Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 7562–7581.
DOI:10.3934/math.2020484
Received: 25 June 2020
Accepted: 23 September 2020
Published: 24 September 2020

Research article

On Reidemeister torsion of flag manifolds of compact semisimple Lie groups

Cenap Özel[1,*], Habib Basbaydar[2], Yasar Sözen[3], Erol Yilmaz[2], Jung Rye Lee[4] and Choonkil Park[5,*]

[1] Department of Mathematics, King Abdulaziz University, 21589 Jeddah, Makkah, Saudi Arabia
[2] Department of Mathematics, AIBU Izzet Baysal University, Bolu, Turkey
[3] Department of Mathematics, Hacettepe University, Ankara, Turkey
[4] Department of Mathematics, Daejin University, Kyunggi 11159, Korea
[5] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Email: cenap.ozel@gmail.com, baak@hanyang.ac.kr; Tel: +8222200892; Fax: +8222810019.

Abstract: In this paper we calculate Reidemeister torsion of flag manifold K/T of compact semi-simple Lie group $K = SU_{n+1}$ using Reidemeister torsion formula and Schubert calculus, where T is maximal torus of K. We find that this number is 1. Also we explicitly calculate ring structure of integral cohomology algebra of flag manifold K/T of compact semi-simple Lie group $K = SU_{n+1}$ using root data, and Groebner basis techniques.

Keywords: Reidemeister torsion; flag manifolds; Weyl groups; Schubert calculus; Groebner-Shirshov bases; graded inverse lexicographic order
Mathematics Subject Classification: 57Q10, 16Z05, 14M15, 14N15, 22E67

1. Introduction

Reidemeister torsion is a topological invariant and was introduced by Reidemeister in 1935. Up to PL equivalence, he classified the lens spaces S^3/Γ, where Γ is a finite cyclic group of fixed point free orthogonal transformations [20]. In [11], Franz extended the Reidemeister torsion and classified the higher dimensional lens spaces S^{2n+1}/Γ, where Γ is a cyclic group acting freely and isometrically on the sphere S^{2n+1}.

In 1964, the results of Reidemeister and Franz were extended by de Rham to spaces of constant curvature +1 [10]. Kirby and Siebenmann proved the topological invariance of the Reidemeister torsion for manifolds in 1969 [14]. Chapman proved for arbitrary simplicial complexes [7, 8]. Hence, the classification of lens spaces of Reidemeister and Franz was actually topological (i.e., up to homeomorphism).

Using the Reidemeister torsion, Milnor disproved *Hauptvermutung* in 1961. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. He identified in 1962 the Reidemeister torsion with Alexander polynomial which plays an important role in knot theory and links [16, 18].

In [21], Sözen explained the claim mentioned in [27, p. 187] about the relation between a symplectic chain complex with $\omega-$compatible bases and the Reidemeister torsion of it. Moreover, he applied the main theorem to the chain-complex

$$0 \to C_2(\Sigma_g; \mathrm{Ad}_\varrho) \stackrel{\partial_2 \otimes id}{\to} C_1(\Sigma_g; \mathrm{Ad}_\varrho) \stackrel{\partial_1 \otimes id}{\to} C_0(\Sigma_g; \mathrm{Ad}_\varrho) \to 0,$$

where Σ_g is a compact Riemann surface of genus $g > 1$, where ∂ is the usual boundary operator, and where $\varrho : \pi_1(\Sigma_g) \to \mathrm{PSL}_2(\mathbb{R})$ is a discrete and faithful representation of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g [21]. Now we will give his description of Reidemesister torsion and explain why it is not unique by a result of Milnor in [17].

Let $\mathcal{H}_p(C_*) = \mathcal{Z}_p(C_*)/\mathcal{B}_p(C_*)$ denote the homologies of the chain complex $(C_*, \partial_*) = (C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0)$ of finite dimensional vector spaces over field $\mathbb{C}$ or $\mathbb{R}$, where $\mathcal{B}_p = \mathrm{Im}\{\partial_{p+1} : C_{p+1} \to C_p\}$, $\mathcal{Z}_p = \ker\{\partial_p : C_p \to C_{p-1}\}$, respectively.

Consider the short-exact sequences:

$$0 \to \mathcal{Z}_p \hookrightarrow C_p \twoheadrightarrow \mathcal{B}_{p-1} \to 0 \tag{1.1}$$

$$0 \to \mathcal{B}_p \hookrightarrow \mathcal{Z}_p \twoheadrightarrow \mathcal{H}_p \to 0, \tag{1.2}$$

where (1.1) is a result of 1$^{\mathrm{st}}$-Isomorphism Theorem and (1.2) follows simply from the definition of $\mathcal{H}_p$. Note that if $\mathfrak{b}_p$ is a basis for $\mathcal{B}_p$, $\mathfrak{h}_p$ is a basis for $\mathcal{H}_p$, and $\ell_p : \mathcal{H}_p \to \mathcal{Z}_p$ and $s_p : \mathcal{B}_{p-1} \to C_p$ are sections, then we obtain a basis for C_p. Namely, $\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1})$.

If, for $p = 0, \cdots, n$, $\mathfrak{c}_p$, $\mathfrak{b}_p$, and $\mathfrak{h}_p$ are bases for C_p, $\mathcal{B}_p$ and $\mathcal{H}_p$, respectively, then the alternating product

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \prod_{p=0}^n \left[\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1}), \mathfrak{c}_p\right]^{(-1)^{(p+1)}} \tag{1.3}$$

is called the *Reidemeister torsion of the complex* C_* with respect to bases $\{\mathfrak{c}_p\}_{p=0}^n$, $\{\mathfrak{h}_p\}_{p=0}^n$, where $\left[\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1}), \mathfrak{c}_p\right]$ denotes the determinant of the change-base matrix from $\mathfrak{c}_p$ to $\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1})$.

Milnor [17] proved that torsion does not depend on neither the bases $\mathfrak{b}_p$, nor the sections s_p, ℓ_p. Moreover, if $\mathfrak{c}'_p, \mathfrak{h}'_p$ are other bases respectively for C_p and $\mathcal{H}_p$, then there is the change-base-formula:

$$\mathrm{Tor}(C_*, \{\mathfrak{c}'_p\}_{p=0}^n, \{\mathfrak{h}'_p\}_{p=0}^n) = \prod_{p=0}^n \left(\frac{[\mathfrak{c}'_p, \mathfrak{c}_p]}{[\mathfrak{h}'_p, \mathfrak{h}_p]}\right)^{(-1)^p} \cdot \mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n). \tag{1.4}$$

Let M be a smooth $n-$manifold, $\mathbf{K}$ be a cell-decomposition of M with for each $p = 0, \cdots, n$, $\mathfrak{c}_p = \{e_1^p, \cdots, e_{m_p}^p\}$, called the *geometric basis* for the $p-$cells $C_p(\mathbf{K};\mathbb{Z})$. Hence, we have the chain-complex associated to M

$$0 \to C_n(\mathbf{K}) \xrightarrow{\partial_n} C_{n-1}(\mathbf{K}) \to \cdots \to C_1(\mathbf{K}) \xrightarrow{\partial_1} C_0(\mathbf{K}) \to 0, \tag{1.5}$$

where ∂_p denotes the boundary operator. Then $\mathrm{Tor}(C_*(\mathbf{K}), \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ is called the *Reidemeister torsion* of M, where $\mathfrak{h}_p$ is a basis for $\mathcal{H}_p(\mathbf{K})$.

In [23], oriented closed connected $2m-$manifolds ($m \geq 1$) are considered and he proved the following formula for computing the Reidemeister torsion of them. Namely,

Theorem 1.1. *Let M be an oriented closed connected $2m-$manifold ($m \geq 1$). For $p = 0, \ldots, 2m$, let $\mathbf{h}_p$ be a basis of $H_p(M)$. Then the Reidemeister torsion of M satisfies the following formula:*

$$\left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p=0}^{m-1} \left|\det H_{p,2m-p}(M)\right|^{(-1)^p} \sqrt{\left|\det H_{m,m}(M)\right|}^{\,(-1)^m},$$

where $\det H_{p,2m-p}(M)$ *is the determinant of the matrix of the intersection pairing* $(\cdot,\cdot)_{p,2m-p} : H_p(M) \times H_{2m-p}(M) \to \mathbb{R}$ *in bases* $\mathbf{h}_p, \mathbf{h}_{2m-p}$.

It is well known that Riemann surfaces and Grasmannians have many applications in a wide range of mathematics such as topology, differential geometry, algebraic geometry, symplectic geometry, and theoretical physics (see [2, 3, 5, 6, 12, 13, 22, 24–26] and the references therein). They also applied Theorem 1.1 to Riemann surfaces and Grasmannians.

In this work we calculate Reidemeister torsion of compact flag manifold K/T for $K = SU_{n+1}$, where K is a compact simply connected semi-simple Lie group and T is maximal torus [28].

The content of the paper is as follows. In Section 2 we give all details of cup product formula in the cohomology ring of flag manifolds which is called Schubert calculus [15, 19]. In the last section we calculate the Reidemesiter torsion of flag manifold SU_{n+1}/T for $n \geq 3$.

The results of this paper were obtained during M.Sc studies of Habib Basbaydar at Abant Izzet Baysal University and are also contained in his thesis [1].

2. Schubert calculus and cohomology of flag manifold

Now, we will give the important formula equivalent to the cup product formula in the cohomology of G/B where G is a Kač-Moody group. The fundamental references for this section are [15, 19]. To do this we will give a relation between the complex nil Hecke ring and $H^*(K/T, \mathbb{C})$. Also we introduce a multiplication formula and the actions of reflections and Berstein-Gelfand-Gelfand type BGG operators A_i on the basis elements in the nil Hecke ring.

Proposition 2.1.

$$\xi^u \cdot \xi^v = \sum_{u,v \leqslant w} p_{u,v}^w \xi^w,$$

where $p_{u,v}^w$ is a homogeneous polynomial of degree $\ell(u) + \ell(v) - \ell(w)$.

Proposition 2.2.

$$r_i\xi^w = \begin{cases} \xi^w & \text{if } r_iw > w, \\ -(w^{-1}\alpha_i)\xi^{r_iw} + \xi^w - \sum\limits_{r_iw \xrightarrow{\gamma} w'} \alpha_i(\gamma^\vee)\xi^{w'} & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let $u, v \in W$. We write $w^{-1} = r_{i_1} \cdots r_{i_n}$ as a reduced expression.*

$$p^w_{u,v} = \sum_{\substack{j_1 < \cdots < j_m \\ r_{j_1} \cdots r_{j_m} = v^{-1}}} A_{i_1} \circ \cdots \circ \hat{A}_{i_{j_1}} \circ \cdots \circ \hat{A}_{i_{j_m}} \circ \cdots \circ A_{i_n}(\xi^u)(e)$$

where $m = \ell(v)$ and the notation $\hat{A}_i$ means that the operator A_i is replaced by the Weyl group action r_i.

Let $\mathbb{C}_0 = S/S^+$ be the S-module where S^+ is the augmentation ideal of S. It is 1-dimensional as $\mathbb{C}$-vector space. Since Λ is a S-module, we can define $\mathbb{C}_0 \otimes_S \Lambda$. It is an algebra and the action of $\mathcal{R}$ on Λ gives an action of $\mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$. The elements $\sigma^w = 1 \otimes \xi^w \in \mathbb{C}_0 \otimes_S \Lambda$ is a $\mathbb{C}$-basis form of $\mathbb{C}_0 \otimes_S \Lambda$.

Proposition 2.4. $\mathbb{C}_0 \otimes_S \Lambda$ *is a graded algebra associated with the filtration of length of the element of the Weyl group W.*

Proposition 2.5. *The complex linear map $f : \mathbb{C}_0 \otimes_S \Lambda \to \mathrm{Gr}\,\mathbb{C}\{W\}$ is a graded algebra homomorphism.*

Theorem 2.6. *Let K be the standard real form of the group G associated to a symmetrizable Kač-Moody Lie algebra* **g** *and let T denote the maximal torus of K. Then the map*

$$\theta : H^*(K/T, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda$$

defined by $\theta(\varepsilon^w) = \sigma^w$ for any $w \in W$ is a graded algebra isomorphism. Moreover, the action of $w \in W$ and A^w on $H^*(K/T, \mathbb{C})$ corresponds respectively to that δ_w and $x_w \in \mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$.

Corollary 2.7. *The operators A^i on $H^*(K/T, \mathbb{C})$ generate the nil-Hecke algebra.*

Corollary 2.8. *We can use Proposition 2.1 and Theorem 2.3 to determine the cup product $\varepsilon^u\varepsilon^v$ in terms of the Schubert basis $\{\varepsilon^w\}_{w\in W}$ of $H^*(K/T, \mathbb{Z})$.*

3. The Reidemeister torsion of compact flag manifold K/T for $K = SU_{n+1}$

This section includes our calculations about Reidemeister torsion of flag manifolds using Theorem 1.1 and Proposition 2.1 because $\chi(SU_{n+1}/T) = |W| = n!$ is always an even number.

We know that the Weyl group W of K acts on the Lie algebra of the maximal torus T. lt is a finite group of isometries of the Lie algebra **t** of the maximal torus T. lt preserves the coweight lattice T^v. For each simple root α, the Weyl group W contains an element r_α of order two represented by $e^{((\pi/2)(e_\alpha + e_{-\alpha}))}$ in $N(T)$. Since the roots α can be considered as the linear functionals on the Lie algebra **t** of the maximal torus T, the action of r_α on **t** is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \quad \text{for} \quad \xi \in \mathbf{t},$$

where h_α is the coroot in **t** corresponding to simple root α.Also, we can give the action of r_α on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \quad \text{for} \quad \alpha, \beta \in \mathbf{t}^*,$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element r_α is the reflection in the hyperplane H_α of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections r_α generate the Weyl group W.

Set $\alpha_1, \alpha_2, \ldots, \alpha_n$ be roots of Weyl Group of SU_{n+1}. Since the Cartan Matrix of Weyl Group of SU_{n+1} is

$$M_{ij} = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$r_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_i, & i = j \\ \alpha_i + \alpha_j, & |i-j| = 1 \\ \alpha_j, & \text{otherwise.} \end{cases}$$

Proposition 3.1. *The Weyl group W of SU_{n+1} is isomorphic to Coxeter Group A_n given by generators $s_1, s_2, \ldots, s_n$ and relations*

(i) $s_i^2 = 1 \qquad i = 1, 2, \ldots, n;$

(ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad i = 1, 2, \ldots, n-1;$

(iii) $s_i s_j = s_j s_i \qquad 1 \le i < j - 1 < n.$

Proof. **(i)**

$$\begin{aligned} r_{\alpha_i} \circ r_{\alpha_i}(\beta) &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i) \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\beta - <\alpha_i, \beta> \alpha_i, \alpha_i> \alpha_i \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\beta, \alpha_i> \alpha_i + <\alpha_i, \beta><\alpha_i, \alpha_i> \alpha_i \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\alpha_i, \beta> \alpha_i + 2 <\alpha_i, \beta> \alpha_i \\ &= \beta. \end{aligned}$$

(ii)

$$\begin{aligned} r_{\alpha_i} \circ r_{\alpha_{i+1}} \circ r_{\alpha_i}(\beta) &= r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta - <\alpha_i, \beta> \alpha_i) \\ &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta - <\alpha_i, \beta> \alpha_i> \alpha_{i+1}) \\ &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} \\ &\qquad + <\alpha_{i+1}, <\alpha_i, \beta> \alpha_i> \alpha_{i+1}) \\ &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} \\ &\qquad + <\alpha_i, \beta><\alpha_{i+1}, \alpha_i> \alpha_{i+1}) \\ &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} - <\alpha_i, \beta> \alpha_{i+1}) \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} - <\alpha_i, \beta> \alpha_{i+1} \\ &\qquad - <\alpha_i, \beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} \\ &\qquad - <\alpha_i, \beta> \alpha_{i+1}> \alpha_i \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\alpha_{i+1}, \beta> \alpha_{i+1} - <\alpha_i, \beta> \alpha_{i+1} \end{aligned}$$

$$
\begin{aligned}
&\quad - < \alpha_i,\beta > \alpha_i + < \alpha_i,\beta >< \alpha_i,\alpha_i > \alpha_i \\
&\quad + < \alpha_{i+1},\beta >< \alpha_{i+1},\alpha_i > \alpha_i + < \alpha_i,\beta >< \alpha_{i+1},\alpha_i > \alpha_i \\
&= \beta - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_{i+1} \\
&\quad - < \alpha_i,\beta > \alpha_i + 2 < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i \\
&\quad - < \alpha_i,\beta > \alpha_i \\
&= \beta - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_{i+1} \\
&\quad - < \alpha_i,\beta > \alpha_{i+1} \\
&= \beta - (< \alpha_i,\beta > + < \alpha_{i+1},\beta >)(\alpha_i + \alpha_{i+1}).
\end{aligned}
$$

$$
\begin{aligned}
r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta) &= r_{\alpha_{i+1}} \circ r_{\alpha_i}(\beta - < \alpha_{i+1},\beta > \alpha_{i+1}) \\
&= r_{\alpha_{i+1}}(\beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta - < \alpha_{i+1},\beta > \alpha_{i+1} > \alpha_i) \\
&= r_{\alpha_{i+1}}(\beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i \\
&\quad + < \alpha_{i+1},\beta >< \alpha_i,\alpha_{i+1} > \alpha_i) \\
&= r_{\alpha_{i+1}}(\beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i) \\
&= \beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i \\
&\quad - < \alpha_{i+1},\beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i \\
&\quad - < \alpha_{i+1},\beta > \alpha_i > \alpha_{i+1} \\
&= \beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_{i+1},\beta > \alpha_i - < \alpha_i,\beta > \alpha_i \\
&\quad - < \alpha_{i+1},\beta > \alpha_{i+1} + < \alpha_i,\beta >< \alpha_{i+1},\alpha_i > \alpha_{i+1} \\
&\quad + < \alpha_{i+1},\beta >< \alpha_{i+1},\alpha_i > \alpha_{i+1} \\
&\quad + < \alpha_{i+1},\beta >< \alpha_{i+1},\alpha_{i+1} > \alpha_{i+1} \\
&= \beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i \\
&\quad - < \alpha_{i+1},\beta > \alpha_{i+1} + 2 < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_{i+1} \\
&\quad - < \alpha_{i+1},\beta > \alpha_{i+1} \\
&= \beta - < \alpha_{i+1},\beta > \alpha_{i+1} - < \alpha_i,\beta > \alpha_i - < \alpha_{i+1},\beta > \alpha_i \\
&\quad - < \alpha_i,\beta > \alpha_{i+1} \\
&= \beta - (< \alpha_{i+1},\beta > + < \alpha_i,\beta >)(\alpha_{i+1} + \alpha_i).
\end{aligned}
$$

Hence $r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta) = r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta)$.

(iii)

$$
\begin{aligned}
r_{\alpha_i} \circ r_{\alpha_j}(\beta) &= r_{\alpha_i} \circ (\beta - < \alpha_j,\beta > \alpha_j) \\
&= \beta - < \alpha_j,\beta > \alpha_j - < \alpha_i,\beta - < \alpha_j,\beta > \alpha_j > \alpha_i \\
&= \beta - < \alpha_j,\beta > \alpha_j - < \alpha_i,\beta > \alpha_i + < \alpha_j,\beta >< \alpha_i,\alpha_j > \alpha_i \\
&= \beta - < \alpha_j,\beta > \alpha_j - < \alpha_i,\beta > \alpha_i.
\end{aligned}
$$

$$
r_{\alpha_j} \circ r_{\alpha_i}(\beta) = r_{\alpha_j} \circ (\beta - < \alpha_i,\beta > \alpha_i)
$$

$$= \beta - <\alpha_i,\beta>\alpha_i - <\alpha_j, \beta - <\alpha_i,\beta>\alpha_i>\alpha_j$$
$$= \beta - <\alpha_i,\beta>\alpha_i - <\alpha_j,\beta>\alpha_j + <\alpha_i,\beta><\alpha_j,\alpha_i>\alpha_j$$
$$= \beta - <\alpha_i,\beta>\alpha_i - <\alpha_j,\beta>\alpha_j.$$

Hence $r_{\alpha_i} \circ r_{\alpha_j}(\beta) = r_{\alpha_j} \circ r_{\alpha_i}(\beta)$.

□

After this point s_i will represent r_{α_i}.
Let us define the word

$$s_{i,j} = \begin{cases} s_i s_{i+1} \cdots s_j & i < j \\ s_i & i = j \\ 1 & i > j. \end{cases}$$

Theorem 3.2. *[4, Theorem 3.1] The reduced Gröbner-Shirshov basis of the coxeter group A_n consists of relation*

$$s_{i,j} s_i = s_{i+j} s_{i,j} \quad 1 \le i < j \le n$$

together with defining relations of A_n.

The following lemma is equivalent of [4, Lemma 3.2]. The only difference is the order of generators $s_1 > s_2 > \ldots s_n$ in our setting.

Lemma 3.3. *Using elimination of leading words of relations, the reduced elements of A_n are in the form*
$s_{n+1,j_{n+1}} s_{n,j_n} s_{n-1,j_{n-1}} \cdots s_{i,j_i} \cdots s_{1,j_1} \quad 1 \le i \le j_i + 1 \le n + 1.$
Notice that $j_{n+1} + 1 = n + 1 \Longrightarrow j_{n+1} = n$ and $s_{n+1,n} = 1$.

Algorithm 3.1. (Finding Inverse) Let $w = s_{n,j_n} s_{n-1,j_{n-1}} \cdots s_{1,j_1}$. The inverse of w can be found using following algorithm.

$Invw = \{\}$;
$Conw = Reverse(w)$;
For $k = 1$ to $k = n$
  Find maximum sequence in $Conw$;
  $list = \{s_k, s_{k+1}, s_{k+2}, \ldots, s_{k+j}\}$;
  $Invw = list \cup Invw$;
End For.

Example 3.4. *Let $s_{4,6} s_{3,5} s_{2,5} s_{1,3}$. The inverse of its is $S_3 s_2 s_1 s_5 s_4 s_3 s_2 s_5 s_4 s_3 s_6 s_5 s_4$.*
$Invw = s_{1,4}$
$S_3 s_2 s_5 s_4 s_3 s_5 s_4 s_6 s_5$
$Invw = s_{2,5} s_{1,4}$
$S_3 s_5 s_4 s_5 s_6$
$Invw = s_{3,5} s_{2,5} s_{1,4}$
$s_5 s_6$
$Invw = s_{5,6} s_{3,5} s_{2,5} s_{1,4}$.

Lemma 3.5. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}})\cdots(s_{i+1,\,j_{i+1}})(s_{i,j_i})\cdots(s_{1,j_1})$ *and*
$s_iw = (s_{n,\overline{j_n}})(s_{n-1,\,\overline{j_{n-1}}})\cdots(s_{i+1,\,\overline{j_{i+1}}})(s_{i,\overline{j_i}})\cdots(s_{1,\overline{j_1}})$*, where*

$$s_iw = \begin{cases} \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1} & if \quad j_i < j_{i+1} \\ \overline{j_{i+1}} = j_i,\ \overline{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1} \\ \overline{j_k} = j_k & if \quad k \neq i, i+1 \end{cases}$$

Here if $i = n$*, then we assume* $j_{n+1} = n$*.*

Corollary 3.6. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}})\cdots(s_{i+1,j_{i+1}})(s_{i,j_i})\cdots(s_{1,j_1})$ *and*
$s_{i-1}(s_iw) = (s_{n,\hat{j}_n})(s_{n-1,\widehat{j_{n-1}}})\cdots(s_{i+1,\widehat{j_{i+1}}})(s_{i,\widehat{j_i}})\cdots(s_{1,\widehat{j_1}})$*, where*

$$s_{i-1}(s_iw) = \begin{cases} \widehat{j_{i+1}} = j_i + 1,\ \widehat{j_i} = j_{i-1} + 1,\ \widehat{j_{i-1}} = j_{i+1} & if \quad j_i < j_{i+1},\ j_{i-1} < j_{i+1} \\ \widehat{j_{i+1}} = j_i + 1,\ \widehat{j_i} = j_{i-1},\ \widehat{j_{i-1}} = j_{i+1} - 1 & if \quad j_i < j_{i+1},\ j_{i-1} \geq j_{i+1} \\ \widehat{j_{i+1}} = j_i,\ \widehat{j_i} = j_{i-1} + 1,\ \widehat{j_{i-1}} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_{i-1} < j_{i+1} - 1 \\ \widehat{j_{i+1}} = j_i,\ \widehat{j_i} = j_{i-1},\ \widehat{j_{i-1}} = j_{i+1} - 2 & if \quad j_i \geq j_{i+1},\ j_{i-1} \geq j_{i+1} - 1 \\ \widehat{j_k} = j_k & if \quad k \neq i-1,\ i,\ i+1. \end{cases}$$

Proof. Let $\overline{w} = s_iw = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}})\cdots(s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}})\cdots(s_{1,\overline{j_1}})$. Then

$$s_{i-1}(\overline{w}) = \begin{cases} \widehat{j_i} = \overline{j_{i-1}} + 1,\ \widehat{j_{i-1}} = \overline{j_i} & if \quad \overline{j_{i-1}} < \overline{j_i} \\ \widehat{j_i} = \overline{j_{i-1}},\ \widehat{j_{i-1}} = \overline{j_i} - 1 & if \quad \overline{j_{i-1}} \geq \overline{j_i} \\ \widehat{j_k} = \overline{j_k} & if \quad k \neq i-1, i. \end{cases}$$

(i) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1}$ So $\overline{j_{i-1}} < \overline{j_i} \Rightarrow j_{i-1} < j_{i+1},\ \widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1,$
$\widehat{j_i} = \overline{j_{i-1}} + 1 = j_{i-1} + 1,\ \widehat{j_{i-1}} = \overline{j_i} = j_{i+1}.$

(ii) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1}$ So $\overline{j_{i-1}} \geq \overline{j_i} \Rightarrow j_{i-1} \geq j_{i+1},\ \widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1,$
$\widehat{j_i} = \overline{j_{i-1}} = j_{i-1},\ \widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 1.$

(iii) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i\ ,\ \overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i-1}} < \overline{j_i} \Rightarrow j_{i-1} < j_{i+1},\ \widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1,$
$\widehat{j_i} = \overline{j_{i-1}} = j_{i-1},\ \widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 1.$

(iv) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i,\ \overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i-1}} \geq \overline{j_i} \Rightarrow j_{i-1} \geq j_{i+1} - 1,\ \widehat{j_{i+1}} = \overline{j_{i+1}} = j_i\ ,$
$\widehat{j_i} = \overline{j_{i-1}} = j_{i-1},\ \widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 2.$

□

Corollary 3.7. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}})\cdots(s_{i+1,j_{i+1}})(s_{i,j_i})\cdots(s_{1,j_1})$ *and*
$s_{i+1}(s_iw) = (s_{n,\hat{j}_n})(s_{n-1,\widehat{j_{n-1}}})\cdots(s_{i+1,\widehat{j_{i+1}}})(s_{i,\widehat{j_i}})\cdots(s_{1,\widehat{j_1}})$*. Then*

$$s_{i+1}(s_iw) = \begin{cases} \widehat{j_{i+2}} = j_i + 2,\ \widehat{j_{i+1}} = j_{i+2},\ \widehat{j_i} = j_{i+1} & if \quad j_i < j_{i+1},\ j_{i+1} < j_{i+2} \\ \widehat{j_{i+2}} = j_i + 1,\ \widehat{j_{i+1}} = j_{i+2} - 1,\ \widehat{j_i} = j_{i+1} & if \quad j_i < j_{i+1},\ j_i + 1 \geq j_{i+2} \\ \widehat{j_{i+2}} = j_i + 1,\ \widehat{j_{i+1}} = j_{i+2},\ \widehat{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_i < j_{i+2} \\ \widehat{j_{i+2}} = j_i,\ \widehat{j_{i+1}} = j_{i+2} - 1,\ \widehat{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_i \geq j_{i+2} \\ \widehat{j_k} = j_k & if \quad k \neq i, i+1, i+2. \end{cases}$$

Proof. Let $\overline{w} = s_i w = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}})\cdots(s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}})\cdots(s_{1,\overline{j_1}})$. Then

$$s_{i+1}(\overline{w}) = \begin{cases} \widehat{j_{i+2}} = \overline{j_{i+1}} + 1,\ \widehat{j_{i+1}} = \overline{j_{i+2}} & if \quad \overline{j_{i+1}} < \overline{j_{i+2}} \\ \widehat{j_{i+2}} = \overline{j_{i+1}},\ \widehat{j_{i+1}} = \overline{j_{i+2}} - 1 & if \quad \overline{j_{i+1}} \geq \overline{j_{i+2}} \\ \widehat{j_k} = \overline{j_k} & if \quad k \neq i+1, i+2. \end{cases}$$

(i) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1}$ So $\overline{j_{i+1}} < \overline{j_{i+2}} \Rightarrow j_i + 1 < j_{i+2},\ \widehat{j_{i+2}} = \overline{j_{i+1}} + 1 = j_i + 2$, $\widehat{j_{i+1}} = \overline{j_{i+2}} = j_{i+2},\ \widehat{j_i} = \overline{j_i} = j_{i+1}$.

(ii) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1}$ So $\overline{j_{i+1}} \geq \overline{j_{i+2}} \Rightarrow j_i + 1 \geq j_{i+2}$, $\widehat{j_{i+2}} = \overline{j_{i+1}} = j_i + 1$, $\widehat{j_{i+1}} = \overline{j_{i+2}} - 1 = j_{i+2} - 1,\ \widehat{j_i} = \overline{j_i} = j_{i+1}$.

(iii) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i,\ \overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i+1}} < \overline{j_{i+2}} \Rightarrow j_i < j_{i+2}$, $\widehat{j_{i+2}} = \overline{j_{i+1}} + 1 = j_i + 1$, $\widehat{j_{i+1}} = \overline{j_{i+2}} = j_{i+2},\ \widehat{j_i} = \overline{j_i} - 1 = j_{i+1} - 1$.

(iv) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i,\ \overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i+1}} \geq \overline{j_{i+2}} \Rightarrow j_i \geq j_{i+2},\ \widehat{j_{i+2}} = \overline{j_{i+1}} = j_i,\ \widehat{j_{i+1}} = \overline{j_{i+2}} - 1 = j_{i+2} - 1,\ \widehat{j_i} = \overline{j_i} = j_{i+1} - 1$.

□

Using Lemma 3.3 and definitions of A^i and r_i operators, we can obtain the followings.

Lemma 3.8. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}})\cdots(s_{i+1,\, j_{i+1}})(s_{i,j_i})\cdots(s_{1,j_1})$. *Then*

$$A^i(\varepsilon^w) = \begin{cases} \varepsilon^{w_1} & if \quad j_i \geq j_{i+1} \\ 0 & if \quad j_i < j_{i+1}, \end{cases}$$

where $w_1 = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}})\cdots(s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}})\cdots(s_{1,\overline{j_1}})$ *with* $\overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ *and* $\overline{j_k} = j_k$ *if* $k \neq i, i+1$.

Lemma 3.9. $r_i(\varepsilon^{s_j}) = \begin{cases} \varepsilon^{s_{i-1}} - \varepsilon^{s_i} - \varepsilon^{s_{i+1}} & if \quad i = j \\ \varepsilon^{s_j} & if \quad i \neq j. \end{cases}$

The integral cohomology of SU_{n+1}/T is generated by Schubert classes indexed

$$W = \{s_{nj_n} s_{n-1,j_{n-1}} \dots s_{1j_1} : \quad j_i = 0 \quad \text{or} \quad i \leq j_i \leq n\}.$$

Let $x_i = \varepsilon^{r_i} \in H^2(SU_{n+1}/T, \mathbb{Z})$. We define an order between generators of the integral cohomology of SU_{n+1}/T. Since each element $\varepsilon^{s_{nj_n} s_{n-1,j_{n-1}} \cdots s_{ij_i} \cdots s_{1j_1}}$ can be represented by an n-tuple $(j_n - n + 1, j_{n-1} - (n-1) + 1, \dots, j_i - i + 1, \dots, j_1 - 1 + 1)$, we can define an order between n-tuples.

Definition 3.10. (Graded Inverse Lexicographic Order) Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha > \beta$ if $|\alpha| = \alpha_1 + \alpha_2 + \dots \alpha_n > |\beta| = \beta_1 + \beta_2 + \dots \beta_n$ or $|\alpha| = |\beta|$ and in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the right-most nonzero entry is positive. We will write $\varepsilon^{s_{nj_n} s_{n-1,j_{n-1}} \cdots s_{ij_i} \cdots s_{1j_1}} > \varepsilon^{s_{nk_n} s_{n-1,j_{k-1}} \cdots s_{ik_i} \cdots s_{1j_1}}$ if $(j_n - n + 1, j_{n-1} - (n-1) + 1, \dots, j_i - i - 1, \dots, j_1 - 1 + 1) > (k_n - n + 1, k_{n-1} - (n-1) + 1, \dots, k_i - i - 1, \dots, k_1 - 1 + 1)$.

Example 3.11. $\varepsilon^{s_{35} s_{23} s_{14}} > \varepsilon^{s_{35} s_{24} s_{13}}$ *since* $(3, 2, 4) > (3, 3, 3)$ *in graded inverse lexicographic order.*

We will try to find a quotient ring $\mathbb{Z}[x_1, x_2, \dots, x_n]/I$ which is isomorphic to $H^*(SU_{n+1}/T, \mathbb{Z})$. We also define an order between monomials as follows.

Definition 3.12. We say $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} > x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$ if $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n > |\beta| = \beta_1+\beta_2+\cdots+\beta_n$ or $|\alpha| = |\beta|$ and in the vector difference $\alpha-\beta \in \mathbb{Z}^n$ the left-most non-zero entry is negative.

Example 3.13. $x_1^4x_2^2x_3^3 < x_1^3x_2^3x_3^3$, *since* $(4,2,3)-(3,3,3) = (1,-1,0)$.

Lemma 3.14. $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\cdots s_{i\alpha_i}\cdots s_{1\alpha_1}}$ + lower terms.

Proof. To prove this, we use induction on degree of the monomials. By definition $x_i = \varepsilon^{s_i}$. Let us compute $x_ix_j = \varepsilon^{s_i}\varepsilon^{s_j}$. Here we may assume that $i \le j$. If $j-i > 1$, the inverse of s_is_j is s_is_j. Hence

$$P_{s_is_j}^{s_js_i} = r_jA^i(\varepsilon^{s_i}) = r_j(1) = 1$$

in the cup product. If $j = i+1$, the inverse of $s_{i+1}s_i$ is s_is_{i+1}. In this case

$$P_{s_i,s_{i+1}} = A^ir_{i+1}(\varepsilon^{s_i}) = A^i(\varepsilon^{s_i}) = \varepsilon^{\{\}} = 1.$$

If $i = j$, then we have to consider the word $s_{i,i+1}$. Its inverse $s_{i+1}s_i$ and

$$P_{s_is_i}^{s_{i,i+1}} = r_{i+1}A^i(\varepsilon^{s_i}) = r_{i+1}(1) = 1.$$

Now we have to show that $P_{s_is_j}^{s_ks_l} = 0$ if $\varepsilon^{s_ks_l} > \varepsilon^{s_js_i}$. By definition of cup product the coefficient of $\varepsilon^{s_ks_l}$ is not zero only if $s_i \to s_ks_l$ and $s_j \to s_ks_l$. However, this is possible only if $s_ks_l = s_js_i$ or $s_ks_l = s_{i,i+1}$ when $j = i+1$. Clearly $\varepsilon^{s_is_{i+1}} < \varepsilon^{s_{i+1}s_i}$. Hence $\varepsilon^{s_i}\varepsilon^{s_{i+1}} = \varepsilon^{s_{i+1}s_i}$ + lower terms and $\varepsilon^{s_i}\varepsilon^{s_j} = \varepsilon^{s_j}\varepsilon^{s_i}$ if $j-i > 1$. In the case $i = j$, we have to look elements s_is_k and s_ks_i. The inverse of s_ks_i is equal to s_ks_i itself if $k-i > 1$, hence

$$P_{s_is_j}^{s_ks_i} = A^kr_i(\varepsilon^{s_i}) = A^k(\varepsilon^{s_{i-1}} - \varepsilon^{s_i} + \varepsilon^{s_{i+1}}) = 0$$

since $k-i > 1$. Clearly $\varepsilon^{s_is_k} < \varepsilon^{s_is_{i+1}}$ if $k < i$. Hence $\varepsilon^{s_i}\varepsilon^{s_i} = \varepsilon^{s_is_{i+1}}$ + lower terms.

Assume that $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\cdots s_{i\alpha_i}\cdots s_{1\alpha_1}}$ + lower terms.

We have to show $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_i^{\alpha_i+1}\ldots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\cdots s_{i\alpha_i+1}\cdots s_{1\alpha_1}}$ + lower terms by Bruhat ordering.

$s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\ldots s_{i\alpha_i+1}\ldots s_{1\alpha_1} \to w'$ only if $w' = s_{n\overline{\alpha_n}}s_{n-1,\overline{\alpha_{n-1}}}\ldots s_{i\overline{\alpha_i}}\ldots s_{1\overline{\alpha_1}}$ where there exists an index j for which $\overline{\alpha_j} = \alpha_j+1$ and $\overline{\alpha_k} = \alpha_k$ if $k \ne j$.

By given ordering

$$w' = s_{n\overline{\alpha_n}}s_{n-1,\overline{\alpha_{n-1}}}\ldots s_{i\overline{\alpha_i}}\ldots s_{1\overline{\alpha_1}} > s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\ldots s_{i\alpha_i+1}\ldots s_{1\alpha_1}.$$

If $j > i$, then, by Algorithm 3.1, in w'^{-1}, we will not have a subsequence $s_{j-1}, s_{j-2}\ldots s_i$ after the elements s_j. Hence in the cup product before applying A^j we will not have the term ε^{s_j}. It means $P_{s_i,w}^{w'} = 0$.

If $j = i$, then, again by Algorithm 3.1, in w'^{-1} we will not have a subsequence $s_{j-1}, s_{j-2}\ldots s_i$ after the elements s_j. Hence in the cup product before applying A^j we will not have the term ε^{s_j}. It means $P_{s_i,w}^{w'} = 1$ if and only if $j > i$. □

Example 3.15. *Let* $l = 3$*,*

$x_1x_2x_3 = \varepsilon^{s_3s_2s_1}$ + lower terms.

$x_1^2x_2x_3 = \varepsilon^{s_3s_2s_{12}}$ + lower terms.

Then we have $\varepsilon^{s_3s_{23}s_1} > \varepsilon^{s_3s_2s_{12}} > \varepsilon^{s_{23}s_{12}} > \varepsilon^{s_3s_{13}} > \varepsilon^{s_2s_{13}}$*. Since the inverse of* $s_3s_{23}s_1$ *is* s_3s_{13} *and the inverse of* $s_3s_2s_1$ *is* s_{13}*,* $A_3r_1r_2r_3(\varepsilon^{s_1}) = A_3r_1(\varepsilon^{s_1}) = A_3(-\varepsilon^{s_1}+\varepsilon^{s_2}) = 0$*.*

Similarly, since the inverse of $s_3s_2s_{12}$ *is* s_2s_{13}*,* $A_2r_1r_2r_3(\varepsilon^{s_1}) = A_2r_1(\varepsilon^{s_1}) = A_2(-\varepsilon^{s_1}+\varepsilon^{s_2}) = 1$*.*

Before finding the quotient ring $\mathbb{Z}[x_1, \ldots, x_n]/I$, we give some information about ring $\Bbbk[x_1, \ldots, x_n]/I$ where $\Bbbk$ is a field. Fix a monomial ordering on $\Bbbk[x_1, \ldots, x_n]$. Let $f \in \Bbbk[x_1, \ldots, x_n]$. The leading monomial of f, denoted by $LM(f)$, is the highest degree monomial of f. The coefficient of $LM(f)$ is called leading coefficient of f and denoted by $LC(f)$. The leading term of f, $LT(f) = LC(f)LM(f)$.

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Define $LT(I) = \{LT(f) : f \in I\}$. Let $< LT(I) >$ be an ideal generated by $LT(I)$.

Proposition 3.16. *[9, Section 5.3, Propostions 1 and 4]*

(i) *Every $f \in \Bbbk[x_1, \ldots, x_n]$ is congruent modulo I to a unique polynomial r which is a $\Bbbk$-linear combination of the monomials in the complement of $< LT(I) >$.*

(ii) *The elements of $\{x^\alpha : x^\alpha \notin < LT(I) >\}$ are linearly independent modulo I.*

(iii) *$\Bbbk[x_1, \ldots, x_n]/I$ is isomorphic as a $\Bbbk-$vector space to*

$$S = \text{Span}\{x^\alpha : x^\alpha \notin < LT(I) >\}.$$

Theorem 3.17. *[9, Section 5.3, Theorem 6] Let $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be an ideal.*

(i) *The $\Bbbk$-vector space $\Bbbk[x_1, \ldots, x_n]/I$ is finite dimensional.*

(ii) *For each i, $1 \leq i \leq n$, there is a polynomial $f_i \in I$ such that $LM(f_i) = x_i^{m_i}$ for some positive integer m_i.*

Theorem 3.18. *$H^*(SU_{n+1}/T, \mathbb{Z})$ isomorphic to $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ where $LT(f_i) = x_i^{n-i+2}$ with respect to monomial order given by Definition 3.12.*

Proof. Let I be the ideal such that $H^*(SU_{n+1}/T, \mathbb{R}) \cong \mathbb{R}[\alpha_1, \alpha_2, \ldots, \alpha_n]/I$. Since we found one to one correspondence between length l elements of $H^*(SU_{n+1}/T, \mathbb{Z})$ and monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_1 + \alpha_2 + \cdots \alpha_n = l$ and for each i, $1 \leq i \leq n$, $\alpha_i \leq n - i + 1$, there should be a polynomial $f_i \in I$ such that $LT(f_i) = x_i^{n-i+2}$. □

Example 3.19. *Let $n = 3$. Then we have*
$\alpha_i \leq n - i + 1,\ i = 1, 2, 3;$
$\alpha_1 \leq 3,\ \alpha_2 \leq 2,\ \alpha_3 \leq 1.$
For $l = 1$; $x_1,\ x_2,\ x_3$; and
for $l = 2$; $x_1^2,\ x_1x_2,\ x_1x_3,\ x_2x_3,\ x_2^2$. So we must have a polynomial f_3 with $LM(f_3) = x_3^2$.
For $l = 3$; $x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2x_3,\ x_1x_2^2,\ x_2^2x_3$, so
we must have a polynomial f_2 with $LM(f_2) = x_2^3$.
For $l = 4$; $x_1^3x_2,\ x_1^3x_3,\ x_1^2x_2x_3,\ x_1^2x_2^2,\ x_1x_2^2x_3$, so
we must have a polynomial f_1 with $LM(f_1) = x_1^4$.

The complex dimension of SU_{n+1}/T is equal to $(n + 1)n/2$. So the highest element has length of $(n + 1)n/2$.

Since the unique highest element has length of $\frac{n(n+1)}{2}$, we now give the result about the multiplication of elements of length k and of length $\frac{n(n+1)}{2} - k$.

Theorem 3.20. *Let $A = \varepsilon^{s_{nj_n}s_{n-1,j_{n-1}}\cdots s_{1j_1}}$ be an element of length k and $B = \varepsilon^{s_{np_n}s_{n-1,p_{n-1}}\cdots s_{1p_1}}$ be an element of length $\frac{n(n+1)}{2} - k$. The corresponding polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ has leading monomials*
$x_1^{j_1-1+1}x_2^{j_2-2+1}\cdots x_i^{j_i-i+1}\cdots x_1^{j_n-n+1}$ and $x_1^{p_1-1+1}x_2^{p_1-2+1}\cdots x_1^{p_n-n+1}$, respectively. Then

$$A \cdot B = \begin{cases} \varepsilon^{s_{n,n}s_{n-1,n},\ldots,s_{in},\ldots,s_{1n}}, & if\ j_i + p_i + 1 = n + i; \\ 0, & if\ j_i + p_i + 1 \neq n + i. \end{cases}$$

Proof. The unique highest degree monomial in $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ is $x_1^n x_2^{n-1}\cdots x_i^{n-i+1}\cdots x_n$. The multiplication of leading monomials of corresponding monomials of A and B produce the monomial

$$x_1^{j_1+p_1}x_2^{j_2+p_2-2}\cdots x_i^{j_i+p_i-2i+2}\cdots x_n^{j_n+p_n-2n+2}.$$

If $j_i + p_i - 2i + 2 = n - i + 1 \rightarrow j_i + p_i + 1 = n + i$ for each i, $i \leq 1 \leq n$, then the multiplication gives the $x_1^n x_2^{n-1}\ldots x_n$. Since this monomial correspondence the element $\varepsilon^{s_{n,n}s_{n-1,n}\cdots s_{in}\cdots s_{1n}}$, $A \cdot B = \varepsilon^{s_{n,n}s_{n-1,n}\cdots s_{1n}}$. If $j_i + p_i + 1 \neq n + i$, then the leading monomial and the monomials of lower degree must reduce to zero modulo $< f_1, f_2, \ldots, f_n >$ in $\mathbb{k}[x_1, x_2, \ldots, x_n]$ when we apply the division algorithm. Hence $A \cdot B = 0$. □

Now we can give the whole computation of the quotient ring $\mathbb{Z}[x_1, x_2, x_3]/ < f_1, f_2, f_3 >$.

Example 3.21. *Let $x_1 = \varepsilon^{s_1}$, $x_2 = \varepsilon^{s_2}$, $x_3 = \varepsilon^{s_3}$.*

For $l = 2$, we have

$$\begin{aligned} x_2x_3 &= \varepsilon^{s_3s_2} + \varepsilon^{s_2s_3} \\ x_2^2 &= \varepsilon^{s_2s_3} + \varepsilon^{s_2s_1} \\ x_1x_3 &= \varepsilon^{s_3s_1} \\ x_1x_2 &= \varepsilon^{s_2s_1} + \varepsilon^{s_1s_2} \\ x_1^2 &= \varepsilon^{s_1s_2}, \end{aligned}$$

and

$$\begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_3 \\ x_1x_2 \\ x_1^2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2} \\ \varepsilon^{s_2s_3} \\ \varepsilon^{s_3s_1} \\ \varepsilon^{s_2s_1} \\ \varepsilon^{s_1s_2} \end{pmatrix} \quad and \quad \begin{pmatrix} \varepsilon^{s_3s_2} \\ \varepsilon^{s_2s_3} \\ \varepsilon^{s_3s_1} \\ \varepsilon^{s_2s_1} \\ \varepsilon^{s_1s_2} \end{pmatrix} = M^{-1} \begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_3 \\ x_1x_2 \\ x_1^2 \end{pmatrix}, \ where$$

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \ Then\ we\ have$$

$$\begin{aligned} \varepsilon^{s_3s_2} &= x_2x_3 - x_2^2 + x_1x_2 - x_1^2 \\ \varepsilon^{s_2s_3} &= x_2^2 - x_1x_2 + x_2^2 \end{aligned}$$

$$\begin{aligned} \varepsilon^{s_3 s_1} &= x_1 x_3 \\ \varepsilon^{s_2 s_1} &= x_1 x_2 - x_1^2 \\ \varepsilon^{s_1 s_2} &= x_1^2. \end{aligned}$$

Here we must have a relation involving x_3^2 and we have it as

$$x_3^2 = \varepsilon^{s_3 s_2} = x_2 x_3 - x_2^2 + x_1 x_2 - x_1^2.$$

For $l = 3$;

$$\begin{aligned} x_2^2 x_3 &= \varepsilon^{s_3 s_2 s_3} + \varepsilon^{s_3 s_2 s_1} + \varepsilon^{s_2 s_3 s_1} \\ x_1 x_2 x_3 &= \varepsilon^{s_3 s_2 s_1} + \varepsilon^{s_2 s_3 s_1} + \varepsilon^{s_3 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\ x_1 x_2^2 &= \varepsilon^{s_2 s_3 s_1} + \varepsilon^{s_2 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\ x_1^2 x_3 &= \varepsilon^{s_3 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\ x_1^2 x_2 &= \varepsilon^{s_2 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\ x_1^3 &= \varepsilon^{s_1 s_2 s_3} \end{aligned}$$

and

$$\begin{pmatrix} x_2^2 x_3 \\ x_1 x_2 x_3 \\ x_1 x_2^2 \\ x_1^2 x_3 \\ x_1^2 x_2 \\ x_1^3 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3 s_2 s_3} \\ \varepsilon^{s_3 s_2 s_1} \\ \varepsilon^{s_2 s_3 s_1} \\ \varepsilon^{s_3 s_1 s_2} \\ \varepsilon^{s_2 s_1 s_2} \\ \varepsilon^{s_1 s_2 s_3} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \varepsilon^{s_3 s_2 s_3} \\ \varepsilon^{s_3 s_2 s_1} \\ \varepsilon^{s_2 s_3 s_1} \\ \varepsilon^{s_3 s_1 s_2} \\ \varepsilon^{s_2 s_1 s_2} \\ \varepsilon^{s_1 s_2 s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_2^2 x_3 \\ x_1 x_2 x_3 \\ x_1 x_2^2 \\ x_1^2 x_3 \\ x_1^2 x_2 \\ x_1^3 \end{pmatrix}, \quad \textit{where}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \varepsilon^{s_3 s_2 s_3} &= x_2^2 x_3 - x_1 x_2 x_3 + x_1^2 x_3 \\ \varepsilon^{s_3 s_2 s_1} &= x_1 x_2 x_3 - x_1 x_2^2 - x_1^2 x_3 + x_1^2 x_2 \\ \varepsilon^{s_2 s_3 s_1} &= x_1 x_2^2 - x_1^2 x_2 \\ \varepsilon^{s_3 s_1 s_2} &= x_1^2 x_3 - x_1^3 \\ \varepsilon^{s_2 s_1 s_2} &= x_1^2 x_2 - x_1^3 \\ \varepsilon^{s_1 s_2 s_3} &= x_1^3. \end{aligned}$$

Here we must have a relation involving x_2^3 and we now we have it as

$$x_2^3 = 2\varepsilon^{s_2 s_3 s_1} = 2(x_1 x_2^2 - x_1^2 x_2).$$

For $l = 4$; we have

$$\begin{aligned}
x_1x_2^2x_3 &= \varepsilon^{s_3s_2s_3s_1} + \varepsilon^{s_3s_2s_1s_2} + 2\varepsilon^{s_2s_3s_1s_2} + 2\varepsilon^{s_3s_1s_2s_3}\\
x_1^2x_2x_3 &= \varepsilon^{s_3s_2s_1s_2} + \varepsilon^{s_2s_3s_1s_2} + \varepsilon^{s_3s_1s_2s_3} + \varepsilon^{s_2s_1s_2s_3}\\
x_1^2x_2^2 &= \varepsilon^{s_2s_3s_1s_2} + \varepsilon^{s_2s_1s_2s_3}\\
x_1^3x_3 &= \varepsilon^{s_3s_1s_2s_3}\\
x_1^3x_2 &= \varepsilon^{s_2s_1s_2s_3}
\end{aligned}$$

and

$$\begin{pmatrix} x_1x_2^2x_3\\ x_1^2x_2x_3\\ x_1^2x_2^2\\ x_1^3x_3\\ x_1^3x_2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1}\\ \varepsilon^{s_3s_2s_1s_2}\\ \varepsilon^{s_2s_3s_1s_2}\\ \varepsilon^{s_3s_1s_2s_3}\\ \varepsilon^{s_2s_1s_2s_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1}\\ \varepsilon^{s_3s_2s_1s_2}\\ \varepsilon^{s_2s_3s_1s_2}\\ \varepsilon^{s_3s_1s_2s_3}\\ \varepsilon^{s_2s_1s_2s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_1x_2^2x_3\\ x_1^2x_2x_3\\ x_1^2x_2^2\\ x_1^3x_3\\ x_1^3x_2 \end{pmatrix}, \quad \text{where}$$

$$M = \begin{pmatrix} 1 & 1 & 2 & 2 & 0\\ 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 & 2\\ 0 & 1 & -1 & -1 & 0\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\varepsilon^{s_3s_2s_3s_1} &= x_1x_2^2x_3 - x_1^2x_2x_3 - x_1^2x_2^2 - x_1^3x_3 + 2x_1^3x_2\\
\varepsilon^{s_3s_2s_1s_2} &= x_1^2x_2x_3 - x_1^2x_2^2 - x_1^3x_3\\
\varepsilon^{s_2s_3s_1s_2} &= x_1^2x_2^2 - x_1^3x_2\\
\varepsilon^{s_3s_1s_2s_3} &= x_1^3x_3\\
\varepsilon^{s_2s_1s_2s_3} &= x_1^3x_2.
\end{aligned}$$

We must have a relation involving x_1^4, which is $x_1x_1^3 = \varepsilon^{s_1}.\varepsilon^{s_1s_2s_3} = 0$.
For $l = 5$;

$$\begin{aligned}
x_1^2x_2^2x_3 &= \varepsilon^{s_3s_2s_3s_1s_2} + \varepsilon^{s_3s_2s_1s_2s_3} + \varepsilon^{s_2s_3s_1s_2s_3}\\
x_1^3x_2x_3 &= \varepsilon^{s_3s_2s_1s_2s_3} + \varepsilon^{s_2s_3s_1s_2s_3}\\
x_1^3x_2^2 &= \varepsilon^{s_2s_3s_1s_2s_3}
\end{aligned}$$

and

$$\begin{pmatrix} x_1^2x_2^2x_3\\ x_1^3x_2x_3\\ x_1^3x_2^2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1s_2}\\ \varepsilon^{s_3s_2s_1s_2s_3}\\ \varepsilon^{s_2s_3s_1s_2s_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1s_2}\\ \varepsilon^{s_3s_2s_1s_2s_3}\\ \varepsilon^{s_2s_3s_1s_2s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_1^2x_2^2x_3\\ x_1^3x_2x_3\\ x_1^3x_2^2 \end{pmatrix}, \quad \text{where}$$

$$M = \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1 \end{pmatrix}. \text{ So}$$

$$\begin{aligned}
\varepsilon^{s_3s_2s_3s_1s_2} &= x_1^2x_2^2x_3 - x_1^3x_2x_3\\
\varepsilon^{s_3s_2s_1s_2s_3} &= x_1^3x_2x_3 - x_1^3x_2^2\\
\varepsilon^{s_2s_3s_1s_2s_3} &= x_1^3x_2^2.
\end{aligned}$$

Hence we don't have any relation.

For $l = 6$*;*

$x_1^3x_2^2x_3 = \varepsilon^{s_3s_2s_3s_1s_2s_3}$ *and* $\varepsilon^{s_3s_2s_3s_1s_2s_3} = x_1^3x_2^2x_3$.

Now let us multiple elements with lengths of k and $6-k$*.*

First $M_0 = 1$ *and* $|\det(M_0)| = 1$.

$\underline{Degree \quad 1 * Degree \quad 5}$

Elements	*Leading Monomial in Polynomial Ring*
ε^{s_1}	x_1
ε^{s_2}	x_2
ε^{s_3}	x_3
$\varepsilon^{s_3s_{23}s_{12}}$	$x_1^2x_2^2x_3$
$\varepsilon^{s_3s_2s_{13}}$	$x_1^3x_2x_3$
$\varepsilon^{s_{23}s_{13}}$	$x_1^3x_2^2$

$\varepsilon^{s_3} * \varepsilon^{s_3s_{23}s_{12}}$	$x_1^2x_2^2x_3^2$	0
$\varepsilon^{s_3} * \varepsilon^{s_3s_2s_{13}}$	$x_1^3x_2x_3^2$	0
$\varepsilon^{s_3} * \varepsilon^{s_{23}s_{13}}$	$x_1^3x_2^2x_3$	1

$\varepsilon^{s_2} * \varepsilon^{s_3s_{23}s_{12}}$	$x_1^2x_2^3x_3$	0
$\varepsilon^{s_2} * \varepsilon^{s_3s_2s_{13}}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_2} * \varepsilon^{s_{23}s_{13}}$	$x_1^3x_2^3$	0

$\varepsilon^{s_1} * \varepsilon^{s_3s_{23}s_{12}}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_1} * \varepsilon^{s_3s_2s_{13}}$	$x_1^4x_2x_3$	0
$\varepsilon^{s_1} * \varepsilon^{s_{23}s_{13}}$	$x_1^4x_2^2$	0

Now we will calculate Reidemeister torsion of SU_4/T *by using above multiplication. From multiplication of the second cohomology, we have* $M_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$ *and* $|\det(M_2)| = 1$.

$\underline{Degree \quad 2 * Degree \quad 4}$

Elements	*Leading Monomial in Polynomial Ring*
$\varepsilon^{s_3 s_2}$	$x_2 x_3$
$\varepsilon^{s_{23}}$	x_2^2
$\varepsilon^{s_3 s_1}$	$x_1 x_3$
$\varepsilon^{s_2 s_1}$	$x_1 x_2$
$\varepsilon^{s_{12}}$	x_1^2
$\varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2 x_3$
$\varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^2$
$\varepsilon^{s_3 s_{13}}$	$x_1^3 x_3$
$\varepsilon^{s_2 s_{13}}$	$x_1^3 x_2$

$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^3 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_{13}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_2 s_{13}}$	$x_1^3 x_2^2 x_3$	1

$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^4 x_3$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^4$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_{13}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{23}} * \varepsilon^{s_2 s_{13}}$	$x_1^3 x_2^3$	0

$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_{23} s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_{13}}$	$x_1^4 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_2 s_{13}}$	$x_1^4 x_2 x_3$	0

$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_2 s_1} * \varepsilon^{s_{23} s_{12}}$	$x_1^3 x_2^3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_{13}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_2 s_{13}}$	$x_1^4 x_2^2$	0

$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_{23} s_{12}}$	$x_1^4 x_2^2$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_{13}}$	$x_1^5 x_3$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_2 s_{13}}$	$x_1^5 x_2$	0

To calculate Reidemeister torsion of SU_4/T we need multiplication of fourth cohomology bases elements and then we have $M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $|\det(M_4)| = 1$.

$\underline{Degree \quad 3 * Degree \quad 3}$

Elements	*Leading Monomial in Polynomial Ring*
$\varepsilon^{s_3 s_{23}}$	$x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_1}$	$x_2^2 x_3$
$\varepsilon^{s_{23} s_1}$	$x_1 x_2^2$
$\varepsilon^{s_3 s_{12}}$	$x_1^2 x_3$
$\varepsilon^{s_2 s_{12}}$	$x_1^2 x_2$
$\varepsilon^{s_{13}}$	x_1^3
$\varepsilon^{s_3 s_{23}}$	$x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_1}$	$x_1 x_2 x_3$
$\varepsilon^{s_{23} s_1}$	$x_1 x_2^2$
$\varepsilon^{s_3 s_{12}}$	$x_1^2 x_3$
$\varepsilon^{s_2 s_{12}}$	$x_1^2 x_2$
$\varepsilon^{s_{13}}$	x_1^3

$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_{23}}$	$x_2^4 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_2 s_1}$	$x_1 x_2^3 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_{23} s_1}$	$x_1 x_2^4 x_3$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_{12}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_2 s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_{13}}$	$x_1^3 x_2^2 x_3$	1

$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_3 s_{23}}$	$x_1 x_2^3 x_3^2$	0
$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_3 s_2 s_1}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_{23} s_1}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_3 s_{12}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_2 s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_3 s_2 s_1} * \varepsilon^{s_{13}}$	$x_1^4 x_2 x_3$	0

$\varepsilon^{s_{23} s_1} * \varepsilon^{s_3 s_{23}}$	$x_1 x_2^4 x_3$	0
$\varepsilon^{s_{23} s_1} * \varepsilon^{s_3 s_2 s_1}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_{23} s_1} * \varepsilon^{s_{23} s_1}$	$x_1^2 x_2^4$	0
$\varepsilon^{s_{23} s_1} * \varepsilon^{s_3 s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{23} s_1} * \varepsilon^{s_2 s_{12}}$	$x_1^3 x_2^3$	0
$\varepsilon^{s_{23} s_1} * \varepsilon^{s_{13}}$	$x_1^4 x_2^2$	0

$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_3 s_{23}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_3 s_2 s_1}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_{23} s_1}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_3 s_{12}}$	$x_1^4 x_3^2$	0
$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_2 s_{12}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_3 s_{12}} * \varepsilon^{s_{13}}$	$x_1^5 x_3$	0

$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_3 s_{23}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_3 s_2 s_1}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_{23} s_1}$	$x_1^3 x_2^3$	0
$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_3 s_{12}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_2 s_{12}}$	$x_1^4 x_2^2$	0
$\varepsilon^{s_2 s_{12}} * \varepsilon^{s_{13}}$	$x_1^5 x_2$	0

$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_{23}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_2 s_1}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_{23} s_1}$	$x_1^4 x_2^2$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_{12}}$	$x_1^5 x_3$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_2 s_{12}}$	$x_1^5 x_2$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_{13}}$	x_1^6	0

To calculate Reidemeister torsion of SU_4/T we need multiplication of sixth cohomology bases elements and then we have $M_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $|\det(M_6)| = 1$.

In general the matrix M_k represents the intersection pairing between the homology classes of degrees k and $(n+1)n-k$ with real coefficient. So in general $|\det(M_{\frac{n(n+1)}{2}})| = 1$. Hence the Reidemeister torsion of SU_4/T is 1 by the Reidmeister torsion formula for manifolds.

By Theorems 1.1, 3.18 and 3.20, we obtain the following result.

Theorem 3.22. *The Reidemeister torsion of SU_{n+1}/T is always 1 for any positive integer n with $n \geq 3$.*

Remark 3.23. We should note that we found this result by Schubert calculus. But, we choose any basis to define Reidemeister torsion. There are many bases for the Reidemeister torsion to be 1. Why we focus on this basis to compute the Reidemeister torsion is that we can use Schubert calculus and we have cup product formula in this algebra in terms of Schubert differential forms. Otherwise these computations are not easy. Also by Groebner techniques we can find the normal form of all elenents of Weyl group indexing our basis. So computations in this algebra is avaliable.

Remark 3.24. In our work, we consider flag manifold SU_{n+1}/T for $n \geq 3$. Then we consider the Schubert cells $\{\mathfrak{c}_p\}$ and the corresponding homology basis a $\{\mathfrak{h}_p\}$ associated to $\{\mathfrak{c}_p\}$. We caculated that $\mathrm{Tor}(C_*(\mathbf{K}), \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = 1$.

If we consider the same cell-decomposition but other homology basis $\{\mathfrak{h}'_p\}$ then by the change-base-formula (1.4), then we have

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}'_p\}_{p=0}^n) = \prod_{p=0}^{n} \left(\frac{1}{[\mathfrak{h}'_p, \mathfrak{h}_p]} \right)^{(-1)^p} \cdot \mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n).$$

Remark 3.25. In the presented paper $M = K/T$ is a flag manifold, where $K = SU_{n+1}$ and T is the maximal torus of K. Clearly, M is a smooth orientable even dimensional(complex) closed manifold. So there is Poincaré (or Hodge) duality. Therefore, we can apply Theorem 1.1 for $M = K/T$.

Acknowledgments

We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

Conflict of interest

The authors declare that they have no competing interests.

References

1. H. Basbaydar, *Calculation of Reidmesster torsion of flag manifolds of compact semi-simple Lie groups*, Master Thesis, Abant Izzet Baysal University, 2013.
2. J. M. Bismut and H. Gillet, C. Soulé, *Analytic torsion and holomorphic determinant bundles I. Bott-Chern forms and analytic torsion*, Commu. Math. Phys., **115** (1988), 49–78.
3. J. M. Bismut and F. Labourie, *Symplectic geometry and the Verlinde formulas*, Surv. Differ. Geom., **5** (1999), 97–311.
4. L. A. Bokut and L. S. Shiao, *Gröbner-Shirshov bases for coxeter groups*, Comm. Algebra, **29** (2001), 4305–4319.
5. A. S. Buch, A. Kresch, H. Tamvakis, *Gromov-Witten Invariants on Grassmannaians*, J. Amer. Math. Soc., **16** (2003), 901–915.
6. A. S. Buch, A. Kresch, H. Tamvakis, *Quantum Pieri rules for isotropic Grassmannians*, Invent. Math., **178** (2009), 345–405.
7. T. A. Chapman, *Hilbert cube manifolds and the invariance of Whitehead torsion*, Bull. Am. Math. Soc., **79** (1973), 52–56.
8. T. A. Chapman, *Topological invariance of Whitehead torsion*, Am. J. Math., **96** (1974), 488–497.
9. D. Cox, J. Little, D. O'Shea, *Ideals, Varieties and Algorithms*, Springer-Verleg, New York, 1992.
10. G. de Rham, *Reidemeister's torsion invariant and rotation of S^n*,, Differential Analysis, Bombay Colloq, (1964), 27–36.
11. W. Franz, *Über die Torsion einer Überdeckung*, J. Reine Angew. Math., **173** (1935), 245–254.
12. W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics, Vol. 1689, Springer-Verlag, Berlin, 1998.
13. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Willey Library Edition, New York, 1994.
14. R. C. Kirby and L. C. Siebenmann, *On triangulation of manifolds and Haupvermutung*, Bull. Am. Math. Soc., **75** (1969), 742–749.
15. B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. Math., **62** (1986), 187–237.
16. J. Milnor, *A duality theorem for Reidemeister torsion*, Ann. Math., **76** (1962), 137–147.

17. J. Milnor, *Whitehead torsion*, Bull. Am. Soc., **72** (1966), 358–426.
18. J. Milnor, *Infinite cyclic covers*, Topology of Manifolds, in Michigan, Prindle, Weber & Schmidt, Boston, (1967), 115–133.
19. C. Özel, *On the cohomology ring of the infinite flag manifold LG/T*, Turk. J. Math., **22** (1998), 415–448.
20. K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 102–109.
21. Y. Sözen, *On Reidemeister torsion of a symplectic complex*, Osaka J. Math., **45** (2008), 1–39.
22. Y. Sözen, *On Fubini-Study form and Reidemeister torsion*, Topol. Appl., **156** (2009), 951–955.
23. Y. Sözen, *Symplectic chain complex and Reidemeister torsion of compact manifolds*, Math. Scand., **111** (2012), 65–91.
24. Y. Sözen, *A note on Reidemeister torsion and period matrix of Riemann surfaces*, Math. Slovaca, (to appear).
25. H. Tamvakis, *Quantum cohomology of isotropic Grassmannians*, Geometric Methods in Algebra and Number Theory, Progress in Math, Birkhäuser, Boston, **235** (2005), 311–338.
26. H. Tamvakis, *Gromov-Witten invariants and quantum cohomology of Grassmannians*, Topics in Cohomological Studies of Algebraic Varieties, Trends in Math., Birkhäuser, Boston, (2005), 271–297.
27. E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys., **141** (1991), 153–209.
28. W. Zhe-Xian, *Introduction to Kač-Moody Algebra*, World Scientific, Singapore, 1991.