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Research article

Coupled common best proximity point theorems for nonlinear contractions in partially ordered metric spaces

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Abstract: In this paper, we first introduce the concept of mixed γ-proximally monotone property type mappings and investigate the existence of the coupled proximally coincidence point for such mappings in partially ordered complete metric spaces. Furthermore, we prove the existence and uniqueness of coupled common best proximity points. Our results extend, improve and generalize several known results in the literature.

Keywords: p-property; coupled coincidence point; coupled fixed point; coupled common fixed point; partially ordered set; mixed monotone mapping
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1. Introduction

The concept of coupled fixed point for partially ordered set was introduced by Guo and Lakshmikantham [6], to investigate the solutions of the initial value problems of ordinary differential equations with discontinuous right-hand sides. See also [1] for more information on partially ordered metric spaces. Bhaskar and Lakshmikantham [4] introduced the mixed monotone mappings and proved existence result for coupled fixed point for mappings satisfying the mixed monotone property, which gives the existence and uniqueness of solution for the periodic boundary value problems.

Later, many researchers derived results on coupled fixed point for various types of maps and studied it's applications in other branch of Mathematics. For more on coupled fixed point results, we refer the reader [3, 7, 11, 12, 14] and references therein. Recently, Kumam et al. [9] have introduced proximally coupled contractions and proved the existence and uniqueness of coupled best proximity point for mappings satisfying the proximally coupled contraction condition in a complete ordered metric space.

In [10], Lakshmikantham and Ćirić introduced the mixed γ-monotone mappings and have proved the theorems on coupled coincidence point and coupled common fixed point for nonlinear contractive mappings in partially ordered complete metric space. In [2], Abkar et al. introduced the notion of proximally γ-Meir-Keeler type mappings and proved existence and uniqueness results for coupled best proximity points for these type of mappings. For more theorems on coupled best proximity point, one can see [5, 8, 13] and references therein.

In this paper, by observing the ideas of [10], we introduce the concept of mixed γ-proximally monotone mappings, through this notion, we give theorems on coupled proximally coincidence point and coupled common best proximity point in partially ordered complete metric space.

2. Mathematical preliminaries

The following notions are used subsequently: Let M, N be two nonempty subsets of a metric space (X, d).

$$dist(M, N) = d(M, N) = inf\{d(a, b) : a \in M, b \in N\};$$
$$M_0 = \{a \in M : d(a, b') = dist(M, N) \text{ for some } b' \in N\};$$
$$N_0 = \{b \in N : d(a', b) = dist(M, N) \text{ for some } a' \in M\}.$$

First, we recall that if $(X, \leq)$ is partially ordered set and a function $\Gamma : X \to X$ is said to be non-decreasing if for every $a, b \in X$ with $a \leq b$ then $\Gamma(a) \leq \Gamma(b)$. In the same way, one can recall that definition of non-increasing mapping. Throughout this paper, it is understandable that $a \leq b$ if and only if $b \geq a$.

Here, we collect some definitions and results from [10].

Definition 2.1. [10] Let $(X, \leq)$ be a partially ordered set and $\Gamma : X \times X \to X$ and $\gamma : X \to X$. The mapping Γ has the mixed γ-monotone property if Γ is monotone γ-non-decreasing in its first argument and is monotone γ-non-increasing in its second argument, that is, for any $a, b \in X$,

$$a_1, a_2 \in X,\ \gamma(a_1) \leq \gamma(a_2) \Rightarrow \Gamma(a_1, b) \leq \Gamma(a_2, b)$$

and

$$b_1, b_2 \in X,\ \gamma(b_1) \leq \gamma(b_2) \Rightarrow \Gamma(a, b_1) \geq \Gamma(a, b_2).$$

Definition 2.2. [10] An element $(a, b) \in X \times X$ is called a coupled coincidence point of mappings $\Gamma : X \times X \to X$ and $\gamma : X \to X$ if

$$\Gamma(a, b) = \gamma(a),\ \Gamma(b, a) = \gamma(b).$$

Definition 2.3. [10] An element $(a, b) \in X \times X$ is called a coupled common fixed point of mappings $\Gamma : X \times X \to X$ and $\gamma : X \to X$ if

$$\Gamma(a, b) = \gamma(a) = a,\ \Gamma(b, a) = \gamma(b) = b.$$

Definition 2.4. [10] Let X be a non-empty set and $\Gamma : X \times X \to X$ and $\gamma : X \to X$. The mappings Γ and γ are commutative if

$$\gamma(\Gamma(a,b)) = \Gamma(\gamma(a),\gamma(b))$$

for all $a, b \in X$.

Definition 2.5. Any two elements a and b of a partially ordered set $(X, \preceq)$ are comparable when either $a \preceq b$ or $b \preceq a$.

The following results provide existence of coupled coincidence point and coupled common fixed point in partially ordered complete metric space.

Theorem 2.6. *[10] Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t+} \varphi(r) < t$ for each $t > 0$ and also suppose $\Gamma : X \times X \to X$ and $\gamma : X \to X$ are such that Γ has the mixed γ-monotone property and*

$$d(\Gamma(a,b),\Gamma(u,v)) \leq \varphi\Big(\frac{d(\gamma(a),\gamma(u)) + d(\gamma(b),\gamma(v))}{2}\Big)$$

for all $a, b, u, v \in X$ for which $\gamma(a) \preceq \gamma(u)$ and $\gamma(b) \succeq \gamma(v)$. Suppose $\Gamma(X \times X) \subseteq \gamma(X)$, γ is continuous and commutes with Γ and also suppose either

(a) Γ is continuous or
(b) X has the following property:
  (i) if a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preceq a$ for all n,
  (ii) if a non-increasing sequence $\{b_n\} \to b$, then $b_n \succeq b$ for all n.

If there exist $a_0, b_0 \in X$ such that

$$\gamma(a_0) \preceq \Gamma(a_0, b_0) \text{ and } \gamma(b_0) \succeq \Gamma(b_0, a_0),$$

then there exist $a, b \in X$ such that

$$\gamma(a) = \Gamma(a,b) \text{ and } \gamma(b) = \Gamma(b,a),$$

that is, Γ and γ have a coupled coincidence point.

Theorem 2.7. *[10] In addition to the hypotheses of Theorem 2.6, suppose that for every $(a,b), (a^*, b^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(\Gamma(u,v), \Gamma(v,u))$ is comparable to $(\Gamma(a,b), \Gamma(b,a))$ and $(\Gamma(a^*,b^*), \Gamma(b^*,a^*))$. Then Γ and γ have a unique coupled common fixed point, that is, there exists a unique $(a,b) \in X \times X$ such that*

$$a = \gamma(a) = \Gamma(a,b) \text{ and } b = \gamma(b) = \Gamma(b,a).$$

3. Main results

In this section, we extend some definitions for non-self mappings to the corresponding definitions in [10] and we extend the results to non-self mappings.

Definition 3.1. Let M, N be two nonempty subsets of a metric space (X, d) with partial order $\preceq$ and $\Gamma : M \times M \to N$ and $\gamma : M \to N$. The mapping Γ has mixed γ-proximally monotone property, if Γ is γ-proximally non-decreasing in first co-ordinate and is γ-proximally non-increasing in its second co-ordinate, that is, for any $a, b \in M$,

$$\begin{cases} d(a_1^*, \gamma(a_1)) = d(M, N) \\ d(a_2^*, \gamma(a_2)) = d(M, N) \\ a_1^* \preceq a_2^* \end{cases} \Rightarrow \Gamma(a_1, b) \preceq \Gamma(a_2, b),$$

and

$$\begin{cases} d(b_1^*, \gamma(b_1)) = d(M, N) \\ d(b_2^*, \gamma(b_2)) = d(M, N) \\ b_1^* \preceq b_2^* \end{cases} \Rightarrow \Gamma(a, b_2) \preceq \Gamma(a, b_1),$$

where $a_1, a_2, b_1, b_2, a_1^*, a_2^*, b_1^*, b_2^* \in M$.

Example 3.2. Let $X = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and define the partial order $(a_1, a_2) \preceq (b_1, b_2)$ as $a_1 \leq b_1$, $a_2 \leq b_2$, and we assume $M = \{(-1, 0), (0, 1)\}$ and $N = \{(0, -1), (1, 0)\}$. Then $d(M, N) = \sqrt{2}$. Now define $\Gamma : M \times M \to N$ by $\Gamma((a_1, a_2), (b_1, b_2)) = (a_2, a_1)$ and $\gamma : M \to N$ by $\gamma(a_1, a_2) = (a_2, a_1)$. Now we show that Γ has mixed γ-proximally monotone property through the following possibilities. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in M$. First we check Γ has γ-proximally non-decreasing in first co-ordinate,

$$\begin{cases} d((-1, 0), \gamma(-1, 0)) = \sqrt{2} \\ d((0, 1), \gamma(0, 1)) = \sqrt{2} \\ (-1, 0) \preceq (0, 1) \end{cases}$$

$\Rightarrow \Gamma((-1, 0), b) = (0, -1) \preceq \Gamma((0, 1), b) = (1, 0)$, and

$$\begin{cases} d((-1, 0), \gamma(-1, 0)) = \sqrt{2} \\ d((-1, 0), \gamma(-1, 0)) = \sqrt{2} \\ (-1, 0) \preceq (-1, 0) \end{cases}$$

$\Rightarrow \Gamma((-1, 0), b) = (0, -1) \preceq \Gamma((-1, 0), b) = (0, -1)$, and

$$\begin{cases} d((0, 1), \gamma(0, 1)) = \sqrt{2} \\ d((0, 1), \gamma(0, 1)) = \sqrt{2} \\ (0, 1) \preceq (0, 1) \end{cases}$$

$\Rightarrow \Gamma((0, 1), b) = (1, 0) \preceq \Gamma((0, 1), b) = (1, 0)$.

Now one can easily verify that Γ has γ-proximally non-increasing in second co-ordinate. Therefore, Γ has mixed γ-proximally monotone property.

Here, we introduce coupled proximally coincidence point and coupled common best proximity point.

Definition 3.3. Let M, N be two nonempty subsets of a metric space (X, d) and $\Gamma : M \times M \to N$ and $\gamma : M \to N$. An element $(a, b) \in M \times M$ is called coupled proximally coincidence point of mappings Γ and γ, if

$$\begin{cases} d(u, \Gamma(a, b)) = d(u, \gamma(a)) = d(M, N), \\ d(v, \Gamma(b, a)) = d(v, \gamma(b)) = d(M, N), \end{cases}$$

for some $u, v \in M$.

Definition 3.4. Let M, N be two nonempty subsets of a metric space (X, d) and $\Gamma : M \times M \to N$ and $\gamma : M \to N$. An element $(a, b) \in M \times M$ is called coupled common best proximity point of mappings Γ and γ, if

$$\begin{cases} d(a, \Gamma(a, b)) = d(a, \gamma(a)) = d(M, N), \\ d(b, \Gamma(b, a)) = d(b, \gamma(b)) = d(M, N). \end{cases}$$

We support our definition by the following example.

Example 3.5. Let $X = \mathbb{R}^2$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and consider $M = \{(0, b) : 0 \leq b \leq 1\}$ and $N = \{(1, b) : 0 \leq b \leq 1\}$. We define $\Gamma : M \times M \to N$ by $\Gamma((0, b_1), (0, b_2)) = (1, b_1 b_2)$ and $\gamma : M \to N$ by $\gamma(0, b) = (1, b/2)$. Then clearly the points $((0, 0), (0, 0)), ((0, 1/2), (0, 1/2)) \in M \times M$ are coupled proximally coincidence of Γ and γ. Moreover, the point $((0, 0), (0, 0)) \in M \times M$ is coupled common best proximity point of Γ and γ.

Here, we define the notion proximally commutative for two non-self mappings.

Definition 3.6. Let M, N be two nonempty subsets of a metric space (X, d) and $\Gamma : M \times M \to N$ and $\gamma : M \to N$. We say Γ and γ are proximally commutative if

$$\begin{cases} d(u, \Gamma(a, b)) = d(M, N), \\ d(v, \gamma(a)) = d(w, \gamma(b)) = d(M, N), \end{cases}$$

then $\gamma(u) = \Gamma(v, w)$, where $a, b, u, v, w \in M$.

Example 3.7. Let $X = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and we consider $M = \{(-1, 0), (0, 1)\}, N = \{(0, -1), (1, 0)\}$. Then $d(M, N) = \sqrt{2}$. Now define $\Gamma : M \times M \to N$ by $\Gamma((a_1, a_2), (b_1, b_2)) = (a_2, a_1)$ and $\gamma : M \to N$ by $\gamma(a, b) = (-a, -b)$. Now we justify proximally commutativity of Γ and γ, via following possibilities: If

$$\begin{cases} d((-1, 0), \Gamma((-1, 0), (0, 1))) = \sqrt{2}, \\ d((0, 1), \gamma((-1, 0))) = d((-1, 0), \gamma((0, 1))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((-1,0)) = (1,0) = \Gamma((0,1),(-1,0))$ and if

$$\begin{cases} d((0,1),\Gamma((0,1),(-1,0))) = \sqrt{2}, \\ d((-1,0),\gamma((0,1))) = d((0,1),\gamma((-1,0))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((0,1)) = (0,-1) = \Gamma((-1,0),(0,1))$ and if

$$\begin{cases} d((-1,0),\Gamma((-1,0),(-1,0))) = \sqrt{2}, \\ d((0,1),\gamma((-1,0))) = d((0,1),\gamma((-1,0))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((-1,0)) = (1,0) = \Gamma((0,1),(0,1))$ and if

$$\begin{cases} d((0,1),\Gamma((0,1),(0,1))) = \sqrt{2}, \\ d((-1,0),\gamma((0,1))) = d((-1,0),\gamma((0,1))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((0,1)) = (0,-1) = \Gamma((-1,0),(-1,0))$. Then Γ and γ are proximally commutative.

Definition 3.8. Let (M,N) be a pair of nonempty subsets of a metric space (X,d) with partial order $\leq$ and $M_0 \neq \emptyset$. The pair is said to have weak P-monotone property if for any $a_1, a_2 \in M_0$, and $b_1, b_2 \in N_0$,

$$\begin{cases} d(a_1,b_1) = d(M,N) \\ d(a_2,b_2) = d(M,N) \end{cases} \Rightarrow d(a_1,a_2) \leq d(b_1,b_2),$$

furthermore, $b_1 \geq b_2$ implies $a_1 \geq a_2$.

Definition 3.9. Let (X,d) be a metric space and let A be a subset of X. Then the closure of A, denoted by $\bar{A}$, is the union of A and the set of all its limit points.

We use the following notions in our proof: Let (X,d) be a metric space. Let M,N be closed subsets of X and also suppose $\Gamma : M \times M \to N$ and $\gamma : M \to N$. First we define $P_{M_0}b = \{a \in M : d(a,b) = d(M,N)\}$. Clearly, $P_{M_0}b \subseteq M_0$. Then, we can set

$$P_{M_0} : \Gamma(\bar{M}_0 \times \bar{M}_0) \to M_0 \text{ and } P_{M_0} : \gamma(\bar{M}_0) \to M_0$$

by $P_{M_0}b = \{a \in M : d(a,b) = d(M,N)\}$.

Now, we prove our main result:

Theorem 3.10. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume there is a function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t+} \varphi(r) < t$ for each $t > 0$. Let M,N be closed subsets of X and the pair (M,N) has weak P-monotone property and also suppose $\Gamma : M \times M \to N$ and $\gamma : M \to N$ are such that Γ has mixed γ-proximally monotone property and*

$$d(\Gamma(a,b),\Gamma(u,v)) \leq \varphi\Big(\frac{d(v_1,v_2) + d(w_1,w_2)}{2}\Big) \tag{3.1}$$

provided that

$$\begin{cases} d(v_1,\gamma(a)) = d(w_1,\gamma(b)) = d(M,N), \\ d(v_2,\gamma(u)) = d(w_2,\gamma(v)) = d(M,N), \end{cases}$$

for all $a, b, u, v, v_1, v_2, w_1, w_2 \in M$ *for which* $v_1 \preceq v_2$ *and* $w_1 \succeq w_2$. *Suppose* $\Gamma(\bar{M}_0 \times \bar{M}_0) \subseteq \gamma(\bar{M}_0)$, *where* $\bar{M}_0$ *is closure of* $M_0, \Gamma(M_0 \times M_0) \subseteq \gamma(M_0) \subseteq N_0$, γ *is continuous and proximally commutes with* Γ *and also suppose either*

(a) Γ *is continuous or*

(b) $\bar{M}_0$ *has the following property :*

  (i) *if a non-decreasing sequence* $\{a_n\} \to a$, *then* $a_n \preceq a$ *for all* n,

  (ii) *if a non-increasing sequence* $\{b_n\} \to b$, *then* $b_n \succeq b$ *for all* n.

If there exist $a_0, b_0 \in M_0$ *such that*

$$\gamma(a_0) \preceq \Gamma(a_0, b_0) \text{ and } \gamma(b_0) \succeq \Gamma(b_0, a_0),$$

then there exist $a, b \in M_0$ *such that*

$$\begin{cases} d(u, \Gamma(a, b)) = d(u, \gamma(a)) = d(M, N), \\ d(v, \Gamma(b, a)) = d(v, \gamma(b)) = d(M, N), \end{cases} \quad for\ some\ u, v \in M_0,$$

that is, Γ *and* γ *have a coupled proximally coincidence point.*

Proof. First we assume that (a) Γ is continuous. Now we prove that N_0 is closed. Let $\{b_n\}$ be a sequence in N_0 such that $b_n \to q \in N$. Since $b_n \in N_0$, there exists $a_n \in M_0$ such that $d(a_n, b_n) = d(M, N)$, for each n. Because of weak P-monotone property, we have

$$d(b_n, b_m) \to 0 \Rightarrow d(a_n, a_m) \to 0,$$

as $n, m \to \infty$. Then $\{a_n\}$ is a Cauchy sequence and converges to a point $p \in M$. By the continuity of the metric d, we get $d(p, q) = d(M, N)$. This implies that $q \in N_0$. Therefore N_0 is closed.

Now we claim that $\Gamma(\bar{M}_0 \times \bar{M}_0) \subseteq N_0$. From the hypothesis, we know that $\Gamma(M_0 \times M_0) \subseteq N_0$. If $a, b \in \bar{M}_0 \backslash M_0$, then there exist sequences $\{a_n\}, \{b_n\} \subseteq M_0$ such that $a_n \to a, b_n \to b$. Since Γ is continuous and N_0 is closed,

$$\Gamma(a, b) = \lim_{n \to \infty} \Gamma(a_n, b_n) \in N_0.$$

Similarly we can prove that $\gamma(\bar{M}_0) \subseteq N_0$.
Define

$$P_{M_0} : \Gamma(\bar{M}_0 \times \bar{M}_0) \to M_0 \text{ and } P_{M_0} : \gamma(\bar{M}_0) \to M_0$$

by $P_{M_0} b = \{a \in M : d(a, b) = d(M, N)\}$. Because of weak P-monotone property, the mapping P_{M_0} is a single valued function. Take $a, b, u, v \in \bar{M}_0$ such that $P_{M_0}\gamma(a) \preceq P_{M_0}\gamma(u)$, $P_{M_0}\gamma(b) \succeq P_{M_0}\gamma(v)$, and then we know that

$$\begin{cases} d(P_{M_0}\gamma(a), \gamma(a)) = d(P_{M_0}\gamma(b), \gamma(b)) = d(M, N), \\ d(P_{M_0}\gamma(u), \gamma(u)) = d(P_{M_0}\gamma(v), \gamma(v)) = d(M, N). \end{cases}$$

From the condition (3.1) of Γ, we obtain

$$\begin{aligned} d(P_{M_0}\Gamma(a, b), & P_{M_0}\Gamma(u, v)) \\ & \leq d(\Gamma(a, b), \Gamma(u, v)) \end{aligned}$$

$$\leq\varphi\Big(\frac{d(P_{M_0}\gamma(a),P_{M_0}\gamma(u))+d(P_{M_0}\gamma(b),P_{M_0}\gamma(v))}{2}\Big).$$

Now we prove that $P_{M_0}\Gamma(\bar{M}_0\times\bar{M}_0)\subseteq P_{M_0}\gamma(\bar{M}_0)$.

For $a\in P_{M_0}\Gamma(\bar{M}_0\times\bar{M}_0)$, there exists $(u,v)\in\bar{M}_0\times\bar{M}_0$ such that $P_{M_0}\Gamma(u,v)=a$. Since $\Gamma(\bar{M}_0\times\bar{M}_0)\subseteq\gamma(\bar{M}_0)$, there exists $a\in\bar{M}_0$, such that $\Gamma(u,v)=\gamma(a)$, which implies that $P_{M_0}\gamma(a)=a$. Therefore $a\in P_{M_0}\gamma(\bar{M}_0)$.

Let $a_n,b_n,a,b\in\bar{M}_0$, with $a_n\to a$, $b_n\to b$. Since Γ is continuous, as $n\to\infty$, we get $\Gamma(a_n,b_n)\to\Gamma(a,b)$, and $d(\Gamma(a_n,b_n),\Gamma(a,b))\to 0$, which implies that $d(P_{M_0}\Gamma(a_n,b_n),P_{M_0}\Gamma(a,b))\to 0$. Therefore $P_{M_0}\Gamma(a_n,b_n)\to P_{M_0}\Gamma(a,b)$. So $P_{M_0}\Gamma$ is continuous.

Since γ is continuous, as $n\to+\infty$, we get $\gamma(a_n)\to\gamma(a)$, and $d(\gamma(a_n),\gamma(a))\to 0$, which implies that $d(P_{M_0}\gamma(a_n),P_{M_0}\gamma(a))\to 0$. Therefore $P_{M_0}\gamma(a_n)\to P_{M_0}\gamma(a)$. So $P_{M_0}\gamma$ is continuous.

For any $a,b,u,v\in\bar{M}_0$, with $P_{M_0}\gamma(u)\leq P_{M_0}\gamma(a)$, $P_{M_0}\gamma(b)\geq P_{M_0}\gamma(v)$. Since Γ has mixed γ-proximally monotone property, we have

$$\begin{cases}d(P_{M_0}\gamma(u),\gamma(u))=d(M,N),\\ d(P_{M_0}\gamma(a),\gamma(a))=d(M,N),\end{cases}$$

which implies that $\Gamma(u,q)\preceq\Gamma(a,q),\forall q\in\bar{M}_0$ and

$$\begin{cases}d(P_{M_0}\gamma(v),\gamma(v))=d(M,N),\\ d(P_{M_0}\gamma(b),\gamma(b))=d(M,N),\end{cases}$$

which implies that $\Gamma(p,b)\preceq\Gamma(p,v),\forall p\in\bar{M}_0$. Now since the pair (M,N) has the weak P-monotone property, we have

$$\begin{cases}d(P_{M_0}\Gamma(a,b),\Gamma(a,b))=d(M,N),\\ d(P_{M_0}\Gamma(u,b),\Gamma(u,b))=d(M,N),\end{cases}$$

which implies that $P_{M_0}\Gamma(u,b)\preceq P_{M_0}\Gamma(a,b)$. Similarly, we get

$$\begin{cases}d(P_{M_0}\Gamma(a,b),\Gamma(a,b))=d(M,N),\\ d(P_{M_0}\Gamma(a,v),\Gamma(a,v))=d(M,N),\end{cases}$$

which implies that $P_{M_0}\Gamma(a,b)\preceq P_{M_0}\Gamma(a,v)$. Therefore $P_{M_0}\Gamma$ has mixed $P_{M_0}\gamma$-monotone property.

Now we show that $P_{M_0}\gamma$ commutes with $P_{M_0}\Gamma$. Since Γ and γ are proximally commutes, we have, for $a,b\in\bar{M}_0$,

$$\begin{cases}d(P_{M_0}\Gamma(a,b),\Gamma(a,b))=d(M,N),\\ d(P_{M_0}\gamma(a),\gamma(a))=d(M,N),\\ d(P_{M_0}\gamma(b),\gamma(b))=d(M,N),\end{cases}$$

which implies that $\gamma(P_{M_0}\Gamma(a,b))=\Gamma(P_{M_0}\gamma(a),P_{M_0}\gamma(b))$. Then $P_{M_0}\gamma(P_{M_0}\Gamma(a,b))=P_{M_0}\Gamma(P_{M_0}\gamma(a),P_{M_0}\gamma(b))$.

From the hypothesis, if $a_0,b_0\in M_0$, from

$$\begin{cases}d(P_{M_0}\Gamma(a_0,b_0),\Gamma(a_0,b_0))=d(M,N),\\ d(P_{M_0}\gamma(a_0),\gamma(a_0)=d(M,N),\end{cases}$$

by weak P-monotone property, we get $P_{M_0}\gamma(a_0) \leq P_{M_0}\Gamma(a_0, b_0)$. Similarly, from

$$\begin{cases} d(P_{M_0}\Gamma(b_0, a_0), \Gamma(b_0, a_0)) = d(M, N), \\ d(P_{M_0}\gamma(b_0), \gamma(b_0)) = d(M, N), \end{cases}$$

by weak P-monotone property, we get $P_{M_0}\gamma(b_0) \geq P_{M_0}\Gamma(b_0, a_0)$.

Finally, the mappings $P_{M_0} : \Gamma(\bar{M}_0 \times \bar{M}_0) \to M_0$ and $P_{M_0} : \gamma(\bar{M}_0) \to M_0$ satisfy all the requirements of Theorem 2.6. Therefore, there exist $a, b \in M_0$ such that $P_{M_0}\gamma(a) = P_{M_0}\Gamma(a, b) = a^*$(say) and $P_{M_0}\gamma(b) = P_{M_0}\Gamma(b, a) = b^*$(say). Now from the fact that

$$\begin{cases} d(P_{M_0}\Gamma(a, b), \Gamma(a, b)) = d(P_{M_0}\gamma(a), \gamma(a)) = d(M, N), \\ d(P_{M_0}\Gamma(b, a), \Gamma(b, a)) = d(P_{M_0}\gamma(b), \gamma(b)) = d(M, N), \end{cases}$$

we get

$$\begin{cases} d(a^*, \Gamma(a, b)) = d(a^*, \gamma(a)) = d(M, N), \\ d(b^*, \Gamma(b, a)) = d(b^*, \gamma(b)) = d(M, N), \end{cases}$$

that is, Γ and γ have a coupled proximally coincidence point.

If (b) holds, the result is the same as above without proving the continuity of $P_{M_0}\Gamma$. □

Here, we illustrate the above theorem:

Example 3.11. Let $X = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and define the partial order $(a_1, a_2) \leq (b_1, b_2)$ as $a_1 \leq b_1, a_2 \leq b_2$. We consider $M = \{(-1, 0), (0, 1)\}, N = \{(0, -1), (1, 0)\}$. Clearly, $M_0 = M$, $N_0 = N$. Also $d(M, N) = \sqrt{2}$. Now define $\Gamma : M \times M \to N$ by $\Gamma\big((a_1, a_2), (b_1, b_2)\big) = (b_2, b_1)$ and $\gamma : M \to N$ by $\gamma(a, b) = (-a, -b)$. Now we justify proximally commutativity of Γ and γ, via following possibilities: If

$$\begin{cases} d((0, 1), \Gamma((-1, 0), (0, 1))) = \sqrt{2}, \\ d((0, 1), \gamma((-1, 0))) = d((-1, 0), \gamma((0, 1))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((0, 1)) = (0, -1) = \Gamma((0, 1), (-1, 0))$ and if

$$\begin{cases} d((-1, 0), \Gamma((0, 1), (-1, 0))) = \sqrt{2}, \\ d((-1, 0), \gamma((0, 1))) = d((0, 1), \gamma((-1, 0))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((-1, 0)) = (1, 0) = \Gamma((-1, 0), (0, 1))$ and if

$$\begin{cases} d((-1, 0), \Gamma((-1, 0), (-1, 0))) = \sqrt{2}, \\ d((0, 1), \gamma((-1, 0))) = d((0, 1), \gamma((-1, 0))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((-1, 0)) = (1, 0) = \Gamma((0, 1), (0, 1))$ and if

$$\begin{cases} d((0, 1), \Gamma((0, 1), (0, 1))) = \sqrt{2}, \\ d((-1, 0), \gamma((0, 1))) = d((-1, 0), \gamma((0, 1))) = \sqrt{2}, \end{cases}$$

then we get $\gamma((0,1)) = (0,-1) = \Gamma((-1,0),(-1,0))$. Then Γ and γ are proximally commutative. Now, we show that Γ has mixed γ-proximally monotone property through the following possibilities. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in M$. First we check Γ has γ-proximally non-increasing in second co-ordinate,

$$\begin{cases} d((-1,0),\gamma(0,1)) = \sqrt{2} \\ d((0,1),\gamma(-1,0)) = \sqrt{2} \\ (-1,0) \preceq (0,1) \end{cases}$$

$\Rightarrow \Gamma(a,(-1,0)) = (0,-1) \preceq \Gamma(a,(0,1)) = (1,0)$, and

$$\begin{cases} d((0,1),\gamma(-1,0)) = \sqrt{2} \\ d((0,1),\gamma(-1,0)) = \sqrt{2} \\ (0,1) \preceq (0,1) \end{cases}$$

$\Rightarrow \Gamma(a,(-1,0)) = (0,-1) = \Gamma(a,(-1,0)) = (0,-1)$, and

$$\begin{cases} d((-1,0),\gamma(0,1)) = \sqrt{2} \\ d((-1,0),\gamma(0,1)) = \sqrt{2} \\ (-1,0) \preceq (-1,0) \end{cases}$$

$\Rightarrow \Gamma(a,(0,1)) = (1,0) = \Gamma(a,(0,1)) = (1,0)$.

Now one can easily verify that Γ has γ-proximally non-decreasing in first co-ordinate. Therefore, Γ has mixed γ-proximally monotone property. And, clearly, we have $\Gamma(\bar{M_0} \times \bar{M_0}) \subseteq \gamma(\bar{M_0}) \subseteq \bar{N_0}$. If we choose $(-1,0),(0,1) \in M_0$, then we get

$$\gamma(-1,0) \preceq \Gamma((-1,0),(0,1)) \ and \ \gamma(0,1) \succeq \Gamma((0,1),(-1,0)).$$

Then by Theorem 3.10, there exist $(-1,0),(0,1) \in M_0$ such that

$$\begin{cases} d((0,1),\Gamma((-1,0),(0,1))) = d((0,1),\gamma(-1,0)) = d(M,N), \\ d((-1,0),\Gamma((0,1),(-1,0))) = d((-1,0),\gamma((0,1))) = d(M,N), \end{cases} \quad for\ some\ (0,1),(-1,0) \in M_0.$$

The following example shows that necessity of the assumption (i) if there exist $a_0, b_0 \in M_0$ such that

$$\gamma(a_0) \preceq \Gamma(a_0,b_0) \ and \ \gamma(b_0) \succeq \Gamma(b_0,a_0)$$

in Theorem 3.10.

Example 3.12. Let $X = \mathbb{R}^2$ with usual metric

$$d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$$

and define the partial order $(a_1,a_2) \preceq (b_1,b_2)$ as $a_1 \leq b_1$, $a_2 \leq b_2$. Consider $M = \{(0,b) : b \in \mathbb{R}\}$ and $N = \{(1,b) : b \in \mathbb{R}\}$. Then $M_0 = M$, $N_0 = N$. We define $\Gamma : M \times M \to N$ by $\Gamma((0,b_1),(0,b_2)) = (1, b_1 + 1/2)$ and $\gamma : M \to N$ by $\gamma(0,b) = (1,b)$. First, we justify the mapping Γ has mixed γ-proximally monotone property. Let $(0, a)$, $(0, b) \in M$. Here, we check Γ has γ-proximally non-decreasing in first co-ordinate, for $a_1 \leq a_2$,

$$\begin{cases} d((0, a_1), \gamma(0, a_1)) = 1 \\ d((0, a_2), \gamma(0, a_2)) = 1 \\ (0, a_1) \preceq (0, a_2) \end{cases}$$

$\Rightarrow \Gamma((0, a_1), (0, b)) = (1, a_1 + 1/2) \preceq \Gamma((0, a_2), (0, b)) = (1, a_2 + 1/2)$. Now, we claim Γ has γ-proximally non-increasing in second co-ordinate, for $b_1 \leq b_2$,

$$\begin{cases} d((0, b_1), \gamma(0, b_1)) = 1 \\ d((0, b_2), \gamma(0, b_2)) = 1 \\ (0, b_1) \preceq (0, b_2) \end{cases}$$

$\Rightarrow \Gamma((0, a), (0, b_2)) = (1, a + 1/2) = \Gamma((0, a), (0, b_1))$. Therefore, Γ has mixed γ-proximally monotone property. And, we show γ is proximally commute with Γ. For,

$$\begin{cases} d((0, b + 1/2), \Gamma((0, a), (0, b))) = 1, \\ d((0, a), \gamma((0, a))) = d((0, b), \gamma((0, b))) = 1. \end{cases}$$

Then, we have $\gamma(0, b + 1/2) = (1, b + 1/2)$ and $\Gamma((0, a), (0, b)) = (1, b + 1/2)$.

And, clearly, we have $\Gamma(\bar{M}_0 \times \bar{M}_0) \subseteq \gamma(\bar{M}_0) \subseteq \bar{N}_0$.

But, there is no points $(0, a_0)$, $(0, b_0) \in M_0$ such that

$$\gamma(0, a_0) \preceq \Gamma((0, a_0), (0, b_0)) \text{ and } \gamma(0, b_0) \succeq \Gamma((0, b_0), (0, a_0)).$$

Hence $\gamma(0, b_0) = (1, b_0) \prec (1, b_0 + 1/2) = \Gamma((0, b_0), (0, a_0))$.

So we cannot apply Theorem 3.10 and there is no coupled proximally coincidence point in M.

The following example shows that necessity of the assumptions (i) the mapping Γ has mixed $\gamma-$proximally monotone property, (ii) $\Gamma(\bar{M}_0 \times \bar{M}_0) \subseteq \gamma(\bar{M}_0)$ in Theorem 3.10.

Example 3.13. Let $X = \mathbb{R}^2$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and define the partial order $(a_1, a_2) \preceq (b_1, b_2)$ as $a_1 \leq b_1$, $a_2 \leq b_2$. Consider $M = \{(0, b) : 0 \leq b \leq 1\}$ and $N = \{(1, b) : 0 \leq b \leq 3\}$. We define $\Gamma : M \times M \to N$ by $\Gamma((0, b_1), (0, b_2)) = (1, b_1 + b_2 + 1)$ and $\gamma : M \to N$ by $\gamma(0, b) = (1, b)$. First, we justify the mapping γ is proximally commutes with Γ. There is only one possibility that

$$\begin{cases} d((0, 1), \Gamma((0, 0), (0, 0)) = d(M, N), \\ d((0, 0), \gamma(0, 0)) = d((0, 0), \gamma(0, 0)) = d(M, N). \end{cases}$$

Also, $\gamma(0, 1) = (1, 1) = \Gamma((0, 0), (0, 0))$. We prove that Γ is not γ-proximally non-increasing in its second co-ordinate. If we choose $(0, 1/4), (0, 1/2)$, we have

$$\begin{cases} d((0, 1/4), \gamma(0, 1/4)) = d(M, N) \\ d((0, 1/2), \gamma(0, 1/2)) = d(M, N) \\ (0, 1/4) \preceq (0, 1/2). \end{cases}$$

But, for any $(0, a)$ in M, we get $\Gamma((0, a), (0, 1/4)) = (1, a + \frac{1}{4} + 1) \prec (1, a + \frac{1}{2} + 1) = \Gamma((0, a), (0, 1/2))$. And, if we choose $((0, 1), (0, 1)) \in \bar{M}_0 \times \bar{M}_0$, then $\Gamma((0, 1), (0, 1)) = (1, 3) \notin \gamma(\bar{M}_0)$. So, we cannot apply Theorem 3.10 and there is no coupled proximally coincidence point in M.

The following example shows that necessity of the assumptions (i) the mapping γ is proximally commutes with Γ, (ii) $\Gamma(\bar{M}_0 \times \bar{M}_0) \subseteq \gamma(\bar{M}_0)$ in Theorem 3.10.

Example 3.14. Let $X = \mathbb{R}^2$ with usual metric

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

and define the partial order $(a_1, a_2) \leq (b_1, b_2)$ as $a_1 \leq b_1$, $a_2 \leq b_2$. Consider $M = \{(0, b) : 1/2 \leq b < +\infty\}$ and $N = \{(1, b) : b \in \mathbb{R}\}$. We define $\Gamma : M \times M \to N$ by $\Gamma((0, b_1), (0, b_2)) = (1, b_1 - b_2 + 1/2)$ and $\gamma : M \to N$ by $\gamma(0, b) = (1, b + 1/4)$. First, we justify the mapping Γ has mixed γ-proximally monotone property. Let $(0, a)$, $(0, b) \in M$. Here, we check Γ has γ-proximally non-decreasing in first co-ordinate, for $a_1 \leq a_2$,

$$\begin{cases} d((0, a_1 + 1/4), \gamma(0, a_1)) = 1 \\ d((0, a_2 + 1/4), \gamma(0, a_2)) = 1 \\ (0, a_1 + 1/4) \leq (0, a_2 + 1/4) \end{cases}$$

$\Rightarrow \Gamma((0, a_1), (0, b)) = (1, a_1 - b + 1/2) \leq \Gamma((0, a_2), (0, b)) = (1, a_2 - b + 1/2)$. Now, we claim Γ has γ-proximally non-increasing in second co-ordinate, for $b_1 \leq b_2$,

$$\begin{cases} d((0, b_1 + 1/4), \gamma(0, b_1)) = 1 \\ d((0, b_2 + 1/4), \gamma(0, b_2)) = 1 \\ (0, b_1 + 1/4) \leq (0, b_2 + 1/4) \end{cases}$$

$\Rightarrow \Gamma((0, a), (0, b_2)) = (1, a - b_2 + 1/2) \leq (1, a - b_1 + 1/2) = \Gamma((0, a), (0, b_1))$. Therefore, Γ has mixed γ-proximally monotone property. And, we show γ is not proximally commute with Γ. For,

$$\begin{cases} d((0, 1/2), \Gamma((0, 1/2), (0, 1/2))) = 1, \\ d((0, 3/4), \gamma((0, 1/2))) = d((0, 3/4), \gamma((0, 1/2))) = 1. \end{cases}$$

But we have $\gamma(0, 1/2) = (1, 3/4)$ and $\Gamma((0, 3/4), (0, 3/4)) = (1, 1/2)$.
If we choose $((0, 1/2), (0, 1/2)) \in \bar{M}_0 \times \bar{M}_0$, then $\Gamma((0, 1/2), (0, 1/2)) = (1, 1/2) \notin \gamma(\bar{M}_0)$.

So we cannot apply Theorem 3.10 and there is no coupled proximally coincidence point in M.

Corollary 3.15. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let M, N be subsets of X and the pair (M, N) have weak P-monotone property and also suppose $\Gamma : M \times M \to N$ and $\gamma : M \to N$ are such that Γ has mixed γ-monotone property and*

$$d(\Gamma(a, b), \Gamma(u, v)) \leq \frac{k}{2}\Big(d(v_1, v_2) + d(w_1, w_2)\Big) \tag{3.2}$$

provided that

$$\begin{cases} d(v_1, \gamma(a)) = d(w_1, \gamma(b)) = d(M, N), \\ d(v_2, \gamma(u)) = d(w_2, \gamma(v)) = d(M, N), \end{cases}$$

for all $a, b, u, v, v_1, v_2, w_1, w_2 \in M$ *and* $k \in [0, 1)$ *for which* $v_1 \preceq v_2$ *and* $w_1 \succeq w_2$. *Suppose* $\Gamma(\bar{M_0} \times \bar{M_0}) \subseteq \gamma(\bar{M_0})$, *where* $\bar{M_0}$ *is closure of* $M_0, \Gamma(M_0 \times M_0) \subseteq \gamma(M_0) \subseteq N_0$, γ *is continuous and proximally commutes with* Γ *and also suppose either*

(a) Γ *is continuous or*

(b) $\bar{M_0}$ *has the following property:*

  (i) if a non-decreasing sequence $\{a_n\} \to a$, *then* $a_n \preceq a$ *for all n,*

  (ii) if a non-increasing sequence $\{b_n\} \to b$, *then* $b_n \succeq b$ *for all n.*

If there exist $a_0, b_0 \in M_0$ *such that*

$$\gamma(a_0) \preceq \Gamma(a_0, b_0) \text{ and } \gamma(b_0) \succeq \Gamma(b_0, a_0),$$

then there exist $a, b \in M_0$ *such that*

$$\begin{cases} d(u, \Gamma(a, b)) = d(u, \gamma(a)) = d(M, N), \\ d(v, \Gamma(b, a)) = d(v, \gamma(b)) = d(M, N), \end{cases}$$

for some $u, v \in M_0$, *that is,* Γ *and* γ *have a coupled proximally coincidence point.*

Proof. Taking $\varphi(t) = kt$ with $k \in [0, 1)$ in Theorem 3.10, we get the result. □

Now we prove that the existence and uniqueness theorem of coupled common best proximity point. If $(X, \preceq)$ is partially ordered set, then for $(a, b), (u, v) \in X \times X$, $(a, b) \preceq (u, v)$ if and only if $a \preceq u$, $b \succeq v$.

Theorem 3.16. *In addition to the hypotheses of Theorem 3.10, suppose that for every* $(a, b), (b^*, a^*) \in M \times M$ *there exists* $(u, v) \in M \times M$ *such that* $(\Gamma(u, v), \Gamma(v, u))$ *is comparable to* $(\Gamma(a, b), \Gamma(b, a))$ *and* $(\Gamma(a^*, b^*), \Gamma(b^*, a^*))$. *Then* Γ *and* γ *have a unique coupled common best proximity, that is, there exists a unique* $(a, b) \in M_0 \times M_0$ *such that*

$$\begin{cases} d(a, \Gamma(a, b)) = d(a, \gamma(a)) = d(M, N), \\ d(b, \Gamma(b, a)) = d(b, \gamma(b)) = d(M, N). \end{cases}$$

Proof. Let $a, b \in M_0$. Since

$$\begin{cases} d(P_{M_0}\Gamma(a, b), \Gamma(a, b)) = d(M, N), \\ d(P_{M_0}\Gamma(u, v), \Gamma(u, v)) = d(M, N), \end{cases}$$

by weak P-monotone property, we obtain $P_{M_0}\Gamma(u, v) \preceq P_{M_0}\Gamma(a, b)$. And also

$$\begin{cases} d(P_{M_0}\Gamma(b, a), \Gamma(b, a)) = d(M, N), \\ d(P_{M_0}\Gamma(v, u), \Gamma(v, u)) = d(M, N), \end{cases}$$

by weak P-monotone property, we obtain $P_{M_0}\Gamma(v, u) \succeq P_{M_0}\Gamma(b, a)$. Therefore, $(P_{M_0}\Gamma(u, v), P_{M_0}\Gamma(v, u))$ is comparable to $(P_{M_0}\Gamma(a, b), P_{M_0}\Gamma(b, a))$. In the same way, we can show that $(P_{M_0}\Gamma(u, v), P_{M_0}\Gamma(v, u))$ is comparable to $(P_{M_0}\Gamma(a^*, b^*), P_{M_0}\Gamma(b^*, a^*))$. Therefore $P_{M_0}\Gamma$ and $P_{M_0}\gamma$ satisfy all the requirements of

Theorem 2.7, and so there exists a unique $(w,z) \in M_0 \times M_0$ such that $w = P_{M_0}\gamma(w) = P_{M_0}\Gamma(w,z)$ and $z = P_{M_0}\gamma(z) = P_{M_0}\Gamma(z,w)$. This implies that

$$\begin{cases} d(w,\Gamma(w,z)) = d(w,\gamma(w)) = d(M,N), \\ d(z,\Gamma(z,w)) = d(z,\gamma(z)) = d(M,N), \end{cases}$$

that is, Γ and γ have a unique coupled common best proximity point. □

By the following example, we illustrate our theorem.

Example 3.17. Let $X = \mathbb{R}$ and metric $d(a,b) = |a-b|$ with usual order $\leq$ on $\mathbb{R}$. We take $M = [0,1]$, $N = [2,3]$. And we define two continuous functions as $\Gamma : M \times M \to N$ by $\Gamma(a,b) = \frac{6}{2+a^2}$ and $\gamma : M \to N$ by $\gamma(a) = 3 - a$. Assume any $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t+} \varphi(r) < t$ for each $t > 0$. Then clearly, we have only the following case,

$$\begin{cases} d(1,\gamma(1)) = 1 \\ d(1,\gamma(1)) = 1 \qquad \Rightarrow \Gamma(1,b) \leq \Gamma(1,b), \\ 1 \leq 1 \end{cases}$$

and

$$\begin{cases} d(1,\gamma(1)) = 1 \\ d(1,\gamma(1)) = 1 \qquad \Rightarrow \Gamma(a,1) \geq \Gamma(a,1). \\ 1 \leq 1 \end{cases}$$

Therefore, Γ has mixed γ-proximally monotone property. And also, we have

$$\begin{cases} d(1,\Gamma(1,1)) = 1, \\ d(1,\gamma(1)) = d(1,\gamma(1)) = 1. \end{cases}$$

Then $\Gamma(1,1) = \gamma(1)$ which implies that Γ and γ are proximally commutative. And, clearly $\gamma(1) \leq \Gamma(1,1)$ *and* $\gamma(1) \geq \Gamma(1,1)$, and so by Theorem 3.10, one can get $(1,1)$ is coupled proximally coincidence point, that is,

$$\begin{cases} d(1,\Gamma(1,1)) = d(1,\gamma(1)) = 1, \\ d(1,\Gamma(1,1)) = d(1,\gamma(1)) = 1. \end{cases}$$

Moreover, this example satisfies all the hypothesis of Theorem 3.16, and so the point $(1,1)$ is also unique coupled common best proximity point of Γ and γ.

Finally, we present an example which shows that, in general, a partially ordered metric space does not guarantee uniqueness of coupled common best proximity point.

Example 3.18. Let $X = \{(0,1),(1,0),(-1,0),(0,-1)\}$ with usual metric

$$d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$$

and define the partial order $(a_1,a_2) \leq (b_1,b_2)$ as $a_1 \leq b_1, a_2 \leq b_2$, and we consider $M = \{(0,1),(1,0)\}, N = \{(-1,0),(0,-1)\}$. Then $d(M,N) = \sqrt{2}$. Now define $\Gamma : M \times M \to N$ by $\Gamma((a_1,a_2),(b_1,b_2)) = (-a_2,-a_1)$ and $\gamma : M \to N$ by $\gamma(a,b) = (-a,-b)$. One can note that the only comparable pairs of points in M are $x \leq x$ for $x \in M$. Therefore, the mappings Γ and γ satisfy all the conditions of Theorem 3.10. Also, there are three coupled common best proximity points, $((0,1),(0,1)), ((0,1),(1,0))$ and $((1,0),(1,0))$.

Conclusion

In this paper, we have introduced the concept of mixed γ-proximally monotone property type mappings and investigated the existence of the coupled proximally coincidence point for such mappings in partially ordered complete metric spaces. Furthermore, we have proved the existence and uniqueness of coupled common best proximity points. Our results extend, improve and generalize several known results in the literature.

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Conflict of interest

The authors declare that they have no competing interests.

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