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Research Article

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On a fixed point theorem with application to functional equations

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Abstract: The purpose of this paper is to study behavior of a rational type contraction introduced in [A fixed point theorem for contractions of rational type in partially ordered metric spaces, Ann. Univ. Ferrara, 2013, 59, 251–258] in context of ordered dualistic partial metric spaces and to investigate sufficient conditions for the existence of a fixed point in this space. These results extend various comparable results, existing in the literature. We give examples to explain our findings. We apply our result to prove the existence of the solution of functional equation.

Keywords: fixed point, dualistic partial metric, application

MSC 2010: 47H09; 47H10; 54H25

1 Introduction

The generalization of a metric function has been done by a number of researchers (see [1–5]). The latest in this regard is introduced by Jleli and Samet [6]. It is known as $\mathcal{F}$-metric. In this metric they particularly modified triangle inequality of metric function. For the current paper we describe partial metric and dualistic partial metric in detail. Partial metric was introduced by Matthews [2]. He generalized metric function with an entirely different angle. He modified concept of self-distance which was known to be zero for all elements in the underlying space and introduced that the self-distance may not be ZERO. Matthews called new metric function with non-zero self-distance a partial metric. He applied this function as a suitable mathematical tool for program verification and also generalized Banach Contraction Principle. This led a number of researchers to investigate existence of fixed points, common fixed points and coupled fixed points of self-mappings defined on a partial metric space (see [7–10] and references herein).

Neill [3] extended range set of the partial metric to real numbers and called new metric function a dualistic partial metric. Neill studied various topological properties and examples of dualistic partial metric space in [3]. The dualistic partial metric was first explored for fixed point theory by Oltra *et al.* [11] where authors presented a Banach fixed point theorem in complete dualistic partial metric spaces along with some convergence properties of sequences and Nazam *et al.* [12–14].

Recently, many authors have investigated fixed points of contractions of rational type and contractive mappings in metric and partial metric spaces, for details, see [15–27].

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Cabrera *et al.* [18] studied the following fixed point theorem on rational type contraction.

Theorem 1. *[18] Let $(\Im, \preceq, d)$ be a complete ordered metric space and $T : \Im \to \Im$ be a continuous and nondecreasing mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(T(\sigma), T(\varsigma)) \leq \frac{\alpha d(\varsigma, T(\varsigma))(1 + d(\sigma, T(\sigma)))}{1 + d(\sigma, \varsigma)} + \beta d(\sigma, \varsigma) \tag{1.1}$$

for all $\sigma, \varsigma \in \Im$ with $\sigma \preceq \varsigma$. If there exists $\sigma_0 \in \Im$ such that $\sigma_0 \preceq T(\sigma_0)$, then T has a unique fixed point.

In this paper, we shall establish aforementioned theorem in the context of dualistic partial metric spaces. We then apply our result to prove the existence and uniqueness of the solution of functional equations appearing in dynamic programming.

2 Preliminaries

We recall some mathematical basics to make this paper self-sufficient.

Let $\Im$ be a non-empty set and $T : \Im \to \Im$ be a self-mapping. A point $\sigma^\star \in \Im$ is called a fixed point of T if $\sigma^\star = T(\sigma^\star)$. For the self-mapping T, Picard iterative sequence $\{\sigma_n\}$ in $\Im$, with initial point σ_0, is defined by

$$\sigma_n = T(\sigma_{n-1}), \text{ for all } n \in \mathbb{N}.$$

Matthews [2] defined partial metric space by the following axioms.

Definition 1. [2] A partial metric on a non-empty set $\Im$ is a function $P : \Im \times \Im \to [0, \infty)$ satisfying the following axioms, for all $\sigma, \varsigma, \nu \in \Im$,

(P_1) $\sigma = \varsigma \Leftrightarrow P(\sigma, \sigma) = P(\varsigma, \varsigma) = P(\sigma, \varsigma)$;
(P_2) $P(\sigma, \sigma) \leq P(\sigma, \varsigma)$;
(P_3) $P(\sigma, \varsigma) = P(\varsigma, \sigma)$;
(P_4) $P(\sigma, \nu) \leq P(\sigma, \varsigma) + P(\varsigma, \nu) - P(\varsigma, \varsigma)$.

The pair $(\Im, P)$ is called a partial metric space.

Neill [3] extended notion of partial metric as follows:

Definition 2. [3] A dualistic partial metric on a non-empty set $\Im$ is a function $\eta : \Im \times \Im \to \mathbb{R}$ satisfying the following properties, for all $\sigma, \varsigma, \nu \in \Im$,

(η_1) $\sigma = \varsigma \Leftrightarrow \eta(\sigma, \sigma) = \eta(\varsigma, \varsigma) = \eta(\sigma, \varsigma)$;
(η_2) $\eta(\sigma, \sigma) \leq \eta(\sigma, \varsigma)$;
(η_3) $\eta(\sigma, \varsigma) = \eta(\varsigma, \sigma)$;
(η_4) $\eta(\sigma, \nu) \leq \eta(\sigma, \varsigma) + \eta(\varsigma, \nu) - \eta(\varsigma, \varsigma)$.

The pair $(\Im, \eta)$ is called a dualistic partial metric space.

Remark 1. *It is obvious that every partial metric is a dualistic partial metric but the converse is not true. To support this comment, define $\eta_m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by*

$$\eta_m(\sigma, \varsigma) = \max\{\sigma, \varsigma\} \ \forall\ \sigma, \varsigma \in \mathbb{R}.$$

Clearly, η_m is a dualistic partial metric (see [4]) but η_m is not a partial metric. Indeed, for all $\sigma < 0, \varsigma < 0$ implies $\eta_m(\sigma, \varsigma) < 0 \notin \mathbb{R}_0^+$.

Unlike other metrics, in dualistic partial metric $\eta(\sigma, \varsigma) = 0$ does not imply $\sigma = \varsigma$. Indeed, $\eta_m(-1, 0) = 0$ and $0 \neq -1$. This situation creates a problem in obtaining a fixed point of a self-mapping in dualistic partial metric space. For the solution of this problem we introduce *convergence comparison property* (defined below) and use it along with axioms (η_2) and (η_1) to get a fixed point.

Definition 3. Let $(\Im, \eta)$ be a dualistic partial metric space and $T : \Im \to \Im$ be a mapping. We say that T has a convergence comparison property (CCP) if for every sequence $\{\sigma_n\}$ in $\Im$ such that $\sigma_n \to \sigma$, T satisfies

$$\eta(\sigma, \sigma) \leq \eta(T(\sigma), T(\sigma)).$$

Example 1. *Let $\Im = \mathbb{R}$. Define η_m as in Remark 1. Consider any sequence $\{\sigma_n\}$ converging to σ in $(\Im, \eta_m)$. Define $T : \Im \to \Im$ by $T(\sigma) = e^{\sigma}$. We have $\sigma \leq e^{\sigma}$ for any $\sigma \in \Im$, that is, $\eta_m(\sigma, \sigma) \leq \eta_m(T(\sigma), T(\sigma)$, i.e., T satisfies (CCP).*

We give some examples to explain dualistic partial metric.

Example 2. *If $(\Im, d)$ is a metric space and $c \in \mathbb{R}$ is arbitrary constant, then*

$$\eta(\sigma, \varsigma) = d(\sigma, \varsigma) + c$$

defines a dualistic partial metric on $\Im$.

Example 3. *[3] Let $(\Im, P)$ be a partial metric space. The mapping $\eta : \Im \times \Im \to \mathbb{R}$ defined by*

$$\eta(\sigma, \varsigma) = P(\sigma, \varsigma) - P(\sigma, \sigma) - P(\varsigma, \varsigma) \text{ for all } \sigma, \varsigma \in \Im,$$

satisfies the conditions $(\eta_1) - (\eta_4)$ and hence defines a dualistic partial metric on $\Im$. We note that $\eta(\sigma, \varsigma)$ may have negative values.

Example 4. *Let $\Im = \mathbb{R}$. Define the mapping $\eta : \Im \times \Im \to \mathbb{R}$ by*

$$\eta(\sigma, \varsigma) = \begin{cases} |\sigma - \varsigma| & \text{if } \sigma \neq \varsigma, \\ -b & \text{if } \sigma = \varsigma; b > 0. \end{cases}$$

The axioms (η_1), (η_2) and (η_3) can be proved immediately. We prove axiom (η_4) in details.

Case 1. *If $\sigma \neq \varsigma = \nu$, then*
$\eta(\sigma, \nu) \leq \eta(\sigma, \varsigma) + \eta(\varsigma, \nu) - \eta(\varsigma, \varsigma)$ implies $|\sigma - \nu| = |\sigma - \varsigma|$.

Case 2. *If $\sigma = \varsigma \neq \nu$, then*
$\eta(\sigma, \nu) \leq \eta(\sigma, \varsigma) + \eta(\varsigma, \nu) - \eta(\varsigma, \varsigma)$ implies $|\sigma - \nu| = |\varsigma - \nu|$.

Case 3. *If $\sigma = \varsigma = \nu$, then*
$\eta(\sigma, \nu) \leq \eta(\sigma, \varsigma) + \eta(\varsigma, \nu) - \eta(\varsigma, \varsigma)$ is obious.

Case 4. *If $\sigma \neq \varsigma \neq \nu$, then*
$\eta(\sigma, \nu) \leq \eta(\sigma, \varsigma) + \eta(\varsigma, \nu) - \eta(\varsigma, \varsigma)$ implies $|\sigma - \nu| \leq |\sigma - \varsigma| + |\varsigma - \nu| + b$.

Thus, the axiom (η_4) holds in all cases. Hence $(\Im, \eta)$ is a dualistic partial metric space.

Neill [3] established that each dualistic partial metric η on $\Im$ generates a T_0 topology $\tau[\eta]$ on $\Im$ having base, the family of η-balls $\{B_\eta(\sigma, \epsilon) : \sigma \in \Im, \epsilon > 0\}$ where

$$B_\eta(\sigma, \epsilon) = \{\varsigma \in \Im : \eta(\sigma, \varsigma) < \epsilon + \eta(\sigma, \sigma)\}.$$

If $(\Im, \eta)$ is a dualistic partial metric space, then the function $d_\eta : \Im \times \Im \to \mathbb{R}_0^+$, defined by

$$d_\eta(\sigma, \varsigma) = \eta(\sigma, \varsigma) - \eta(\sigma, \sigma), \tag{2.1}$$

defines a quasi metric on $\Im$ such that $\tau(\eta) = \tau(d_\eta)$ and $d_\eta^s(\sigma, \varsigma) = \max\{d_\eta(\sigma, \varsigma), d_\eta(\varsigma, \sigma)\}$ defines a metric on $\Im$.

Following definition and Lemma describe the convergence criteria established by Oltra *et al.* [11].

Definition 4. [11] Let $(\Im, \eta)$ be a dualistic partial metric space.

(1) A sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to a point σ in $(\Im, \eta)$ if

$$\lim_{n\to\infty} \eta(\sigma_n, \sigma) = \eta(\sigma, \sigma).$$

(2) A sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in $(\Im, \eta)$ is called a Cauchy sequence if

$$\lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m) \text{ exists and is finite.}$$

(3) A dualistic partial metric space $(\Im, \eta)$ is said to be complete if every Cauchy sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in $\Im$ converges, with respect to $\mathcal{T}[\eta]$, to a point $\sigma \in \Im$ such that

$$\eta(\sigma, \sigma) = \lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m).$$

Remark 2. *For a sequence $\{\sigma_n\} \subset \Im$, convergence wrt(with respect to) metric space $(\Im, d)$ may not implies convergence wrt dualistic partial metric space $(\Im, \eta)$.*

Indeed, let $\{\sigma_n = \frac{1}{n} - 1\} \subset \Im$ and define $\eta : \Im \times \Im \to \mathbb{R}$ by

$$\eta(\sigma, \varsigma) = \begin{cases} |\sigma - \varsigma| = d(\sigma, \varsigma) & \text{if } \sigma \neq \varsigma, \\ -1 & \text{if } \sigma = \varsigma. \end{cases}$$

Clearly,

$$\lim_{n\to\infty} d(\sigma_n, -1) = 0.$$

This implies $\sigma_n \to -1$ *wrt* $(\Im, d)$. On the other hand, consider

$$\begin{aligned} \lim_{n\to\infty} \eta(\sigma_n, -1) = \eta(-1, -1) &\Rightarrow \lim_{n\to\infty} |\sigma_n + 1| = -1 \\ \Rightarrow \lim_{n\to\infty} \left|\frac{1}{n}\right| &= -1 \Rightarrow 0 = -1 \text{ (false sentence).} \end{aligned}$$

This shows that $\sigma_n \nrightarrow -1$ *wrt* $(\Im, \eta)$.

Lemma 1. [11] *Let $(\Im, \eta)$ be a dualistic partial metric space.*

(1) Every Cauchy sequence in $(\Im, d_\eta^s)$ is also a Cauchy sequence in $(\Im, \eta)$.
(2) A dualistic partial metric $(\Im, \eta)$ is complete if and only if the induced metric space $(\Im, d_\eta^s)$ is complete.
(3) A sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in $\Im$ converges to a point $\upsilon \in \Im$ with respect to $\mathcal{T}[(d_\eta^s)]$ if and only if

$$\lim_{n\to\infty} \eta(\upsilon, \sigma_n) = \eta(\upsilon, \upsilon) = \lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m).$$

3 Main results

In this section, we shall prove a dualistic partial metric version of Theorem 1. We begin with the following definitions. Let

$$\mathcal{D} = \{(\sigma, \varsigma) \in \Im \times \Im |\ \sigma \preceq \varsigma \ \wedge \ \eta(\sigma, \varsigma) \neq -1\}.$$

Definition 5. *Let $(\Im, \preceq)$ be a partially ordered set and $(\Im, \eta)$ be a dualistic partial metric space. A mapping $T : \Im \to \Im$ is said to be a dualistic contraction of rational type if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that:*

$$|\eta(T(\sigma), T(\varsigma))| \leq \alpha \left| \frac{\eta(\varsigma, T(\varsigma))(1 + \eta(\sigma, T(\sigma)))}{1 + \eta(\sigma, \varsigma)} \right| + \beta |\eta(\sigma, \varsigma)| \tag{3.1}$$

for all $\sigma, \varsigma \in \mathcal{D}$.

Remark 3. The contractive condition (3.1) has some differences with (1.1). Since, for a metric d, $d(\sigma, \sigma) = 0$ for any $\sigma \in \Im$ which ensures that (1.1) holds for all σ such that $\sigma = T(\sigma)$ and conversely. However, from definition of dualistic partial metric we know that, in general, $\eta(\sigma, \sigma) \neq 0$ for any $\sigma \in \Im$. For if σ is a fixed point of T then from (3.1) one can follow that $\eta(\sigma, \sigma) = 0$. So if a self-mapping T has a fixed point σ such that $\eta(\sigma, \sigma) \neq 0$, then $\sigma \preceq \sigma$ but σ does not satisfy (3.1).

We state our main result as follows:

Theorem 2. *Let $(\Im, \preceq)$ be a partially ordered set and $(\Im, \eta)$ be a complete dualistic partial metric space. If T is a nondecreasing, dualistic contraction of rational type satisfying the following conditions:*

(1) there exists $\sigma_0 \in \Im$ such that $\sigma_0 \preceq T(\sigma_0)$;
(2) if $\{\sigma_n\}$ is a nondecreasing sequence in $\Im$ such that $\{\sigma_n\} \to \upsilon$, then $\sigma_n \preceq \upsilon$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Let σ_0 be an initial point of $\Im$ and let us define Picard iterative sequence $\{\sigma_n\}$ by

$$\sigma_n = T(\sigma_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer i such that $\sigma_i = \sigma_{i+1}$, then $\sigma_i = \sigma_{i+1} = T(\sigma_i)$. So σ_i is a fixed point of T. In this case, the proof is complete.

On the other hand, if $\sigma_n \neq \sigma_{n+1}$ for all $n \in \mathbb{N}$, then $\sigma_n \preceq \sigma_{n+1}$. Indeed by $\sigma_0 \preceq T(\sigma_0)$, we obtain $\sigma_0 \preceq \sigma_1$. Since T is nondecreasing, $\sigma_0 \preceq \sigma_1$ implies $T(\sigma_0) \preceq T(\sigma_1)$ and thus $\sigma_1 \preceq \sigma_2$. Continuing in this way, we get

$$\sigma_0 \preceq \sigma_1 \preceq \sigma_2 \preceq \sigma_3 \preceq \cdots \preceq \sigma_n \preceq \sigma_{n+1} \preceq \cdots$$

Since $\sigma_n \preceq \sigma_{n+1}$, using (3.1), we have

$$\begin{aligned}
|\eta(\sigma_n, \sigma_{n+1})| &= |\eta(T(\sigma_{n-1}), T(\sigma_n))| \\
&\leq \alpha \left| \frac{\eta(\sigma_n, T(\sigma_n))(1 + \eta(\sigma_{n-1}, T(\sigma_{n-1})))}{1 + \eta(\sigma_{n-1}, \sigma_n)} \right| + \beta |\eta(\sigma_{n-1}, \sigma_n)| \\
&\leq \alpha \left| \frac{\eta(\sigma_n, \sigma_{n+1})(1 + \eta(\sigma_{n-1}, \sigma_n))}{1 + \eta(\sigma_{n-1}, \sigma_n)} \right| + \beta |\eta(\sigma_{n-1}, \sigma_n)| \\
&\leq \alpha |\eta(\sigma_n, \sigma_{n+1})| + \beta |\eta(\sigma_{n-1}, \sigma_n)| \\
&\leq \frac{\beta}{1 - \alpha} |\eta(\sigma_{n-1}, \sigma_n)|.
\end{aligned}$$

If we set $\lambda = \dfrac{\beta}{1 - \alpha}$, then $0 < \lambda < 1$ and so

$$\begin{aligned}
|\eta(\sigma_n, \sigma_{n+1})| &\leq \lambda |\eta(\sigma_{n-1}, \sigma_n)| \\
&\leq \lambda(\lambda |\eta(\sigma_{n-2}, \sigma_{n-1})|) \\
&\ \ \vdots \\
&\leq \lambda^n |\eta(\sigma_0, \sigma_1)|.
\end{aligned}$$

Further, generalizing the above inequality, we have

$$|\eta(\sigma_{n+k-1}, \sigma_{n+k})| \leq \lambda^{n+k-1} |\eta(\sigma_0, \sigma_1)| \text{ for all } n, k \in \mathbb{N}.$$

To find self distance, since $\sigma_n \preceq \sigma_n$ for all $n \in \mathbb{N}$, again from (3.1), we have

$$\begin{aligned}
|\eta(\sigma_1, \sigma_1)| &= |\eta(T(\sigma_0), T(\sigma_0))| \\
&\leq \alpha \left| \frac{\eta(\sigma_0, T(\sigma_0))(1 + \eta(\sigma_0, T(\sigma_0)))}{1 + \eta(\sigma_0, \sigma_0)} \right| + \beta |\eta(\sigma_0, \sigma_0)|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha\left|\frac{\eta(\sigma_0,\sigma_1)(1+\eta(\sigma_0,\sigma_1))}{1+\eta(\sigma_0,\sigma_0)}\right| + \beta|\eta(\sigma_0,\sigma_0)| \\
&\leq \alpha|\eta(\sigma_0,\sigma_1)(1+\eta(\sigma_0,\sigma_1))| + \beta|\eta(\sigma_0,\sigma_0)| \\
&\leq \alpha h(1+h) + \beta\eta_0,
\end{aligned}$$

where $|\eta(\sigma_0,\sigma_1)| = h$ and $|\eta(\sigma_0,\sigma_0)| = \eta_0$. Repeating the process, we get

$$\begin{aligned}
|\eta(\sigma_2,\sigma_2)| &\leq \alpha\lambda h(1+\lambda h) + \alpha\beta h(1+h) + \beta^2\eta_0, \\
|\eta(\sigma_3,\sigma_3)| &\leq \alpha\lambda^2 h(1+\lambda^2 h) + \alpha\beta\lambda h(1+\lambda h) + \alpha\beta^2 h(1+h) + \beta^3\eta_0. \\
&\vdots \\
|\eta(\sigma_n,\sigma_n)| &\leq \alpha\lambda^{n-1} h(1+\lambda^{n-1} h) + \alpha\beta\lambda^{n-2} h(1+\lambda^{n-2} h) \\
&\quad + \alpha\beta^2\lambda^{n-3} h(1+\lambda^{n-3} h) + \cdots + \alpha\beta^{n-1} h(1+h) + \beta^n\eta_0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}|\eta(\sigma_n,\sigma_n)| = 0,\ \ i.e.,\ \ \lim_{n\to\infty}\eta(\sigma_n,\sigma_n) = 0. \tag{3.2}$$

Now since

$$\begin{aligned}
d_\eta(\sigma_n,\sigma_{n+1}) &= \eta(\sigma_n,\sigma_{n+1}) - \eta(\sigma_n,\sigma_n), \\
d_\eta(\sigma_n,\sigma_{n+1}) + \eta(\sigma_n,\sigma_n) &= \eta(\sigma_n,\sigma_{n+1}), \\
d_\eta(\sigma_n,\sigma_{n+1}) + \eta(\sigma_n,\sigma_n) &\leq |\eta(\sigma_n,\sigma_{n+1})| \leq \lambda^n|\eta(\sigma_0,\sigma_1)|, \\
d_\eta(\sigma_n,\sigma_{n+1})) &\leq \lambda^n|\eta(\sigma_0,\sigma_1)| + |\eta(\sigma_n,\sigma_n)| \\
&\leq \lambda^n h + \alpha\lambda^{n-1} h(1+\lambda^{n-1} h) + \alpha\beta\lambda^{n-2} h(1+\lambda^{n-2} h) \\
&\quad + \alpha\beta^2\lambda^{n-3} h(1+\lambda^{n-3} h) + \cdots + \alpha\beta^{n-1} h(1+h) + \beta^n\eta_0 \\
&\leq \lambda^n h + \mu^n,
\end{aligned}$$

where

$$\begin{aligned}
\mu^n = \alpha\lambda^{n-1} h(1+\lambda^{n-1} h) &+ \alpha\beta\lambda^{n-2} h(1+\lambda^{n-2} h) + \alpha\beta^2\lambda^{n-3} h(1+\lambda^{n-3} h) \\
&+ \cdots + \alpha\beta^{n-1} h(1+h) + \beta^n\eta_0.
\end{aligned}$$

Further, generalization implies

$$d_\eta(\sigma_{n+k-1},\sigma_{n+k}) \leq \lambda^{n+k-1} h + \mu^{n+k-1} \text{ for all } n, k \in \mathbb{N}. \tag{3.3}$$

Next step is to show that $\{\sigma_n\}$ is a Cauchy sequence in $(\Im,\eta)$. For this purpose, first we show that $\{\sigma_n\}$ is a Cauchy sequence in $(\Im, d_\eta^s)$. Since d_η is a quasi metric, by using triangular property and (3.3), we have

$$\begin{aligned}
d_\eta(\sigma_n,\sigma_{n+k}) &\leq d_\eta(\sigma_n,\sigma_{n+1}) + d_\eta(\sigma_{n+1},\sigma_{n+2}) + \cdots + d_\eta(\sigma_{n+k-1},\sigma_{n+k}) \\
&\leq \lambda^n h + \mu^n + \lambda^{n+1} h + \mu^{n+1} + \cdots + \lambda^{n+k-1} h + \mu^{n+k-1} \\
&\leq \frac{\lambda^n}{1-\lambda} h + \frac{\mu^n}{1-\mu}.
\end{aligned}$$

Note that each term in μ^n contains either β^n or λ^n and since $\beta < 1$, $\lambda < 1$ so each term in μ^n vanishes as $n \to \infty$. Thus, $\lim_{n\to\infty}\lambda^n = 0$ and $\lim_{n\to\infty}\mu^n = 0$ and hence

$$\lim_{n\to\infty} d_\eta(\sigma_n,\sigma_{n+k}) = 0. \tag{3.4}$$

Replacing the positions of σ_n and σ_{n+1}, we get

$$\begin{aligned}
d_\eta(\sigma_{n+1},\sigma_n) &= \eta(\sigma_{n+1},\sigma_n) - \eta(\sigma_{n+1},\sigma_{n+1}),\ \text{ implies} \\
d_\eta(\sigma_{n+1},\sigma_n)) &\leq \lambda^n|\eta(\sigma_0,\sigma_1)| + |\eta(\sigma_{n+1},\sigma_{n+1})|
\end{aligned}$$

$$\leq \lambda^n h + \mu^{n+1}.$$

Further, generalization implies

$$d_\eta(\sigma_{n+k}, \sigma_{n+k-1}) \leq \lambda^{n+k-1} h + \mu^{n+k} \text{ for all } n, k \in \mathbb{N}. \tag{3.5}$$

Since d_η is a quasi metric, by using triangular property and (3.5), we have

$$\begin{aligned} d_\eta(\sigma_{n+k}, \sigma_n) &\leq d_\eta(\sigma_{n+k}, \sigma_{n+k-1}) + d_\eta(\sigma_{n+k-1}, \sigma_{n+k-2}) + \cdots + d_\eta(\sigma_{n+1}, \sigma_n) \\ &\leq \lambda^{n+k-1} h + \mu^{n+k} + \lambda^{n+k-2} h + \mu^{n+k-1} + \cdots + \lambda^n h + \mu^{n+1} \\ &\leq \frac{\lambda^n}{1-\lambda} h + \frac{\mu^{n+1}}{1-\mu}. \end{aligned}$$

Since $\lim_{n\to\infty} \lambda^n = 0$ and $\lim_{n\to\infty} \mu^n = 0$, thus, $\lim_{n\to\infty} d_\eta(\sigma_{n+k}, \sigma_n) = 0$. By definition of d_η^s, we deduce that $\lim_{n\to\infty} d_\eta^s(\sigma_n, \sigma_{n+k}) = 0$. Hence $\{\sigma_n\}$ is a Cauchy sequence in $(\Im, d_\eta^s)$. Since $(\Im, \eta)$ is a complete dualistic partial metric space, by Lemma 1, $(\Im, d_\eta^s)$ is also a complete metric space. Thus $\{\sigma_n\}$ converges to a point $\upsilon \in \Im$, i.e., $\lim_{n\to\infty} d_\eta^s(\sigma_n, \upsilon) = 0$. By Lemma 1, we get

$$\lim_{n\to\infty} \eta(\upsilon, \sigma_n) = \eta(\upsilon, \upsilon) = \lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m). \tag{3.6}$$

Now from (3.4) and for $m = n + k$

$$\begin{aligned} 0 &= \lim_{n,m\to\infty} d_\eta(\sigma_n, \sigma_m) = \lim_{n,m\to\infty} [\eta(\sigma_n, \sigma_m) - \eta(\sigma_n, \sigma_n)] \\ &\Rightarrow \lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m) = \lim_{n\to\infty} \eta(\sigma_n, \sigma_n) = 0 \text{ by (3.2)}. \end{aligned}$$

Consequently, $\lim_{n,m\to\infty} \eta(\sigma_n, \sigma_m) = 0$ and so $\{\sigma_n\}$ is a Cauchy sequence in $(\Im, \eta)$. From (3.6), we obtain

$$\lim_{n\to\infty} \eta(\upsilon, \sigma_n) = \eta(\upsilon, \upsilon) = 0. \tag{3.7}$$

This shows that $\{\sigma_n\}$ converges to υ in $(\Im, \eta)$ and $\eta(\upsilon, \upsilon) = 0$. By hypothesis (2) and (η_4), we have

$$\begin{aligned} \eta(\upsilon, T(\upsilon)) &\leq \eta(\upsilon, \sigma_n) + \eta(\sigma_n, T(\upsilon)) - \eta(\sigma_n, \sigma_n) \\ &= \eta(\upsilon, \sigma_n) + \alpha \left| \frac{\eta(\upsilon, T(\upsilon))(1 + \eta(\sigma_{n-1}, \sigma_n))}{1 + \eta(\sigma_{n-1}, \upsilon)} \right| + \beta |\eta(\sigma_{n-1}, \upsilon)| - \eta(\sigma_n, \sigma_n). \end{aligned}$$

Letting $n \to \infty$, we get $\eta(\upsilon, T(\upsilon)) \leq 0$ and by (2.1) we have $\eta(\upsilon, T(\upsilon)) \geq 0$. Thus, $\eta(\upsilon, T(\upsilon)) = 0$. By (CCP), we have

$$0 = \eta(\upsilon, \upsilon) \leq \eta(T(\upsilon), T(\upsilon)).$$

By (η_2), we get

$$\eta(T(\upsilon), T(\upsilon)) \leq \eta(\upsilon, T(\upsilon)) = 0.$$

These arguments imply that $\eta(T(\upsilon), T(\upsilon)) = 0$ and hence

$$\eta(\upsilon, T(\upsilon)) = \eta(T(\upsilon), T(\upsilon)) = \eta(\upsilon, \upsilon).$$

By (η_1), we have $\upsilon = T(\upsilon)$. □

Remark 4. Usually the range of a dualistic partial metric η is $(-\infty, \infty)$ but if we replace $(-\infty, \infty)$ by $[0, \infty)$, then η is identical to a partial metric P and hence Theorem 2 is applicable in the setting of partial metric space. The Theorem 2 also generalizes the results in [18, Theorem 1] [22].

Now, we provide an example of a mapping that satisfies all assumptions of the statement of Theorem 2 but it has several fixed points.

Example 5. Let $\Im = \{0, -1\}$ and consider the partially ordered dualistic partial metric space $(\Im, \eta_m, \preceq)$ with $0 \preceq 0$, $-1 \preceq -1$ and $-1 \preceq 0$. It is clear that $(\Im, \eta_m)$ is complete. Define the increasing mapping $T : \Im \to \Im$ by $T(0) = 0$ and $T(-1) = -1$. Clearly, $-1 \preceq T(-1)$. A straightforward computation shows that

$$|\eta_m(T(\sigma), T(\varsigma))| \leq \alpha \left| \frac{\eta_m(\varsigma, T(\varsigma))(1 + \eta_m(\sigma, T(\sigma)))}{1 + \eta_m(\sigma, \varsigma)} \right| + \beta|\eta_m(\sigma, \varsigma)|.$$

for all $x, y \in \Im$ with $x \preceq y$ and $\eta_m(\sigma, \varsigma) \neq -1$. Besides if a sequence $\{\sigma_n\}$ is increasing in $(\Im, \preceq)$ and $\sigma_n \to \upsilon$ then $\sigma_n \preceq \upsilon$ for all n. It is obvious that given $\sigma, \varsigma \in \Im$ we always have that $\sigma, \varsigma \preceq 0$. Nevertheless, T has two fixed points.

For the uniqueness of fixed point of T in Theorem 2, we have the following theorem.

Theorem 3. *Let $T : \Im \to \Im$ be defined on $(\Im, \preceq, \eta)$ and satisfies conditions assumed in Theorem 2. If there exists an element $\omega \in \Im$ such that it is comparable with every fixed point of T, then T has a unique fixed point in $\Im$.*

Proof. From Theorem 2, it follows that the set of fixed points of T is non-empty. We suppose that υ_1 is also a fixed point of T. Here two cases arise, first, υ and υ_1 are comparable and second, υ and υ_1 are not comparable.

In Case 1, $\upsilon \preceq \upsilon_1$ and using (3.1),

$$\begin{aligned} |\eta(\upsilon, \upsilon_1)| &= |\eta(T(\upsilon), T(\upsilon_1))| \\ &\leq \alpha \left| \frac{\eta(\upsilon_1, T(\upsilon_1))(1 + \eta(\upsilon, T(\upsilon))}{1 + \eta(\upsilon, \upsilon_1)} \right| + \beta|\eta(\upsilon, \upsilon_1)| \\ &\leq \alpha \left| \frac{\eta(\upsilon_1, \upsilon_1)(1 + \eta(\upsilon, T(\upsilon))}{1 + \eta(\upsilon, \upsilon_1)} \right| + \beta|\eta(\upsilon, \upsilon_1)| \\ &\leq \beta|\eta(\upsilon_1, \upsilon_1)|, \quad \text{since } \ \eta(\upsilon_1, \upsilon_1) = 0. \end{aligned}$$

Therefore, $(1 - \beta)|\eta(\upsilon_1, \upsilon_1)| \leq 0$, which is only true if $\eta(\upsilon_1, \upsilon_1) = 0$ and $\eta(\upsilon, \upsilon_1) = 0 = \eta(\upsilon, \upsilon) = \eta(\upsilon_1, \upsilon_1)$. Hence $\upsilon = \upsilon_1$.

In Case 2, there exists an element $\omega \in \Im$ such that it is comparable with υ, υ_1. Without any loss of generality, we assume that $\omega \preceq \upsilon$ and $\omega \preceq \upsilon_1$. Since T is nondecreasing, $T(\omega) \preceq T(\upsilon)$ and $T(\omega) \preceq T(\upsilon_1)$. Moreover, $T^{n-1}(\omega) \preceq T^{n-1}(\upsilon)$ and $T^{n-1}(\omega) \leq T^{n-1}(\upsilon_1)$. Thus

$$\begin{aligned} |\eta(T^n(\omega), T^n(\upsilon))| \leq \alpha & \left| \frac{\eta(T^{n-1}(\upsilon), T^n(\upsilon))(1 + \eta(T^{n-1}(\omega), T^n(\omega))}{1 + \eta(T^{n-1}(\upsilon), T^{n-1}(\omega))} \right| \\ & + \beta|\eta(T^{n-1}(\upsilon), T^{n-1}(\omega))|. \end{aligned}$$

This implies $|\eta(T^n(\omega), \upsilon)| \leq \beta|\eta(\upsilon, T^{n-1}(\omega))|$.
Now using (3.7), we get $\lim_{n\to\infty} \eta(\upsilon, T^n(\omega)) = 0$.

Similarly, we can show that $\lim_{n\to\infty} \eta(\upsilon_1, T^n(\omega)) = 0$. By η_4, we have

$$\begin{aligned} \eta(\upsilon_1, \upsilon) &\leq \eta(\upsilon_1, T^n(\omega)) + \eta(T^n(\omega), \upsilon) - \eta(T^n(\omega), T^n(\omega)) \\ &\leq \eta(\upsilon_1, T^n(\omega)) + \eta(T^n(\omega), \upsilon) - \eta(T^n(\omega), \upsilon) - \eta(\upsilon, T^n(\omega)) + \eta(\upsilon, \upsilon). \end{aligned}$$

Letting $n \to \infty$, we obtain that $\eta(\upsilon_1, \upsilon) \leq 0$. Now $\eta_\eta(\upsilon, \upsilon_1) = \eta(\upsilon, \upsilon_1) - \eta(\upsilon, \upsilon)$ implies that $\eta(\upsilon, \upsilon_1) \geq 0$. Hence $\eta(\upsilon, \upsilon_1) = 0$, which gives that $\upsilon = \upsilon_1$. □

To illustrate our result, we present an easy example.

Example 6. Let $\Im = (-\infty, 0]^2$. Define $\eta_m : \Im \times \Im \to \mathbb{R}$ by $\eta_m(\mathbf{x}, \mathbf{y}) = \sigma_1 m \varsigma_1 = \max\{\sigma_1, \varsigma_1\}$, where $\mathbf{x} = (\sigma_1, \sigma_2)$ and $\mathbf{y} = (\varsigma_1, \varsigma_2)$. Note that $(\Im, \eta_m)$ is a complete dualistic partial metric space. Let $T : \Im \to \Im$ be given by

$$T(\mathbf{x}) = \frac{\mathbf{x}}{2} \text{ for all } \mathbf{x} \in \Im.$$

In $\Im$, we define the relation $\preceq$ in the following way:

$$\mathbf{x} \preceq \mathbf{y} \text{ if and only if } \sigma_1 \leq \varsigma_1, \text{ where } \mathbf{x} = (\sigma_1, \sigma_2) \text{ and } \mathbf{y} = (\varsigma_1, \varsigma_2).$$

Clearly, $\preceq$ is a partial order on $\Im$ and T is a nondecreasing mapping and satisfies (CCP) with respect to $\preceq$. Moreover, $(-1, 0) \preceq T(-1, 0)$.

We shall show that for all $\mathbf{x}, \mathbf{y} \in \Im$ with $\mathbf{x} \preceq \mathbf{y}$, (3.1) is satisfied. For this, consider

$$\begin{aligned} |\eta_m(T(\mathbf{x}), T(\mathbf{y}))| &= \left|\eta_m\left(\frac{\mathbf{x}}{2}, \frac{\mathbf{y}}{2}\right)\right| = \left|\frac{\varsigma_1}{2}\right| \text{ for all } \sigma_1 \leq \varsigma_1, \\ |\eta_m(\mathbf{x}, T(\mathbf{x}))| &= \left|\eta_m\left(\mathbf{x}, \frac{\mathbf{x}}{2}\right)\right| = \left|\frac{\sigma_1}{2}\right|, \\ |\eta_m(\mathbf{y}, T(\mathbf{y}))| &= \left|\eta_m\left(\mathbf{y}, \frac{\mathbf{y}}{2}\right)\right| = \left|\frac{\varsigma_1}{2}\right|, \\ |\eta_m(\mathbf{x}, \mathbf{y})| &= |\varsigma_1| \text{ for all } \sigma_1 \leq \varsigma_1. \end{aligned}$$

Set $\alpha = \frac{1}{3}, \beta = \frac{1}{2}$ and for all $x, y \in \Im$, we find that

$$|\eta_m(T(\mathbf{x}), T(\mathbf{y}))| \leq \alpha \left|\frac{\eta_m(\mathbf{y}, T(\mathbf{y}))(1 + \eta_m(\mathbf{x}, T(\mathbf{x}))}{1 + \eta_m(\mathbf{x}, \mathbf{y})}\right| + \beta|\eta_m(\mathbf{x}, \mathbf{y})|$$

holds for $\mathbf{x} \preceq \mathbf{y}$ if and only if $6|\varsigma_1||1 + \varsigma_1| \leq |\varsigma_1|(2 + |\sigma_1|) + 6|\varsigma_1||1 + \varsigma_1|$.

Thus, all the conditions of Theorem 2 are satisfied. Moreover, $(0, 0)$ is a fixed point of T.

Next example also explains main result and shows its significance.

Example 7. Let $\Im = (-\infty, -2]^2 \cup \{O\}$. Define $\eta : \Im \times \Im \to \mathbb{R}$ by

$$\eta(\sigma, \varsigma) = \begin{cases} d(\sigma, \varsigma) = \sqrt{(\sigma_1 - \sigma_2)^2 + (\varsigma_1 - \varsigma_2)^2} & \text{if } \sigma \neq \varsigma, \\ -2 & \text{if } \sigma = \varsigma. \end{cases}$$

where $\sigma = (\sigma_1, \sigma_2)$ and $\varsigma = (\varsigma_1, \varsigma_2)$. Then $(\Im, \eta)$ is a complete dualistic partial metric space. Let $T : \Im \to \Im$ be given by

$$T(\sigma) = \frac{\sigma}{2} \text{ for all } \sigma \in \Im.$$

In $\Im$, we define the relation $\preceq$ in the following way

$$\sigma \preceq \varsigma \text{ if and only if } \sigma_1 \leq \varsigma_1 \text{ and } \sigma_2 \leq \varsigma_2.$$

Clearly, $\preceq$ is a partial order on $\Im$ and T is a nondecreasing mapping and satisfies (CCP) with respect to $\preceq$. Moreover, $(-1, 0) \preceq T(-1, 0)$. Also we have the following information for all $\sigma \neq \varsigma$ (either $\sigma \preceq \varsigma$ or $\varsigma \preceq \sigma$)

$$\begin{aligned} \eta(T(\sigma), T(\varsigma)) &= \eta\left(\frac{\sigma}{2}, \frac{\varsigma}{2}\right) = d\left(\frac{\sigma}{2}, \frac{\varsigma}{2}\right) = \frac{1}{2}d(\sigma, \varsigma), \\ \eta(\sigma, T(\sigma)) &= \eta\left(\sigma, \frac{\sigma}{2}\right) = d\left(\sigma, \frac{\sigma}{2}\right) = \frac{1}{2}d(\sigma, O); O = (0, 0), \\ \eta(\varsigma, T(\varsigma)) &= \eta\left(\varsigma, \frac{\varsigma}{2}\right) = d\left(\varsigma, \frac{\varsigma}{2}\right) = \frac{1}{2}d(\varsigma, O), \\ \eta(\sigma, \varsigma) &= d(\sigma, \varsigma). \end{aligned}$$

Thus, the contractive condition (3.1) takes the form

$$\frac{1}{2}d(\sigma, \varsigma) \leq \alpha \left(\frac{\frac{1}{2}d(\varsigma, O)\left(1 + \frac{1}{2}d(\sigma, O)\right)}{1 + d(\sigma, \varsigma)}\right) + \beta d(\sigma, \varsigma). \tag{3.8}$$

For $\beta = \frac{2}{3}$, there exists α such that $\alpha + \beta < 1$ which satisfies inequality (3.8).

For all $\sigma = \varsigma$ we have

$$\begin{aligned}\eta(T(\sigma), T(\varsigma)) &= \eta\left(\frac{\sigma}{2}, \frac{\sigma}{2}\right) = -2,\\ \eta(\sigma, T(\sigma)) &= \eta\left(\sigma, \frac{\sigma}{2}\right) = d\left(\sigma, \frac{\sigma}{2}\right) = \frac{1}{2}d(\sigma, O); O = (0,0),\\ \eta(\varsigma, T(\varsigma)) &= \eta\left(\sigma, \frac{\sigma}{2}\right) = d\left(\sigma, \frac{\sigma}{2}\right) = \frac{1}{2}d(\sigma, O),\\ \eta(\sigma, \varsigma) &= \eta(\sigma, \sigma) = -2.\end{aligned}$$

Here, the contractive condition (3.1) takes the following form

$$\frac{8(1-\beta)}{\alpha} \leq d(\sigma, O)(2 + d(\sigma, O)). \tag{3.9}$$

It can be noticed that there exist α and β with $\alpha + \beta < 1$ satisfying (3.9) for all $\sigma \in \Im$. Thus T is the required self-mapping having fixed point O and which fulfills requirements of Theorem 2.

We note that under the dualistic partial metric defined in this example, fixed point theorem established by Oltra and Valero in [11] fails to have fixed point. Indeed, for all $\sigma = \varsigma$

$$2 = |\eta(T(\sigma), T(\sigma))| \leq c|\eta(\sigma, \sigma)| = 2c,$$

which is a contradiction to definition of c used in Said theorem [11].

Following example emphasizes use of absolute value function in contractive condition (3.1).

Example 8. Define the mapping $T_0 : \mathbb{R}^+ \to \mathbb{R}$ by

$$T_0(\sigma) = \begin{cases} 0 & \text{if } \sigma > 1,\\ -5 & \text{if } \sigma = 1.\end{cases}$$

Clearly, for all comparable $\sigma, \varsigma \in \mathbb{R}$, the following contractive condition without absolute value function is satisfied

$$\eta_m(T_0(\sigma), T_0(\varsigma)) \leq \frac{\alpha\eta_m(\varsigma, T_0(\varsigma)) \cdot (1 + \eta_m(\sigma, T_0(\sigma)))}{1 + \eta_m(\sigma, \varsigma)} + \beta\eta_m(\sigma, \varsigma),$$

where η_m is a complete dualistic partial metric on $\mathbb{R}$. Here, T has no fixed point. Thus a fixed point free mapping satisfies this contractive condition.

On the other hand, for all $0 < \alpha + \beta < 1$, we have

$$5 = |\eta_m(-5, -5)| = |\eta_m(T_0(1), T_0(1))| > \alpha\left|\frac{\eta_m(1, T_0(1)) \cdot (1 + \eta(1, T_0(1)))}{1 + \eta_m(1,1)}\right| + \beta|\eta_m(1,1)|.$$

Thus the contractive condition (3.1) does not hold. This situation happens always in dualistic partial metric space and that is why we place absolute value function in contractive condition taken under dualistic partial metric.

If we set $\alpha = 0$ in Theorem 2, then we get the following result as a corollary of our result.

Corollary 1. [11] *Let $(\Im, \eta)$ be a complete dualistic partial metric space and $T : \Im \to \Im$ be a mapping such that there exists $c \in [0, 1[$ satisfying*

$$|\eta(T(\sigma), T(\varsigma))| \leq c|\eta(\sigma, \varsigma)|$$

for all $\sigma, \varsigma \in \Im$ with $\sigma \preceq \varsigma$. Then T has a unique fixed point $\sigma^\star \in \Im$. Moreover, $\eta(\sigma^\star, \sigma^\star) = 0$ and the Picard iterative sequence $\{T^n(\sigma_0)\}_{n\in\mathbb{N}}$ converges to $\sigma^\star$ with respect to $\tau(d^s_\eta)$ for every $x \in \Im$.

Remark 5. *Corollary 1 generalizes the result presented by Oltra and Valero in [11] to ordered dualistic partial metric spaces.*

Corollary 2. *Let the self-mapping T defined on an ordered complete dualistic partial metric space $(\Im, \preceq, \eta)$ be nondecreasing and there exists $\alpha \in [0, 1)$ such that,*

$$|\eta(T(\sigma), T(\varsigma))| \leq \left| \frac{\alpha\eta(\varsigma, T(\varsigma))(1 + \eta(\sigma, T(\sigma)))}{1 + \eta(\sigma, \varsigma)} \right|$$

for all $\sigma, \varsigma \in \mathcal{D}$. Moreover, if

(1) *there exists $\sigma_0 \in \Im$ such that $\sigma_0 \preceq T(\sigma_0)$,*
(2) *if $\{\sigma_n\}$ is a nondecreasing sequence in $\Im$ such that $\{\sigma_n\} \to \upsilon$, then $\sigma_n \preceq \upsilon$ for all $n \in \mathbb{N}$,*

then T has a fixed point υ such that $\eta(\upsilon, \upsilon) = 0$.

Proof. Set $\beta = 0$ in the statement of Theorem 2. □

4 Application

In this section, we apply main result to show the existence of the solution of functional equation. We introduce some notations for the sake of convenience.
Let state space, decision space and space of bounded functions be denoted by $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{B}(\mathcal{S})$ respectively. Also we introduce the following functions
$g: \mathcal{S} \times \mathcal{W} \to \mathbb{R}$,
$F: \mathcal{S} \times \mathcal{W} \times \mathbb{R} \to \mathbb{R}$,
$\phi: \mathcal{S} \times \mathcal{W} \to \mathcal{S}$.

In the following, we shall prove the existence and uniqueness of solution of functional equation (4.1) appearing in dynamic programming (for example, see [28])

$$u(\sigma) = \sup_{\varsigma \in W}\{g(\sigma, \varsigma) + F(\sigma, \varsigma, u(\phi(\sigma, \varsigma)))\} \text{ for all } \sigma \in \mathcal{S}. \tag{4.1}$$

Define order $\preceq$ on $\mathcal{B}(\mathcal{S})$ by $u \preceq v \iff u(\sigma) \leq v(\sigma)$. Then we observe that $(\mathcal{B}(\mathcal{S}), \preceq, \eta)$ is a complete ordered dualistic partial metric space and the distance function in $B(S)$ is defined by

$$d(u, v) = \sup_{\sigma \in \mathcal{S}} |u(\sigma) - v(\sigma)|, \text{ for all } u, v \in \mathcal{B}(\mathcal{S}),$$

where as for dualistic partial metric the distance function is given by

$$\eta(u, v) = d(u, v) + c, \text{ for all } u, v \in \mathcal{B}(\mathcal{S}) \text{ and } c \in \mathbb{R}.$$

Following two lemmas will be helpful in the sequel.

Lemma 2. [28] Let $G, H : \mathcal{S} \to \mathbb{R}$ be two bounded functions. Then

$$|\sup_{\sigma \in \mathcal{S}} G(\sigma) - \sup_{\sigma \in \mathcal{S}} H(\sigma)| \leq \sup_{\sigma \in \mathcal{S}} |G(\sigma) - H(\sigma)|.$$

Lemma 3. [28] Assume that

(1) g, F are bounded functions;
(2) there exists $k > 0$ such that for all $t, r \in \mathbb{R}$, $\sigma \in \mathcal{S}$ and $\varsigma \in \mathcal{W}$

$$|F(\sigma, \varsigma, t) - F(\sigma, \varsigma, r)| \leq k|t - r|.$$

Then the operator $R : \mathcal{B}(\mathcal{S}) \to \mathcal{B}(\mathcal{S})$ defined by

$$(Ru)(\sigma) = \sup_{\varsigma \in \mathcal{W}}\{g(\sigma, \varsigma) + F(\sigma, \varsigma, u(\phi(\sigma, \varsigma)))\}$$

is well-defined.

Now we present our next result.

Theorem 4. *Let all the conditions of Lemma 3 be satisfied and*

$$|F(\sigma,\varsigma,u)-F(\sigma,\varsigma,v)|+|c|\leq\alpha\left|\frac{\eta(u,Ru)(1+\eta(v,Rv))}{1+\eta(u,v)}\right|+\beta|\eta(u,v)| \tag{4.2}$$

for all $u,v\in\mathcal{B}(\mathcal{S})$ *with* $u\preceq v$ *and* $\eta(u,v)\neq-1$. *Then the functional equation (4.1) has a unique solution.*

Proof. Let $R:\mathcal{B}(\mathcal{S})\to\mathcal{B}(\mathcal{S})$ be an operator as defined in Lemma 3. Then R is continuous and nondecreasing with respect to $\preceq$. We shall show that R satisfies the contractive condition (3.1). Indeed, by Lemma 2, for all $u,v\in\mathcal{B}(\mathcal{S})$ with $u\preceq v$,

$$\begin{aligned}|(Ru)(\sigma)-(Rv)(\sigma)| &= |\sup_{\varsigma\in\mathcal{W}}\{g(\sigma,\varsigma)+F(\sigma,\varsigma,u(\phi(\sigma,\varsigma)))\}-\sup_{\varsigma\in\mathcal{W}}\{g(\sigma,\varsigma)+F(\sigma,\varsigma,v(\phi(\sigma,\varsigma)))\}|\\ &\leq \sup_{\varsigma\in\mathcal{W}}|g(\sigma,\varsigma)+F(\sigma,\varsigma,u(\phi(\sigma,\varsigma)))-g(\sigma,\varsigma)-F(\sigma,\varsigma,v(\phi(\sigma,\varsigma)))|\\ &\leq \sup_{\varsigma\in\mathcal{W}}|F(\sigma,\varsigma,u(\phi(\sigma,\varsigma)))-F(\sigma,\varsigma,v(\phi(\sigma,\varsigma)))|.\end{aligned}$$

Therefore,

$$\begin{aligned}|\eta(Ru,Rv)| &= |\sup_{\sigma\in\mathcal{S}}|(Ru)(\sigma)-(Rv)(\sigma)+c|\\ &\leq \sup_{\sigma\in\mathcal{S}}|(Ru)(\sigma)-(Rv)(\sigma)|+|c|\\ &\leq \sup_{\varsigma\in\mathcal{W}}|F(\sigma,\varsigma,u(\phi(\sigma,\varsigma)))-F(\sigma,\varsigma,v(\phi(\sigma,\varsigma)))|+|c|.\\ &\leq \alpha\left|\frac{\eta(u,Ru)(1+\eta(v,Rv))}{1+\eta(u,v)}\right|+\beta|\eta(u,v)|.\end{aligned}$$

Hence R satisfies all the conditions of Theorem 2. Thus there exists a unique solution $u_0\in\mathcal{B}(\mathcal{S})$ of (4.1) such that $Ru_0=u_0$. □

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