Journal of
Mathematical
Inequalities
Volume 13, Number 4 (2019), 1235–1244 doi:10.7153/jmi-2019-13-87

3–VARIABLE DOUBLE ρ–FUNCTIONAL INEQUALITIES OF DRYGAS

WENLONG SUN, YUANFENG JIN, CHOONKIL PARK AND GANG LU

(*Communicated by A. Gilányi*)

Abstract. Drygas introduced the functional equation $f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$ in quasi-inner product spaces. In this paper, we introduce and solve 3-variable double ρ-functional inequalities associated to the functional equation $f(x+y+z)+f(x+y-z)=2f(x)+2f(y)+f(z)+f(-z)$. Moreover, we prove the Hyers-Ulam stability of the 3-variable double ρ-functional inequalities in complex Banach spaces.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following:"When is it true that a function which approximately satisfies a functional equation must be closed to an exact solution of question?". If the problem accepts a solution, we say the equation is stable. The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [13] gave the first affirmative answer to the question of Ulam for additive groups in Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 3, 5, 7, 15, 16, 23]).

Gilányi [11] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\|\leqslant\|f(x+y)\| \tag{1}$$

Mathematics subject classification (2010): Primary 39B62, 39B52, 46B25.
Keywords and phrases: Drygas functional equation, Hyers-Ulam stability, double ρ-functional inequalities, Banach space.

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [21]. Fechner [9] and Gilányi [12] proved the Hyers-Ulam stability of the functional inequality (1).

Park [17, 18] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

To obtain a Jordan and von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [6] considered the functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),$$

whose solution is called a *Drygas mapping*. The general solution of the above functional equation was given by Ebanks, Kannappan and Sahoo [8] as

$$f(x)=Q(x)+A(x)$$

where A is an additive mapping and Q is a quadratic mapping. In [19], Park et al. investigated the following inequalities

$$\begin{aligned}\|f(x)+f(y)+f(z)\| &\leqslant \left\|2f\left(\tfrac{x+y+z}{2}\right)\right\|,\\ \|f(x)+f(y)+f(z)\| &\leqslant \|f(x+y+z)\|,\\ \|f(x)+f(y)+2f(z)\| &\leqslant \left\|2f\left(\tfrac{x+y}{2}+z\right)\right\|\end{aligned}$$

in Banach spaces. Recently, Cho et al. [4] investigated the following functional inequality

$$\|f(x)+f(y)+f(z)\leqslant \left\|Kf\left(\frac{x+y+z}{K}\right)\right\| \qquad (0<|K|<|3|)$$

in non-Archimedean Banach spaces. Lu et al. [14] investigated 3-variable Jensen ρ-functional inequalities associated to the following functional equations

$$\begin{aligned}f(x+y+z)+f(x+y-z)-2f(x)-2f(y)&=0,\\ f(x+y+z)-f(x-y-z)-2f(y)-2f(z)&=0\end{aligned}$$

in complex Banach spaces.

The function equation

$$f(x+y+z)+f(x+y-z)=2f(x)+2f(y)+f(z)+f(-z)$$

is called *3-variable Drygas functional equation*, whose solution is called a *3-variable Drygas mapping*.

In this paper, we introduce double ρ-functional inequalities associated to 3-variable Drygas functional equation, and prove the Hyers-Ulam stability of the double ρ-functional inequalities in complex Banach spaces.

Throughout this paper, assume that X is a complex normed vector space and that Y is a complex Banach space.

2. A double ρ-functional inequality relate to the 3-variable Drygas functional equation I

In this section, we prove the Hyers-Ulam stability of the following 3-variable double ρ-functional inequality

$$\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\
\leqslant &\|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\
&+\|\rho_2(f(x+y-z)-f(x)-f(y)-f(-z))\|
\end{aligned} \tag{2}$$

in complex Banach spaces, where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1|<1$ and $|\rho_1|+|\rho_2|<2$.

LEMMA 2.1. Let $f: X \rightarrow Y$ be a mapping. If it satisfies (2) for all $x,y,z \in X$, then f is additive.

Proof. Letting $x=-y=z$ in (2), we get

$$2\|f(z)+f(-z)\| \leqslant |\rho_1|\|f(-z)+f(z)\|+|\rho_2|\|f(z)+f(-z)\|$$

and so $f(-x)=-f(x)$ for all $x \in X$, and $f(0)=0$.

Letting $z=0$ in (2), we have

$$\begin{aligned}
&\|2f(x+y)-2f(x)-2f(y)\| \leqslant \|\rho_1(f(x+y)-f(x)-f(y))\| \\
&+\|\rho_2(f(x+y)-f(x)-f(y))\| \\
=&(|\rho_1|+|\rho_2|)\|f(x+y)-f(x)-f(y)\|
\end{aligned}$$

and so $f(x+y)=f(x)+f(y)$ for all $x,y \in X$. Hence $f: X \rightarrow Y$ is additive. □

Now we prove the Hyers-Ulam stability of the double ρ-functional inequality (2) in complex Banach spaces.

THEOREM 2.2. Let $f: X \rightarrow Y$ be a mapping. If there is a function $\varphi: X^3 \rightarrow [0,\infty)$ such that

$$\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\
\leqslant &\|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\
&+\|\rho_2(f(x+y-z)-f(x)-f(y)-f(-z))\|+\varphi(x,y,z)
\end{aligned} \tag{3}$$

and

$$\widetilde{\varphi}(x,y,z):=\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, 2^j z\right)<\infty$$

for all $x,y,z \in X$, then there exists a unique additive mapping $A: X \rightarrow Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{1}{2(2-|\rho_1|-|\rho_2|)} \widetilde{\varphi}(x,x,0) \tag{4}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3), we get $\|4f(0)\| \leqslant \|2\rho_1 f(0)\| + \|2\rho_2 f(0)\|$ and so $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (3), we get

$$\|2f(2x) - 4f(x)\| \leqslant |\rho_1|\|f(2x) - 2f(x)\| + |\rho_2|\|f(2x) - 2f(x)\| + \varphi(x,x,0)$$

for all $x \in X$.

Thus

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \leqslant \frac{1}{2 - |\rho_1| - |\rho_2|} \frac{1}{2} \varphi(x,x,0)$$

for all $x \in X$.

Hence one may have the following formula for positive integers m,l with $m > l$,

$$\left\| \frac{1}{(2)^l} f\left((2)^l x\right) - \frac{1}{(2)^m} f\left((2)^m x\right) \right\| \leqslant \frac{1}{2(2 - |\rho_1| - |\rho_2|)} \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi\left(2^i x, 2^i x, 0\right), \quad (5)$$

for all $x \in X$.

It follows from (5) that the sequence $\left\{ \frac{f(2^k x)}{2^k} \right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a Banach space, the sequence $\left\{ \frac{f(2^k x)}{2^k} \right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k \to \infty} \left\{ \frac{f(2^k x)}{2^k} \right\}, \quad \forall x \in X.$$

Taking $l = 0$ and letting m tend to ∞ in (5), we get (4).

It follows from (3) that

$$\begin{aligned}
&\|A(x+y+z) + A(x+y-z) - 2A(x) - 2A(y) - A(z) - A(-z)\| \\
= &\lim_{n \to \infty} \frac{1}{2^n} \|f[2^n(x+y+z)] + f[2^n(x+y-z)] - 2f(2^n x) \\
&\qquad -2f(2^n y) - f(2^n z) - f(-2^n z)\| \\
\leqslant &\lim_{n \to \infty} \frac{1}{2^n} \|\rho_1 (f[2^n(x+y+z)] - f(2^n x) - f(2^n y) - f(2^n z))\| \\
&+ \lim_{n \to \infty} \frac{1}{2^n} \|\rho_2 (f[2^n(x+y-z)] - f(2^n x) - f(2^n y) - f(-2^n z))\| \\
&+ \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) \\
= &\|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&+ \|\rho_2(A(x+y-z) - A(x) - A(y) - A(-z))\|
\end{aligned}$$

for all $x,y,z \in X$. Sot A satisfies (2) and so it is additive by Lemma 2.1.

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (3). Then one has

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|\frac{1}{2^k}A\left(2^kx\right)-\frac{1}{2^k}T\left(2^kx\right)\right\| \\ &\leqslant \frac{1}{2^k}\left(\left\|A\left(2^kx\right)-f\left(2^kx\right)\right\|+\left\|T\left(2^kx\right)-f\left(2^kx\right)\right\|\right) \\ &\leqslant 2\frac{1}{2^k}\widetilde{\varphi}(2^kx,2^kx,0)\end{aligned}$$

which tends to zero as $k\to\infty$ for all $x\in X$. So we can conclude that $A(x)=T(x)$ for all $x\in X$. □

COROLLARY 2.3. Let $r<1$ and θ be nonnegative real numbers, and let $f:X\to Y$ be a mapping such that

$$\begin{aligned}&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\ &\leqslant \|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &\quad+\|\rho_2(f(x+y-z)-f(x)-f(y)+f(z))\|+\theta(\|x\|^r+\|y\|^r+\|z\|^r)\end{aligned} \tag{6}$$

for all $x,y,z\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leqslant\frac{2\theta}{(2-2^r)(2-|\rho_1|-|\rho_2|)}\|x\|^r$$

for all $x\in X$.

THEOREM 2.4. Let $f:X\to Y$ be a mapping with $f(0)=0$. If there is a function $\varphi:X^3\to[0,\infty)$ satisfying (3) such that

$$\widetilde{\varphi}(x,y,z):=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right)<\infty$$

for all $x,y,z\in X$, then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leqslant\frac{1}{2(2-|\rho_1|-|\rho_2|)}\widetilde{\varphi}(x,x,0)$$

for all $x\in X$.

Proof. By a similar method to the proof of Theorem 2.2, we can get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\leqslant\frac{1}{2-|\rho_1|-|\rho_2|}\varphi\left(\frac{x}{2},\frac{y}{2},0\right)$$

for all $x\in X$.

Next, we can prove that the sequence $\{2^nf\left(\frac{x}{2^n}\right)$ is a Cauchy sequence for all $x\in X$, and define a mapping $A:X\to Y$ by

$$A(x):=\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right)$$

for all $x\in X$.

The rest proof is similar to the corresponding part of the proof of Theorem 2.2. □

COROLLARY 2.5. Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ ba a mapping satisfying (6). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{(2^r - 2)(2 - |\rho_1| - |\rho_2|)} \|x\|^r$$

for all $x \in X$.

3. A double ρ-functional inequality relate to the 3-variable Drygas functional equation *II*

In this section, we prove the Hyers-Ulam stability of the following 3-variable double ρ-functional inequality

$$\begin{aligned} &\|f(x+y+z) + f(x+y-z) - 2f(x) - 2f(y) - f(z) - f(-z)\| \\ &\leqslant \|\rho_1(f(x+y-z) + f(x-y+z) - 2f(x) - f(y) - f(-y) - f(z) - f(-z))\| \\ &\quad + \|\rho_2(f(x+y+z) - f(x+z) - f(y))\| \end{aligned}$$

in complex Banach spaces, where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| + |\rho_2| < 1$.

THEOREM 3.1. Let $f : X \to Y$ be a mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ such that

$$\begin{aligned} &\|f(x+y+z) + f(x+y-z) - 2f(x) - 2f(y) - f(z) - f(-z)\| \\ &\leqslant \|\rho_1(f(x+y-z) + f(x-y+z) - 2f(x) - f(y) - f(-y) - f(z) - f(-z))\| \\ &\quad + \|\rho_2(f(x+y+z) - f(x+z) - f(y))\| + \varphi(x,y,z) \end{aligned} \tag{7}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi\left(2^j x, 2^j y, 2^j z\right) \leqslant \infty \tag{8}$$

for all $x, y, z \in X$, then there exists a unique Drygas mapping $A : X \to Y$ such that

$$\|A(x) - f(x) - f(-x)\| \leqslant \frac{1}{4(1 - |\rho_1|)} [\widetilde{\varphi}(x,0,x) + \widetilde{\varphi}(-x,0,x)] \tag{9}$$

Proof. Letting $x = y = z = 0$ in (7), we get $4\|f(0)\| \leqslant (4|\rho_1| + |\rho_2|)\|f(0)\|$ and so $f(0) = 0$.

Letting $y = 0$ in (7), we get

$$\|f(x+z) + f(x-z) - 2f(x) - f(z) - f(-z)\| \leqslant \frac{1}{1 - |\rho_1|} \varphi(x,0,z) \tag{10}$$

for all $x,z \in X$. Letting $z = x$ in (10), we get

$$\|f(2x) - 3f(x) - f(-x)\| \leqslant \frac{1}{1-|\rho_1|}\varphi(x,0,x)$$

for all $x \in X$. Similarly, we get

$$\|f(-2x) - 3f(-x) - f(x)\| \leqslant \frac{1}{1-|\rho_1|}\varphi(-x,0,x)$$

for all $x \in X$. Thus we have

$$\begin{aligned} &\|f(2x) + f(-2x) - 4f(x) - 4f(-x)\| \\ &\leqslant \|f(2x) - 3f(x) - f(-x)\| + \|f(-2x) - 3f(-x) - f(x)\| \\ &\leqslant \frac{1}{1-|\rho_1|}[\varphi(x,0,x) + \varphi(-x,0,x)] \end{aligned}$$

for all $x \in X$. Therefore

$$\left\| \frac{f(2x) + f(-2x)}{4} - (f(x) + f(-x)) \right\| \leqslant \frac{1}{4(1-|\rho_1|)}[\varphi(x,0,x) + \varphi(-x,0,x)]$$

for all $x \in X$.

Hence one may have the following formula for positive integers m,l with $m > l$,

$$\begin{aligned} &\left\| \frac{f\left(2^l x\right) + f(-2^l x)}{4^l} - \frac{f\left(2^m x\right) + f\left(-2^m x\right)}{4^m} \right\| \\ &\leqslant \sum_{i=l}^{m-1} \frac{1}{4^i} \frac{1}{4(1-|\rho_1|)} \left(\varphi(2^i x, 0, 2^i x) + \varphi(-2^i x, 0, 2^i x)\right), \end{aligned} \tag{11}$$

for all $x \in X$.

It follows from (8) that the sequence $\left\{\frac{f(2^k x)+f(-2^k)}{4^k}\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a Banach space, the sequence $\left\{\frac{f(2^k x)+f(-2^k x)}{4^k}\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} \left\{ \frac{f(2^k x) + f(-2^k x)}{4^k} \right\}, \quad \forall x \in X.$$

Taking $l = 0$ and letting m tend to ∞ in (11), we get (9).

It follows from (7) that

$$
\begin{aligned}
&\|A(x+y)+A(x-y)-2A(x)-A(y)-A(-y)\| \\
=&\lim_{n\to\infty}\frac{1}{4^n}\|f[2^n(x+y)]+f[2^n(-x-y)]+f[2^n(x-y)]+f[2^n(-x+y)] \\
&-2f(2^nx)-2f(-2^nx)-f(2^ny)-f(-2^ny)\| \\
\leqslant&\lim_{n\to\infty}\frac{1}{4^n}\|\rho_1(f(2^nx+2^ny)+f(2^nx-2^ny)-2f(2^nx)-f(2^ny)-f(-2^ny))\| \\
&+\lim_{n\to\infty}\frac{1}{4^n}\|\rho_1(f(-2^nx-2^ny)+f(-2^nx+2^ny)-2f(-2^nx)-f(2^ny)-f(-2^ny))\| \\
\leqslant&\lim_{n\to\infty}\frac{1}{4^n}\frac{|\rho_1|}{2(1-|\rho_1|)}\varphi(2^nx,0,2^ny)=0
\end{aligned}
$$

for all $x,y\in X$. So A is a Drygas mapping.

Now, we show that the uniqueness of A. Let $T:X\to Y$ be another Drygas mapping satisfying (9). Then one has

$$
\begin{aligned}
&\|A(x)-T(x)\|=\left\|\frac{1}{4^k}A\left(2^kx\right)-\frac{1}{4^k}T\left(2^kx\right)\right\| \\
\leqslant&\frac{1}{4^k}\left(\left\|A\left(2^kx\right)-f\left(2^kx\right)-f\left(-2^kx\right)\right\|+\left\|T\left(2^kx\right)-f\left(2^kx\right)-f\left(-2^kx\right)\right\|\right) \\
\leqslant&\frac{2}{4(1-|\rho_1|)}\left(\widetilde{\varphi}(2^kx,0,2^kx)+\widetilde{\varphi}(-2^kx,0,2^kx)\right)
\end{aligned}
$$

which tends to zero as $k\to\infty$ for all $x\in X$. So we can conclude that $A(x)=T(x)$ for all $x\in X$. $\square$

COROLLARY 3.2. *Let $r<2$ and θ be nonnegative real numbers, and let $f:X\to Y$ be a mapping such that*

$$
\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\
\leqslant&\|\rho_1(f(x+y+z)-f(x-y-z)-2f(x)-f(y)-f(-y)-f(z)-f(-z))\| \\
&+\|\rho_2(f(x+y+z)-f(x+z)-f(y))\|+\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}
\tag{12}
$$

for all $x,y,z\in X$. Then there exists a unique Drygas mapping $A:X\to Y$ such that

$$\|A(x)-f(x)-f(-x)\|\leqslant\frac{4\theta}{(4-2^r)(1-|\rho_1|)}\|x\|^r$$

for all $x\in X$.

THEOREM 3.3. *Let $f:X\to Y$ be a mapping with $f(0)=0$. If there is a function $\varphi:X^3\to[0,\infty)$ satisfying (7) and*

$$\widetilde{\varphi}(x,y,z):=\sum_{j=1}^{\infty}4^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right)=0$$

for all $x,y,z \in X$, then there exists a unique Drygas mapping $A: X \to Y$ such that

$$\|A(x) - f(x) - f(-x)\| \leqslant \frac{1}{4(1-|\rho_1|)}[\widetilde{\varphi}(x,0,x) + \widetilde{\varphi}(-x,0,x)]$$

for all $x \in X$.

Proof. By a similar method to the proof of Theorem 3.1, we can get

$$\left\|f(x) + f(-x) - 4\left(f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right)\right\| \leqslant \frac{1}{1-|\rho_1|}\left(\varphi(\frac{x}{2},0,\frac{x}{2}) + \varphi(-\frac{x}{2},0,\frac{x}{2})\right)$$

for all $x \in X$.

Next, we can prove that the sequence $\{4^n[f(\frac{x}{2^n}) + f(-\frac{x}{2^n})]\}$ is a Cauchy sequence for all $x \in X$, and define a mapping $A: X \to Y$ by

$$A(x) := \lim_{n\to\infty} 4^n[f(\frac{x}{2^n}) + f(-\frac{x}{2^n})]$$

for all $x \in X$.

The rest proof is similar to the corresponding part of the proof of Theorem 3.1. □

COROLLARY 3.4. Let $r > 2$ and θ be nonnegative real numbers and let $f: X \to Y$ be a mapping satisfying (12). Then there exists a unique Drygas mapping $A: X \to Y$ such that

$$\|f(x) + f(-x) - A(x)\| \leqslant \frac{4\theta}{(2^r-4)(1-|\rho_1|)}\|x\|^r$$

for all $x \in X$.

Acknowledgement. This work was supported by National Natural Science Foundation of China (No. 11761074), the Projection of the Department of Science and Technology of JiLin Province (No. JJKH20170453KJ) and the Education Department of Jilin Province (No. 20170101052JC) and Natural Science fund of Liaoning Province (No. 201602547).

REFERENCES

[1] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] J. ACZEL AND J. DHOMBRES, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.
[3] P. W. CHOLEWA, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[4] Y. CHO, C. PARK AND R. SAADATI, *Functional inequalities in non-Archimedean Banach spaces*, Appl. Math. Lett. **23** (2010), 1238–1242.
[5] Y. CHO, R. SAADATI AND Y. YANG, *Approximation of homomorphisms and derivations on Lie C^*-algebras via fixed point method*, J. Inequal. Appl. **3013**, 2013:415.

[6] H. Drygas, *Quasi-inner products and their applications*, Advances in Multivariate Statistical Analyis, 13–30, Theory Decis. Lib Ser. B: Math. Statist. Methods, Reidel, Dordrecht, 1987.

[7] A. Ebadian, N. Ghobadipour, Th. M. Rassias and M. Eshaghi Gordji, *Functional inequalities associated with Cauchy additive functional equation in non-Archimedean spaces*, Discrete Dyn. Nat. Soc. **2011** (2011), Article ID 929824.

[8] B. R. Ebanks, P. L. Kannappan and P. K. Sahoo, *A common generalization of functional equations charactering normed and quasi-inner-product spaces*, Canad. Math. Bull. **35** (1992), 321–327.

[9] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[10] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl.**184** (1994), 431–436.

[11] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.

[12] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[13] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[14] G. Lu, Q. Liu, Y. Jin and J. Xie, *3-Variable Jensen ρ-functional equations*, J. Nonlinear Sci. Appl. **9** (2016), 5995–6003.

[15] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1312–1316.

[16] G. Lu and C. Park, *Hyers-Ulam stability of general Jensen-type mappings in Banach algebras*, Results Math. **66** (2014), 385–404.

[17] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[18] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[19] C. Park, Y. Cho and M. Han, *Functional inequalities associated with Jordan-von-Neumann-type additive functional equations*, J. Inequal. Appl. **2007** (2007), Article ID 41820.

[20] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[21] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.

[22] S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science ed., Wiley, New York, 1940.

[23] T. Z. Xu, J. M. Rassias and W. X. Xu, *A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces*, Internat. J. Phys. Sci. **6** (2011), 313–324.

(Received February 18, 2018)

Wenlong Sun
Department of Mathematics, School of Science
ShenYang University of Technology
Shenyang 110870, P.R. China
e-mail: 179378033@qq.com

Yuanfeng Jin
Department of Mathematics, Yanbian University
Yanji 133001, P.R. China
e-mail: jinyuanfeng1976@hotmail.com

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences
Hanyang University
Seoul 04763, Republic of Korea
e-mail: baak@hanyang.ac.kr

Gang Lu
Department of Mathematics, School of Science
ShenYang University of Technology
Shenyang 110870, P.R. China
e-mail: lvgang1234@hanmail.net