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ADDITIVE s–FUNCTIONAL INEQUALITIES AND PARTIAL MULTIPLIERS IN BANACH ALGEBRAS

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Abstract. In this paper, we solve the additive s-functional inequalities

$$\|f(x+y-z)-f(x)-f(y)+f(z)\| \leqslant \|s(f(x-y)+f(y-z)-f(x-z))\|, \qquad (0.1)$$

where s is a fixed nonzero complex number with $|s|<1$, and

$$\|f(x-y)+f(y-z)-f(x-z)\| \leqslant \|s(f(x+y-z)-f(x)-f(y)+f(z))\|, \qquad (0.2)$$

where s is a fixed nonzero complex number with $|s|<1$.

Furthermore, we prove the Hyers-Ulam stability of the additive s-functional inequalities (0.1) and (0.2) in complex Banach spaces. This is applied to investigate partial multipliers in Banach $*$-algebras and unital C^*-algebras, associated with the additive s-functional inequalities (0.1) and (0.2).

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [20] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [4] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leqslant \|f(x+y)\| \qquad (1.1)$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

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See also [18]. Fechner [2] and Gilányi [5] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park [14, 15] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [8, 9, 10, 11, 12, 16]).

In [19], Taghavi introduced partial multipliers in complex Banach $*$-algebras as follows.

DEFINITION 1.1. Let A be a complex Banach $*$-algebra. A $\mathbb{C}$-linear mapping $P : A \to A$ is called a *partial multiplier* if P satisfies

$$P \circ P(xy) = P(x)P(y)$$
$$P(x^*) = P(x)^*$$

for all $x, y \in A$.

This paper is organized as follows: In Section 2, we solve the additive s-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive s-functional inequality (0.1) in complex Banach spaces. In Section 3, we solve the additive s-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive s-functional inequality (0.2) in complex Banach spaces. In Section 4, we investigate partial multipliers in C^*-algebras associated with the additive s-functional inequalities (0.1) and (0.2).

Throughout this paper, let X be a complex normed space with norm $\|\cdot\|$, Y a complex Banach space with norm $\|\cdot\|$ and A a complex Banach $*$-algebra with norm $\|\cdot\|$. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Additive s-functional inequality (0.1)

We solve and investigate the additive s-functional inequality (0.1) in complex normed spaces.

LEMMA 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|f(x+y-z) - f(x) - f(y) + f(z)\| \leqslant \|s(f(x-y) + f(y-z) - f(x-z))\| \quad (2.1)$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).
Letting $x = y$ and $z = 0$ in (2.1), we get $f(2x) = 2f(x)$ for all $x \in X$.
Letting $y = -x$ and $z = 0$ in (2.1), we get

$$\|f(x) + f(-x)\| \leqslant \|s(f(2x) + f(-x) - f(x))\| = \|s(f(x) + f(-x))\|$$

and so $f(-x) = -f(x)$ for all $z \in X$, since $|s| < 1$.

Letting $x = 0$ in (2.1), we get

$$\|f(y-z)-f(y)+f(z)\| \leqslant \|s(f(-y)+f(y-z)-f(-z))\| = \|s(f(y-z)-f(y)+f(z))\|$$

and so $f(y-z) = f(y) - f(z)$ for all $y,z \in X$, since $|s| \leqslant 1$. So $f(y+z) = f(y) + f(z)$ for all $y,z \in X$. $\square$

We prove the Hyers-Ulam stability of the additive s-functional inequality (2.1) in complex Banach spaces.

THEOREM 2.2. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} \|f(x+y-z)-f(x)-f(y)+f(z)\| &\leqslant \|s(f(x-y)+f(y-z)-f(x-z))\| \\ &\quad +\theta(\|x\|^r+\|y\|^r+\|z\|^r) \end{aligned} \tag{2.2}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{2\theta}{2^r-2}\|x\|^r \tag{2.3}$$

for all $x \in X$.

Proof. Letting $z = 0$ and $y = x$ in (2.2), we get

$$\|f(2x)-2f(x)\| \leqslant 2\theta\|x\|^r \tag{2.4}$$

for all $x \in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leqslant \frac{2}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leqslant \frac{2}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.5) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.2) that

$$\begin{aligned}
&\|A(x+y-z)-A(x)-A(y)+A(z)\| \\
&= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y-z}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)+f\left(\frac{z}{2^n}\right)\right)\right\| \\
&\leqslant \lim_{n\to\infty}\left\|2^n s\left(f\left(\frac{x-y}{2^n}\right)+f\left(\frac{y-z}{2^n}\right)-f\left(\frac{x-z}{2^n}\right)\right)\right\| \\
&\quad + \lim_{n\to\infty}\frac{2^n}{2^{rn}}\theta(\|x\|^r+\|y\|^r+\|z\|^r) \leqslant \|s(A(x-y)+A(y-z)-A(x-z))\|
\end{aligned}$$

for all $x,y,z\in X$. So

$$\|A(x+y-z)-A(x)-A(y)+A(z)\| \leqslant \|s(A(x-y)+A(y-z)-A(x-z))\|$$

for all $x,y,z\in X$. By Lemma 2.1, the mapping $A: X\to Y$ is additive.

Now, let $T: X\to Y$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right)-2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leqslant \left\|2^q A\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leqslant \frac{4\theta}{2^r-2}\frac{2^q}{2^{qr}}\|x\|^r,
\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in X$. So we can conclude that $A(x)=T(x)$ for all $x\in X$. This proves the uniqueness of A, as desired. □

THEOREM 2.3. *Let $r<1$ and θ be nonnegative real numbers and let $f: X\to Y$ be an odd mapping satisfying* (2.2). *Then there exists a unique additive mapping $A: X\to Y$ such that*

$$\|f(x)-A(x)\| \leqslant \frac{2\theta}{2-2^r}\|x\|^r \tag{2.6}$$

for all $x\in X$.

Proof. It follows from (2.4) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \leqslant \theta\|x\|^r$$

for all $x\in X$. Hence

$$\begin{aligned}
\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\
&\leqslant \sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|^r
\end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.6).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

3. Additive s-functional inequality (0.2)

We solve and investigate the additive s-functional inequality (0.2) in complex normed spaces.

LEMMA 3.1. *If a mapping $f : X \to Y$ satisfies*

$$\|f(x-y)+f(y-z)-f(x-z)\| \leqslant \|s(f(x+y-z)-f(x)-f(y)+f(z))\| \quad (3.1)$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1). Letting $x = y = z = 0$ in (3.1), we get $f(0) = 0$.

Letting $x = z = 0$ in (3.1), we get $\|f(-y)+f(y)\| \leqslant 0$ and so $f(-y) = -f(y)$ for all $y \in X$.

Letting $z = x+y$ in (3.1), we get

$$\|f(x-y)+f(-x)-f(-y)\| \leqslant \|s(f(x+y)-f(x)-f(y))\| \quad (3.2)$$

for all $x, y \in X$. Replacing y and $-y$ in (3.2), we obtain

$$\|f(x+y)-f(x)-f(y)\| \leqslant \|s(f(x-y)-f(x)+f(y))\| \quad (3.3)$$

for all $x, y \in X$. It follows from (3.2) and (3.3) that $f(x+y) = f(x)+f(y)$ for all $x, y \in X$, since $|s| \leqslant 1$. So f is additive. $\square$

We prove the Hyers-Ulam stability of the additive s-functional inequality (3.1) in complex Banach spaces.

THEOREM 3.2. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned}\|f(x-y)+f(y-z)-f(x-z)\| &\leqslant \|s(f(x+y-z)-f(x)-f(y)+f(z))\| \\ &\quad +\theta(\|x\|^r+\|y\|^r+\|z\|^r) \quad (3.4)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{2\theta}{2^r-2}\|x\|^r \quad (3.5)$$

for all $x \in X$.

Proof. Since f is odd, $f(0)=0$.

Letting $z=0$ and $y=-x$ in (3.4), we get

$$\|f(2x)-2f(x)\| \leqslant 2\theta\|x\|^r \tag{3.6}$$

for all $x\in X$. So

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leqslant \sum_{j=l+1}^{m}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned}\tag{3.7}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (3.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x\in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A: X\to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x\in X$. Moreover, letting $l=0$ and passing to the limit $m\to\infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.2. □

THEOREM 3.3. *Let $r<1$ and θ be nonnegative real numbers and let $f: X\to Y$ be an odd mapping satisfying* (3.4)*. Then there exists a unique additive mapping A : $X\to Y$ such that*

$$\|f(x)-A(x)\| \leqslant \frac{2\theta}{2-2^r}\|x\|^r \tag{3.8}$$

for all $x\in X$.

Proof. It follows from (3.6) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \leqslant \theta\|x\|^r$$

for all $x\in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leqslant \sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|^r \end{aligned}\tag{3.9}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A: X\to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

4. Partial multipliers in C^*-algebras

In this section, we investigate partial multipliers in complex Banach $*$-algebras and unital C^*-algebras associated with the additive ρ-functional inequalities (2.1) and (3.1).

THEOREM 4.1. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be an odd mapping such that*

$$\begin{aligned} \|f(\mu(x+y-z)) - \mu(f(x)+f(y)-f(z))\| &\leqslant \|s(f(x-y)+f(y-z)-f(x-z))\| \\ &\quad +\theta(\|x\|^r+\|y\|^r+\|z\|^r) \end{aligned} \tag{4.1}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in A$. Then there exists a unique $\mathbb{C}$-linear mapping $P : A \to A$ such that

$$\|f(x) - P(x)\| \leqslant \frac{2\theta}{2^r - 2}\|x\|^r \tag{4.2}$$

for all $x \in A$.

If, in addition, the mapping $f : A \to A$ satisfies $f(2x) = 2f(x)$ and

$$\|f \circ f(xy) - f(x)f(y)\| \leqslant \theta(\|x\|^r + \|y\|^r), \tag{4.3}$$

$$\|f(x^*) - f(x)^*\| \leqslant \theta\|x\|^r \tag{4.4}$$

for all $x, y \in A$, then the mapping f is a partial multiplier.

Proof. Let $\mu = 1$ in (4.1). By Theorem 2.2, there is a unique additive mapping $P : A \to A$ satisfying (4.2) defined by

$$P(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

Letting $y = z = 0$ in (4.1), we get

$$\|f(\mu x) - \mu f(x)\| \leqslant \theta\|x\|^r$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$\|P(\mu x) - \mu P(x)\| = \lim_{n\to\infty} 2^n \left\|f\left(\mu\frac{x}{2^n}\right) - f\left(\mu\frac{x}{2^n}\right)\right\| \leqslant \lim_{n\to\infty} \frac{2^n}{2^{rn}}\theta\|x\|^r = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $P(\mu x) = \mu P(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $P : A \to A$ is $\mathbb{C}$-linear.

If $f(2x)=2f(x)$ for all $x\in A$, then we can easily show that $P(x)=f(x)$ for all $x\in A$. It follows from (4.3) that

$$\begin{aligned}\|f\circ f(xy)-f(x)f(y)\| &= \|P\circ P(xy)-P(x)P(y)\|\\ &= \lim_{n\to\infty}4^n\left\|f\circ f\left(\frac{xy}{2^n\cdot 2^n}\right)-f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)\right\|\\ &\leqslant \lim_{n\to\infty}\frac{4^n\theta}{2^{rn}}(\|x\|^r+\|y\|^r)=0\end{aligned}$$

for all $x,y\in A$. Thus

$$f\circ f(xy)=f(x)f(y)$$

for all $x,y\in A$.

It follows from (4.4) that

$$\begin{aligned}\|f(x^*)-f(x)^*\| = \|P(x^*)-P(x)^*\| &= \lim_{n\to\infty}2^n\left\|f\left(\frac{x^*}{2^n}\right)-f\left(\frac{x}{2^n}\right)^*\right\|\\ &\leqslant \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}(\|x\|^r+\|x\|^r)=0\end{aligned}$$

for all $x\in A$. Thus

$$f(x^*)=f(x)^*$$

for all $x\in A$. Hence the mapping $f:A\to A$ is a partial multiplier. □

THEOREM 4.2. *Let $r<1$ and θ be nonnegative real numbers, and let $f:A\to A$ be an odd mapping satisfying (4.1). Then there exists a unique $\mathbb{C}$-linear mapping $P:A\to A$ such that*

$$\|f(x)-P(x)\|\leqslant\frac{2\theta}{2-2^r}\|x\|^r \tag{4.5}$$

for all $x\in A$.

If, in addition, the mapping $f:A\to A$ satisfies $f(2x)=2f(x)$ for all $x\in A$, (4.3) and (4.4), then the mapping f is a partial multiplier.

Proof. The proof is similar to the proof of Theorem 4.1. □

Similarly, we can obtain the following results.

THEOREM 4.3. *Let $r>2$ and θ be nonnegative real numbers, and let $f:A\to A$ be an odd mapping such that*

$$\begin{aligned}\|f(\mu(x-y))+f(\mu(y-z))-\mu f(x-z)\| \leqslant{}& \|s(f(x+y-z)-f(x)-f(y)+f(z))\|\\ &+\theta(\|x\|^r+\|y\|^r+\|z\|^r)\end{aligned} \tag{4.6}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$. Then there exists a unique $\mathbb{C}$-linear mapping $P:A\to A$ such that

$$\|f(x)-P(x)\|\leqslant\frac{2\theta}{2^r-2}\|x\|^r \tag{4.7}$$

for all $x \in A$.

If, in addition, the mapping $f : A \to A$ *satisfies* $f(2x) = 2f(x)$ *for all* $x \in A$, (4.3) *and* (4.4)*, then the mapping* f *is a partial multiplier.*

THEOREM 4.4. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be an odd mapping satisfying* (4.6)*. Then there exists a unique* $\mathbb{C}$*-linear mapping* $P : A \to A$ *such that*

$$\|f(x) - P(x)\| \leqslant \frac{2\theta}{2 - 2^r}\|x\|^r \tag{4.8}$$

for all $x \in A$.

If, in addition, the mapping $f : A \to A$ *satisfies* $f(2x) = 2f(x)$ *for all* $x \in A$, (4.3) *and* (4.4)*, then the mapping* f *is a partial multiplier.*

From now on, assume that A is a unital C^*-algebra with norm $\|\cdot\|$ and unitary group $U(A)$.

THEOREM 4.5. *Let* $r > 2$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be an odd mapping satisfying* (4.1)*. Then there exists a unique* $\mathbb{C}$*-linear mapping* $P : A \to A$ *satisfying* (4.2)*.*

If, in addition, the mapping $f : A \to A$ *satisfies* $f(2x) = 2f(x)$ *for all* $x \in A$ *and*

$$\|f \circ f(uv) - f(u)f(v)\| \leqslant 2\theta, \tag{4.9}$$

$$\|f(u^*) - f(u)^*\| \leqslant \theta \tag{4.10}$$

for all $u, v \in U(A)$*, then the mapping* f *is a partial multiplier.*

Proof. By the same reasoning as in the proof of Theorem 4.1, there is a unique $\mathbb{C}$-linear mapping $P : A \to A$ satisfying (4.2) defined by

$$P(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

If $f(2x) = 2f(x)$ for all $x \in A$, then we can easily show that $P(x) = f(x)$ for all $x \in A$.

By the same reasoning as in the proof of Theorem 4.1, $f \circ f(uv) = f(u)f(v)$ and $f(u^*) = f(u)^*$ for all $u, v \in U(A)$.

Since f is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C},\ u_j \in U(A))$,

$$\begin{aligned} f(x^*) &= f(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} f(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} f(u_j)^* = (\sum_{j=1}^m \lambda_j f(u_j))^* = f(\sum_{j=1}^m \lambda_j u_j)^* \\ &= f(x)^* \end{aligned}$$

for all $x \in A$.

Since f and $f\circ f$ are $\mathbb{C}$-linear and each $x,y\in A$ is a finite linear combination of unitary elements, i.e., $x=\sum_{j=1}^{m}\lambda_j u_j$ $(\lambda_j\in\mathbb{C},\ u_j\in U(A))$ and $y=\sum_{k=1}^{n}\beta_k v_k$ $(\beta_k\in\mathbb{C},$ $v_k\in U(A))$,

$$f\circ f(xy)=f\circ f(\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_j\beta_k u_j v_k)=\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_j\beta_k f\circ f(u_j v_k)=\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_j\beta_k f(u_j)f(v_k)$$
$$=f(\sum_{j=1}^{m}\lambda_j u_j)f(\sum_{k=1}^{n}\beta_k v_k)=f(x)f(y)$$

for all $x,y\in A$.

Therefore, the mapping $f:A\to A$ is a partial multiplier. □

THEOREM 4.6. *Let $r<1$ and θ be nonnegative real numbers, and let $f:A\to A$ be an odd mapping satisfying (4.1). Then there exists a unique $\mathbb{C}$-linear mapping $P:A\to A$ satisfying (4.8).*

If, in addition, the mapping $f:A\to A$ satisfies $f(2x)=2f(x)$ for all $x\in A$, (4.9) and (4.10), then the mapping f is a partial multiplier.

Proof. The proof is similar to the proof of Theorem 4.5. □

Similarly, we can obtain the following results.

THEOREM 4.7. *Let $r>2$ and θ be nonnegative real numbers, and let $f:A\to A$ be an odd mapping satisfying (4.6). Then there exists a unique $\mathbb{C}$-linear mapping $P:A\to A$ satisfying (4.7).*

If, in addition, the mapping $f:A\to A$ satisfies $f(2x)=2f(x)$ for all $x\in A$, (4.9) and (4.10), then the mapping f is a partial multiplier.

THEOREM 4.8. *Let $r<1$ and θ be nonnegative real numbers, and let $f:A\to A$ be an odd mapping satisfying (4.6). Then there exists a unique $\mathbb{C}$-linear mapping $P:A\to A$ satisfying (4.8).*

If, in addition, the mapping $f:A\to A$ satisfies $f(2x)=2f(x)$ for all $x\in A$, (4.9) and (4.10), then the mapping f is a partial multiplier.

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