sum_{i=1}^{l} t_i\right) = \sum_{1\leq i<j\leq l} \phi\left(t_i + t_j\right) - l\sum_{i=1}^{l} \phi(t_i) + \sum_{j=1}^{l} 2\phi(t_j) \tag{2.4}$$

for all $t_1, t_2, \cdots, t_l \in P$. Replacing $(t_1, t_2, \cdots, t_l)$ by $(x, y, x, y, 0, \cdots, 0)$ in (2.4), we get

$$\phi(x + y) = \phi(x) + \phi(y)$$

for all $x, y \in P$. Therefore the mapping $\phi : P \to Q$ is additive. □

Lemma 2.2. *If an even mapping $\phi : P \to Q$ satisfies $\phi(0) = 0$ and (1.8) for all $t_1, t_2, \cdots, t_l \in P$, then ϕ is quadratic.*

Proof. In view of the evenness of ϕ, we have $\phi(-t) = \phi(t)$ for all $t \in P$. Now, (1.8) becomes

$$\phi\left(\sum_{i=1}^{l} t_i\right) = \sum_{1\le i<j\le l} \phi\left(t_i + t_j\right) - (l+2)\sum_{i=1}^{l} \phi(t_i) + \sum_{j=1}^{l} \phi(2t_j) \tag{2.5}$$

for all $t_1, t_2, \cdots, t_l \in P$. Replacing $(t_1, t_2, \cdots, t_l)$ by $(t, 0, \cdots, 0)$ in (2.5), we obtain

$$\phi(2t) = 2^2\phi(t) \tag{2.6}$$

for all $t \in P$. Replacing t by $2t$ in (2.6), we have

$$\phi(2^2t) = 2^4\phi(t) \tag{2.7}$$

for all $t \in P$. Replacing t by $2t$ in (2.7), we obtain

$$\phi(2^3t) = 2^6\phi(t)$$

for all $t \in P$. In general, for any positive integer l, we get

$$\phi(2^lt) = 2^{2l}\phi(t)$$

for all $t \in P$. Therefore, (2.5) becomes

$$\phi\left(\sum_{i=1}^{l} t_i\right) = \sum_{1\le i<j\le l} \phi\left(t_i + t_j\right) - (l+2)\sum_{i=1}^{l} \phi(t_i) + \sum_{j=1}^{l} 4\phi(t_j) \tag{2.8}$$

for all $t_1, t_2, \cdots, t_l \in P$. Replacing g $(t_1, t_2, \cdots, t_l)$ by $(x, y, -x, -y, 0, \cdots, 0)$ in (2.8), we get

$$\phi(x+y) + \phi(x-y) = 2\phi(x) + 2\phi(y)$$

for all $x, y \in P$. Therefore the mapping $\phi : P \to Q$ is quadratic. □

Lemma 2.3. *A mapping $\phi : P \to Q$ satisfies $\phi(0) = 0$ and (1.8) for all $t_1, t_2, \cdots, t_l \in P$ if and only if there exist a symmetric bi-additive mapping $B : P \times P \to Q$ and an additive mapping $A : P \to Q$ such that $\phi(t) = B(t, t) + A(t)$ for all $t \in P$.*

Proof. Let ϕ with $\phi(0) = 0$ satisfy (1.8). We decompose ϕ into the even part and odd part by putting

$$\phi_e = \frac{1}{2}\left(\phi(t) + \phi(-t)\right) \quad \textit{and} \quad \phi_o(t) = \frac{1}{2}\left(\phi(t) - \phi(-t)\right)$$

for all $t \in P$. It is clear that $\phi(t) = \phi_e(t) + \phi_o(t)$ for all $t \in P$. It is easy to show that the mappings ϕ_e and ϕ_o satisfy (1.8). Hence by Lemmas 2.1 and 2.2, we obtain that ϕ_e and ϕ_o are quadratic and additive, respectively. Therefore, there exists a symmetric bi-additive mapping $B : P \times P \to Q$ such that $\phi_e(t) = B(t, t)$ for all $t \in P$. So $\phi(t) = B(t, t) + A(t)$ for all $t \in P$, where $A(t) = \phi_o(t)$ for all $t \in P$.

Conversely, assume that there exist a symmetric bi-additive mapping $B : P\times P \to Q$ and an additive mapping $A : P \to Q$ such that $\phi(t) = B(t, t) + A(t)$ for all $t \in P$. By a simple computation one can show that the mappings $t \mapsto B(t, t)$ and A satisfy the functional Eq (1.8). So the mapping ϕ satisfies (1.8). □

3. Hyers-Ulam stability of (1.8)

Throughout this section, assume that E is a quasi-Banach space with quasi-norm $\|\cdot\|$ and that F is a $p-$Banach space with p-norm $\|\cdot\|$. Let K be the modulus of concavity of $\|\cdot\|$.

In this section, using an idea of Gavruta we prove the Hyers-Ulam stability of the functional Eq (1.8) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given mapping $\phi : E \to F$:

$$D\phi(t_1, t_2, \cdots, t_l) := \phi\left(\sum_{i=1}^{l} t_i\right) - \sum_{1\le i<j\le l} \phi\left(t_i + t_j\right) + (l+2)\sum_{i=1}^{l}\left[\frac{\phi(t_i) + \phi(-t_i)}{2}\right] + l\sum_{i=1}^{l}\left[\frac{\phi(t_i) - \phi(-t_i)}{2}\right] - \sum_{j=1}^{l}\phi(2t_j)$$

for all $t_1, t_2, \cdots, t_l \in E$.

We will use the following lemma in this section.

Lemma 3.1. *[21] Let $0 \le p \le 1$ and let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \le \sum_{i=1}^{n} x_i^p.$$

Theorem 3.2. *Let $v \in \{-1, 1\}$ be fixed and let $\chi : E^l \to [0, \infty)$ be a function such that*

$$\lim_{l\to\infty} 2^{2lv}\chi\left(\frac{t_1}{2^{lv}}, \frac{t_2}{2^{lv}}, \cdots, \frac{t_l}{2^{lv}}\right) = 0 \tag{3.1}$$

for all $t_1, t_2, \cdots, t_l \in E$ and

$$\tilde{\psi}_e(t) := \sum_{g=\frac{1+v}{2}}^{\infty} 2^{2gvp}\chi^p\left(\frac{t}{2^{gv}}, 0, \cdots, 0\right) < \infty \tag{3.2}$$

for all $t \in E$. Suppose that an even mapping $\phi : E \to F$ with $\phi(0) = 0$ satisfies the inequality

$$\|D\phi(t_1, t_2, \cdots, t_l)\| \le \chi(t_1, t_2, \cdots, t_l) \tag{3.3}$$

for all $t_1, t_2, \cdots, t_l \in E$. Then the limit

$$\Phi(t) := \lim_{l\to\infty} 2^{2lv}\phi\left(\frac{t}{2^{lv}}\right) \tag{3.4}$$

exists for all $t \in E$ and $\Phi : E \to F$ is a unique quadratic mapping satisfying

$$\|\phi(t) - \Phi(t)\| \le \frac{K}{2^2}[\tilde{\psi}_e(t)]^{\frac{1}{p}} \tag{3.5}$$

for all $t \in E$.

Proof. Let $v = 1$. Replacing $(t_1, t_2, \cdots, t_l)$ by $(t, 0, \cdots, 0)$ in (3.3), we obtain

$$\left\|\phi(2t) - 2^2\phi(t)\right\| \leq \chi(t, 0, \cdots, 0) \tag{3.6}$$

for all $t \in E$. Let us take $\psi_e(t) = \chi(t, 0, \cdots, 0)$ for all $t \in E$. Then by (3.6), we have

$$\left\|\phi(2t) - 2^2\phi(t)\right\| \leq \psi_e(t) \tag{3.7}$$

for all $t \in E$. If we replace t by $\frac{t}{2^{l+1}}$ in (3.7) and multiply both sides of (3.7) by 2^{2l}, then we get

$$\left\|2^{2(l+1)}\phi\left(\frac{t}{2^{l+1}}\right) - 2^{2l}\phi\left(\frac{t}{2^l}\right)\right\| \leq K2^{2l}\psi_e\left(\frac{t}{2^{l+1}}\right) \tag{3.8}$$

for all $t \in E$ and all nonnegative integers l. Since F is a p-Banach space, by (3.8) we obtain

$$\left\|2^{2(l+1)}\phi\left(\frac{t}{2^{l+1}}\right) - 2^{2k}\phi\left(\frac{t}{2^k}\right)\right\|^p \leq \sum_{g=k}^{l}\left\|2^{2(g+1)}\phi\left(\frac{t}{2^{g+1}}\right) - 2^{2g}\phi\left(\frac{t}{2^g}\right)\right\|^p \leq K^p\sum_{g=k}^{l}2^{2gp}\psi_e^p\left(\frac{t}{2^{g+1}}\right) \tag{3.9}$$

for all nonnegative integers l and k with $l \geq k$ and all $t \in E$. Since $\psi_e^p(t) = \chi^p(t, 0, \cdots, 0)$ for all $t \in E$, by (3.2), we have

$$\sum_{g=1}^{\infty}2^{2gp}\psi_e^p\left(\frac{t}{2^g}\right) < \infty \tag{3.10}$$

for all $t \in E$. Therefore, it follows from (3.9) and (3.10) that the sequence $\{2^{2l}\phi\left(\frac{t}{2^l}\right)\}$ is a Cauchy sequence for each $t \in E$. Since F is complete, the sequence $\{2^{2l}\phi\left(\frac{t}{2^l}\right)\}$ converges for each $t \in E$. So one can define the mapping $\Phi : E \to F$ given by (3.4) for all $t \in E$. Letting $k = 0$ and passing the limit $l \to \infty$ in (3.9), we have

$$\|\phi(t) - \Phi(t)\|^p \leq K^p\sum_{g=0}^{\infty}2^{2gp}\psi_e^p\left(\frac{t}{2^{g+1}}\right) = \frac{K^p}{2^{2p}}\sum_{g=1}^{\infty}2^{2gp}\psi_e^p\left(\frac{t}{2^g}\right) \tag{3.11}$$

for all $t \in E$. Therefore, (3.5) follows from (3.2) and (3.11). Now, we show that Φ is quadratic. It follows from (3.1), (3.3) and (3.4) that

$$\|D\Phi(t_1, t_2, \cdots, t_l)\| = \lim_{l\to\infty}2^{2l}\left\|D\phi\left(\frac{t_1}{2^l}, \frac{t_2}{2^l}, \cdots, \frac{t_l}{2^l}\right)\right\| \leq \lim_{l\to\infty}2^{2l}\chi\left(\frac{t_1}{2^l}, \frac{t_2}{2^l}, \cdots, \frac{t_l}{2^l}\right) = 0$$

for all $t_1, t_2, \cdots, t_l \in E$. Therefore, the mapping $\Phi : E \to F$ satisfies (1.8). Since ϕ is an even mapping, (3.4) implies that the mapping $\Phi : E \to F$ is even. Therefore, by Lemma 2.2, we get that the mapping $\Phi : E \to F$ is quadratic.

To prove the uniqueness of Φ, let $\Phi' : E \to F$ be another quadratic mapping satisfying (3.5). Since

$$\lim_{l\to\infty}2^{2lp}\sum_{g=1}^{\infty}2^{2gp}\chi^p\left(\frac{t}{2^{g+l}}, 0, \cdots, 0\right) = \lim_{l\to\infty}\sum_{g=l+1}^{\infty}2^{2gp}\chi^p\left(\frac{t}{2^g}, 0, \cdots, 0\right) = 0$$

for all $t \in E$,

$$\lim_{l\to\infty} 2^{2lp}\tilde{\psi}_e\left(\frac{t}{2^l}\right) = 0$$

for all $t \in E$. Therefore, it follows from (3.5) and the last equation that

$$\left\|\Phi(t) - \Phi^{'}(t)\right\|^p = \lim_{l\to\infty} 2^{2lp}\left\|\phi\left(\frac{t}{2^l}\right) - \Phi^{'}\left(\frac{t}{2^l}\right)\right\|^p \le \frac{K^p}{2^{2p}}\lim_{l\to\infty} 2^{2lp}\tilde{\psi}_e\left(\frac{t}{2^l}\right) = 0$$

for all $t \in E$. Hence $\Phi = \Phi^{'}$.

For $v = -1$, we can prove this theorem by a similar manner. □

Corollary 3.3. *Let λ and $r_1, r_2, \cdots, r_l$ be nonnegative real numbers such that $r_1, r_2, \cdots, r_l > 2$ or $0 \le r_1, r_2, \cdots, r_l < 2$. Suppose that an even mapping $\phi : E \to F$ with $\phi(0) = 0$ satisfies the inequality*

$$\|D\phi(t_1, t_2, \cdots, t_l)\| \le \lambda\left(\|t_1\|^{r_1} + \|t_2\|^{r_2} + \cdots + \|t_l\|^{r_l}\right), \tag{3.12}$$

for all $t_1, t_2, \cdots, t_l \in E$. Then there exists a unique quadratic mapping $\phi : E \to F$ satisfying

$$\|\phi(t) - \Phi(t)\| \le K\lambda\left(\frac{\|t\|^{r_1 p}}{|2^{2p} - 2^{r_1 p}|}\right)^{\frac{1}{p}}$$

for all $t \in E$.

Proof. It follows from Theorem 3.2. □

Theorem 3.4. *Let $v \in \{-1, 1\}$ be fixed and let $\chi : E^l \to [0, \infty)$ be a function such that*

$$\lim_{l\to\infty} 2^{lv}\chi\left(\frac{t_1}{2^{lv}}, \frac{t_2}{2^{lv}}, \cdots, \frac{t_l}{2^{lv}}\right) = 0 \tag{3.13}$$

for all $t_1, t_2, \cdots, t_l \in E$ and

$$\tilde{\psi}_o(t) := \sum_{g=\frac{1+v}{2}}^{\infty} 2^{gvp}\chi^p\left(\frac{t}{2^{gv}}, 0, \cdots, 0\right) < \infty \tag{3.14}$$

for all $t \in E$. Suppose that an odd mapping $\phi : E \to F$ satisfies the inequality

$$\|D\phi(t_1, t_2, \cdots, t_l)\| \le \chi(t_1, t_2, \cdots, t_l) \tag{3.15}$$

for all $t_1, t_2, \cdots, t_l \in E$. Then the limit

$$\Psi(t) := \lim_{l\to\infty} 2^{lv}\phi\left(\frac{t}{2^{lv}}\right) \tag{3.16}$$

exists for all $t \in E$ and $\Psi : E \to F$ is a unique additive mapping satisfying

$$\|\phi(t) - \Psi(t)\| \le \frac{K}{2}[\tilde{\psi}_o(t)]^{\frac{1}{p}} \tag{3.17}$$

for all $t \in E$.

Proof. Let $v = 1$. Replacing $(t_1, t_2, \cdots, t_l)$ by $(t, 0, \cdots, 0)$ in (3.15), we obtain

$$\|\phi(2t) - 2\phi(t)\| \le \chi(t, 0, \cdots, 0) \tag{3.18}$$

for all $t \in E$. Let us take $\psi_o(t) = \chi(t, 0, \cdots, 0)$ for all $t \in E$. Then by (3.18), we have

$$\|\phi(2t) - 2\phi(t)\| \le \psi_o(t) \tag{3.19}$$

for all $t \in E$. If we replace t by $\frac{t}{2^{l+1}}$ in (3.19) and multiply both sides of (3.19) by 2^l, then we get

$$\left\|2^{(l+1)}\phi\left(\frac{t}{2^{l+1}}\right) - 2^l\phi\left(\frac{t}{2^l}\right)\right\| \le K2^l\psi_o\left(\frac{t}{2^{l+1}}\right) \tag{3.20}$$

for all $t \in E$ and all nonnegative integers l. Since F is a p-Banach space, by (3.20), we obtain

$$\left\|2^{(l+1)}\phi\left(\frac{t}{2^{l+1}}\right) - 2^k\phi\left(\frac{t}{2^k}\right)\right\|^p \le \sum_{g=k}^{l}\left\|2^{(g+1)}\phi\left(\frac{t}{2^{g+1}}\right) - 2^g\phi\left(\frac{t}{2^g}\right)\right\|^p \le K^p\sum_{g=k}^{l}2^{gp}\psi_o^p\left(\frac{t}{2^{g+1}}\right) \tag{3.21}$$

for all nonnegative integers l and k with $l \ge k$ and all $t \in E$. Since $\psi_o^p(t) = \chi^p(t, 0, \cdots, 0)$ for all $t \in E$, by (3.14) we have

$$\sum_{g=1}^{\infty}2^{gp}\psi_o^p\left(\frac{t}{2^g}\right) < \infty \tag{3.22}$$

for all $t \in E$. Therefore, it follows from (3.21) and (3.22) that the sequence $\{2^l\phi\left(\frac{t}{2^l}\right)\}$ is a Cauchy sequence for all $t \in E$. Since F is complete, the sequence $\{2^l\phi\left(\frac{t}{2^l}\right)\}$ converges for all $t \in E$. So one can define the mapping $\Psi : E \to F$ given by (3.16) for all $t \in E$. Letting $k = 0$ and passing the limit $l \to \infty$ in (3.21), we have

$$\|\phi(t) - \Psi(t)\|^p \le K^p\sum_{g=0}^{\infty}2^{gp}\psi_o^p\left(\frac{t}{2^{g+1}}\right) = \frac{K^p}{2^p}\sum_{g=1}^{\infty}2^{gp}\psi_o^p\left(\frac{t}{2^g}\right) \tag{3.23}$$

for all $t \in E$. Therefore, (3.17) follows from (3.14) and (3.23). Now, we show that Ψ is additive. It follows from (3.20), (3.22) and (3.17) that

$$\|\Psi(2t) - 2\Psi(t)\| = \lim_{l\to\infty}\left\|2^{l+1}\phi\left(\frac{t}{2^{l+1}}\right) - 2^l\phi\left(\frac{t}{2^l}\right)\right\| \le K\lim_{l\to\infty}2^l\psi_o\left(\frac{t}{2^{l+1}}\right) = 0$$

for all $t \in E$. So $\Psi(2t) = 2\Psi(t)$ for all $t \in E$. On the other hand, it follows from (3.13), (3.15) and (3.16) that

$$\|D\Psi(t_1, t_2, \cdots, t_l)\| = \lim_{l\to\infty}2^l\left\|D\phi\left(\frac{t_1}{2^l}, \frac{t_2}{2^l}, \cdots, \frac{t_l}{2^l}\right)\right\| \le \lim_{l\to\infty}2^l\chi\left(\frac{t_1}{2^l}, \frac{t_2}{2^l}, \cdots, \frac{t_l}{2^l}\right) = 0$$

for all $t_1, t_2, \cdots, t_l \in E$. Therefore, the mapping $\Psi : E \to F$ satisfies (1.8). Since ϕ is an odd mapping, (3.16) implies that the mapping $\Psi : E \to F$ is odd. Therefore, by Lemma 2.1, we get that the mapping $\psi : E \to F$ is additive.

To prove the uniqueness of Ψ, let $\Psi^{'} : E \to F$ be another additive mapping satisfying (3.17). Since

$$\lim_{l\to\infty} 2^{lp} \sum_{g=1}^{\infty} 2^{gp} \chi^p \left(\frac{t}{2^{g+l}}, 0, \cdots, 0\right) = \lim_{l\to\infty} \sum_{g=l+1}^{\infty} 2^{gp} \chi^p \left(\frac{t}{2^g}, 0, \cdots, 0\right) = 0$$

for all $t \in E$,

$$\lim_{l\to\infty} 2^{lp} \tilde{\psi}_o \left(\frac{t}{2^l}\right) = 0$$

for all $t \in E$. Therefore, it follows from (3.17) and the last equation that

$$\left\|\Psi(t) - \Psi^{'}(t)\right\|^p = \lim_{l\to\infty} 2^{lp} \left\|\phi\left(\frac{t}{2^l}\right) - \Psi^{'}\left(\frac{t}{2^l}\right)\right\|^p \le \frac{K^p}{2^p} \lim_{l\to\infty} 2^{lp} \tilde{\psi}_o \left(\frac{t}{2^l}\right) = 0$$

for all $t \in E$. Hence $\Psi = \Psi^{'}$.

For $v = -1$, we can prove this theorem by a similar manner. □

Corollary 3.5. *Let λ and $r_1, r_2, \cdots, r_l$ be nonnegative real numbers such that $r_1, r_2, \cdots, r_l > 1$ or $0 \le r_1, r_2, \cdots, r_l < 1$. Suppose that an odd mapping $\phi : E \to F$ satisfies the inequality*

$$\|D\phi(t_1, t_2, \cdots, t_l)\| \le \lambda\left(\|t_1\|^{r_1} + \|t_2\|^{r_2} + \cdots + \|t_l\|^{r_l}\right),$$

for all $t_1, t_2, \cdots, t_l \in E$. Then there exists a unique additive function $\phi : E \to F$ satisfying

$$\|\phi(t) - \Psi(t)\| \le K\lambda \left(\frac{\|t\|^{r_1 p}}{|2^p - 2^{r_1 p}|}\right)^{\frac{1}{p}}$$

for all $t \in E$.

Proof. It follows from Theorem 3.4. □

Proposition 3.6. *Let $\chi : E^l \to [0, \infty)$ be a function which satisfies (3.1) and (3.2) for all $t_1, t_2, \cdots, t_l \in E$ and satisfies (3.13) and (3.14) for all $t_1, t_2, \cdots, t_l \in E$. Suppose that a mapping $\phi : E \to F$ with $\phi(0) = 0$ satisfies the inequality (3.3) for all $t_1, t_2, \cdots, t_l \in E$. Then there exist a unique quadratic mapping $\Phi : E \to F$ and a unique additive mapping $\Psi : E \to F$ satisfying (1.8) and*

$$\|\phi(t) - \Phi(t) - \Psi(t)\| \le \frac{K^3}{8} \left\{\left[\tilde{\psi}_e(t) + \tilde{\psi}_e(-t)\right]^{\frac{1}{p}} + 2\left[\tilde{\psi}_o(t) + \tilde{\psi}_o(-t)\right]^{\frac{1}{p}}\right\}$$

for all $t \in E$, where $\tilde{\psi}_e(t)$ and $\tilde{\psi}_o(t)$ were defined in (3.2) and (3.14), respectively, for all $t \in E$.

Proof. Let $\phi_o(t) = \frac{\phi(t)-\phi(-t)}{2}$ for all $t \in E$. Then

$$\|D\phi_o(t_1, t_2, \cdots, t_l)\| \le \frac{1}{2}\left\{\|D\phi(t_1, t_2, \cdots, t_l)\| + \|D\phi(-t_1, -t_2, \cdots, -t_l)\|\right\}$$

for all $t_1, t_2, \cdots, t_l \in E$. And let $\phi_e(t) = \frac{\phi(t)+\phi(-t)}{2}$ for all $t \in E$. Then

$$\|D\phi_e(t_1, t_2, \cdots, t_l)\| \le \frac{1}{2}\left\{\|D\phi(t_1, t_2, \cdots, t_l)\| + \|D\phi(-t_1, -t_2, \cdots, -t_l)\|\right\}$$

for all $t_1, t_2, \cdots, t_l \in E$. Let us define

$$\phi(t) = \phi_e(t) + \phi_o(t)$$

for all $t \in E$. Now,

$$\|\phi(t) - \Phi(t) - \Psi(t)\| = \|\phi_e(t) + \phi_o(t) - \Phi(t) - \Psi(t)\| \leq \|\phi_e(t) - \Phi(t)\| + \|\phi_o(t) - \Psi(t)\|.$$

Using Theorems 3.2 and Theorem 3.4, we can prove the remaining proof of the theorem. □

Corollary 3.7. *Let λ and $r_1, r_2, \cdots, r_l$ be nonnegative real numbers such that $r_1, r_2, \cdots, r_l \neq 2$ or $r_1, r_2, \cdots, r_l \neq 1$. Suppose that a mapping $\phi : E \to F$ with $\phi(0) = 0$ satisfies the inequality (3.12) for all $t_1, t_2, \cdots, t_l \in E$. Then there exists a unique quadratic mapping $\Phi : E \to F$ and a unique additive mapping $\Psi : E \to F$ satisfying (1.8) and*

$$\|\phi(t) - \Phi(t) - \Psi(t)\| \leq K^3 \lambda \left[\left(\frac{\|t\|^{r_1 p}}{|2^{2p} - 2^{r_1 p}|} \right)^{\frac{1}{p}} + \left(\frac{\|t\|^{r_1 p}}{|2^{p} - 2^{r_1 p}|} \right)^{\frac{1}{p}} \right]$$

for all $t \in E$.

4. Conclusions

We have introduced the mixed type finite variable additive-quadratic functional Eq (1.8) and have obtained the general solution of the mixed type finite variable additive-quadratic functional Eq (1.8) in quasi-Banach spaces. Furthermore, we have proved the Hyers-Ulam stability for the mixed type finite variable additive-quadratic functional Eq (1.8) in quasi-Banach spaces.

Conflict of interest

The authors declare that they have no competing interests.

References

1. J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.
2. D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Basel, 1986.
3. Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society, 2000.
4. P. W. Cholewa, *Remarks on the stability of functional equations*, Aequations Math., **27** (1984), 76–86.
5. S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Hamburg, **62** (1992), 59–64.

6. E. Elqorachi, M. Th. Rassias, *Generalized Hyers-Ulam stability of trigonometric functional equations*, Mathematics, **6** (2018), 1–11.
7. M. E. Gordji, H. Khodaei, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, Nonlinear Anal-Theor., **71** (2009), 5629–5643.
8. Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431–434.
9. P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.
10. V. Govindan, C. Park, S. Pinelas, et al. *Solution of a 3-D cubic functional equation and its stability*, AIMS Mathematics, **5** (2020), 1693–1705.
11. A. Grabiec, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48** (1996), 217–235.
12. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224.
13. P. Jordan, J. Neumann, *On inner products in linear metric spaces*, Ann. Math., **36** (1935), 719–723.
14. K. Jun, H. Kim, *Ulam stability problem for a mixed type of cubic and additive functional equation*, B. Belg. Math. Soc-Sim., **13** (2006), 271–285.
15. S. M. Jung, D. Popa, M. T. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, J. Global Optim., **59** (2014), 165–171.
16. P. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., **27** (1995), 368–372.
17. T. M. Kim, C. Park, S. H. Park, *An AQ-functional equation in paranormed spaces*, J. Comput. Anal. Appl., **15** (2013), 1467–1475.
18. Y. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics, **7** (2019), 1–15.
19. Y. H. Lee, S. M. Jung, *A general theorem on the stability of a class of functional equations including quartic-cubic-quadratic-additive equations*, Mathematics, **6** (2018), 1–24.
20. Y. Lee, S. Jung, M. T. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal., **12** (2018), 43–61.
21. A. Najati, M. B. Moghimi, *Stability of a functional equation deriving from quadratic and additive function in quasi-Banach spaces*, J. Math. Anal. Appl., **337** (2008), 399–415.
22. A. Najati, G. Z. Eskandani, *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., **342** (2008), 1318–1331.
23. T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc., **72** (1978), 297–300.
24. P. Saha, T. K. Samanta, N. C. Kayal, et al. *Hyers-Ulam-Rassias stability of set valued additive and cubic functional equations in several variables*, Mathematics, **7** (2019), 1–11.
25. F. Skof, *Proprieta' locali e approssimaziones di operatori*, Seminario Mat. e. Fis. di Milano, **53** (1983), 113–129.

26. J. Tabor, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Pol. Math., **83** (2004), 243–255.

27. S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, Inc., 1964.

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