

Acoustic black holes in curved spacetime and the emergence of analogue Minkowski spacetime

Xian-Hui Ge,^{1,†} Mikio Nakahara,^{1,2,*} Sang-Jin Sin,^{3,‡} Yu Tian,^{4,§} and Shao-Feng Wu^{1,||}

¹College of Science, Shanghai University, Shanghai 200444, China

²Research Institute for Science and Technology, Kindai University, Higashi-Osaka, 577-8502, Japan

³Department of Physics, Hanyang University, Seoul 133-791, Korea

⁴School of Physics, University of Chinese Academy of Sciences, Beijing, 100049, China



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Gravity is not only able to be mimicked in flat spacetimes, but also in curved spacetimes. We study analogue gravity models in curved spacetime by considering the relativistic Gross-Pitaevskii theory and Yang-Mills theory in the fixed background spacetime geometry. The results show that acoustic metrics can be emergent from curved spacetimes yielding a Hadamard product of a real metric tensor and an analogue metric tensor. Taking quantum vortices as test particles, we evaluate their released energy ratio during the “gravitational binding.” The $(2 + 1)$ -dimensional flat Minkowski metric is derived from the $(3 + 1)$ -dimensional anti-de Sitter space by considering perturbations of the Yang-Mills field, which implies that Minkowski spacetime can also be simulated and the derivations presented here have some deep connections with the holographic principle.

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I. INTRODUCTION

Analogue models of gravity have become an active field in recent years because they provide important connections between astrophysical phenomena, such as the Hawking radiation of black holes [1], the Penrose process of rotating black holes, and the Kibble mechanism of topological defect deformation in the early Universe, with tabletop experiments. The recent experimental realizations of acoustic black holes reported were conducted in a Bose-Einstein condensate [2–4] and optical medium [5], and the observation of the Kibble-Zureck mechanism was proposed in various systems [6–11].

Being able to mimic kinematic aspects of general relativity only, the analogue models of gravity were originally proposed to explore the quantum nature of gravity. In 1968, Sakharov argued that gravity might be induced since it would not be fundamental from the particle physics point of view [12]. Although the Weinberg-Witten no-go theorem

states that a spin-2 graviton cannot be a composite particle in a relativistic quantum field theory [13], the holographic principle and the later discovered AdS/CFT correspondence build a bridge between a $(d + 1)$ -dimensional gravity theory and a d -dimensional quantum field theory. Therefore, via the AdS/CFT correspondence, gravity can be studied in a system in the absence of gravity. The gauge/gravity duality and analogue gravity are quite different but they share one property: spacetime is emergent and gravity may not be the fundamental force. In desktop systems, one can mimic black holes by using the fluid mechanics (i.e., acoustic black holes) in the absence of gravity. In most of the previous literature, the analogue black hole metric was derived from flat Minkowski spacetime. However, acoustic black holes can be embedded in a curved spacetime in general. What is more, a real black hole surrounded by an acoustic horizon is equivalent to a dumb black hole: at the acoustic horizon sound cannot escape while at the event horizon light cannot escape.¹ In this paper, we are going to reverse the logic by first producing an analogue gravity metric in generalized background and then try to mimic the Minkowski metric from curved spacetimes.

We first derive the analogue gravity metric in curved spacetime by considering a relativistic Gross-Pitaevskii equation and then the Yang-Mills equation. The reason comes from the following observations:

¹Sound waves can propagate only in a medium. Outside the Schwarzschild black hole in vacuum, sound waves are not well defined. Here, we consider a black hole immersed in the cosmic microwave background so that sound waves propagate.

*Corresponding author.
nakahara@shu.edu.cn
†gexh@shu.edu.cn
‡sjsin@hanyang.ac.kr
§ytian@ytian@ucas.ac.cn
||sfwu@shu.edu.cn

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- (i) A real black hole, the Schwarzschild black hole, for example, in the bath of the cosmological microwave background is a natural candidate for an acoustic black hole. At the location $r_s = 6GM$, the escape velocity is $v_s = \frac{1}{\sqrt{3}}$ and in the region $r < r_s$, the relativistic sound waves of thermal radiation cannot escape² [14].
- (ii) For astrophysical black holes, accretion disks around black holes at the centers of galaxies play a central role in explaining active galactic nuclei such as quasars and those include the most energetic steady sources of radiation in the Universe. Relativistic and transonic accretion onto astrophysical black holes is a unique example of analogue gravity realized in nature [15–18].
- (iii) In addition to accretion disks, black holes may be surrounded by some quantum superfluids. There have long been some proposals that dark matter might be some kind of superfluid [19]. The transonic accretion and the condensation of those quantum fluids also provide a scenario realizing analogue gravity.

This paper is organized as follows. Firstly, we consider a superfluid described by the Gross-Pitaevskii equation in the curved spacetime. Secondly, we show that a three-dimensional analogue Minkowski metric can emerge from the combination of two SU(2) gauge fields in four-dimensional Yang-Mills theory in the curved spacetime. Throughout this paper, we work in the probe limit, where the background spacetime is considered as a rigid frame without dynamics. Acoustic black holes are formed only at next-to-leading order level. Both the Cartesian coordinates (t, x, y, z) and the spherical coordinates (t, r, ϑ, ϕ) will be used and switched frequently in this paper. We also adopt the units $c = G = \hbar = 1$ in what follows.

II. ACOUSTIC BLACK HOLES FROM GROSS-PITAEVSKII THEORY IN CURVED SPACETIME

Let us consider a nonlinear complex scalar field, obeying the Gross-Pitaevskii equation. The matter field has no backreaction to the background spacetime in the probe limit. The Gross-Pitaevskii theory yields the action [20,21]

$$S = \int d^4x \sqrt{-g} \left(|\partial_\mu \varphi|^2 + m^2 |\varphi|^2 - \frac{b}{2} |\varphi|^4 \right), \quad (1)$$

where φ is a complex scalar order parameter. In the following, we consider fluctuations of the complex scalar field. The equation of motion for φ is written as

²The escape velocity for an observer maintaining a stationary position at the Schwarzschild coordinate radius r_s obeys the relation $v_{\text{escape}} = \sqrt{\frac{2GM}{r_s}}$. At the event horizon $r_H = 2GM$, the escape velocity is exactly the speed of light. The speed of sound for thermal radiation is $v_s = \frac{1}{\sqrt{3}}$, corresponding to $r_s = 6GM$.

$$\square \varphi + m^2 \varphi - b |\varphi|^2 \varphi = 0, \quad (2)$$

where m^2 is a temperature-dependent parameter. Because Eq. (2) is the relativistic version of the theory describing the second-order phase transition and φ is the corresponding complex order parameter [22], it is natural to assume the temperature dependence $m^2 \sim (T - T_c)$. For temperatures above the critical temperature $T > T_c$, the phenomenological parameter m^2 is positive and it becomes vanishing at $T = T_c$ and negative at $T < T_c$. The background spacetime metric is fixed as a static one:

$$ds^2 = g_{tt} dt^2 + g_{xx} dx^2 + g_{yy} dy^2 + g_{zz} dz^2. \quad (3)$$

In the Madelung representation $\phi = \sqrt{\rho(\vec{x}, t)} e^{i\theta(\vec{x}, t)}$, we obtain

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} g^{tt} \partial_t \sqrt{\rho}) + \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ii} \partial_i \sqrt{\rho}) \\ &\quad - g^{tt} \sqrt{\rho} (\partial_t \theta)^2 - g^{ii} \sqrt{\rho} (\partial_i \theta)^2 + m^2 \rho^{\frac{1}{2}} - b \rho^{\frac{3}{2}}, \\ 0 &= \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} g^{tt} \rho \partial_t \theta) + \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ii} \rho \partial_i \theta). \end{aligned} \quad (4)$$

A dumb black hole metric can be obtained by considering perturbations around the background (ρ_0, θ_0) :

$$\rho = \rho_0 + \rho_1 \quad \text{and} \quad \theta = \theta_0 + \theta_1. \quad (5)$$

Working in the long-wavelength limit, thus neglecting the quantum potential terms, we obtain a relativistic wave equation governing the propagation of the phase fluctuation of the weak excitations in a homogeneous stationary condensate:

$$\partial_\mu \left(\sqrt{-\mathcal{G}} \mathcal{G}^{\mu\nu} \partial_\nu \theta_1 \right) = 0, \quad (6)$$

with $\mathcal{G} = \det(\mathcal{G}_{\mu\nu})$ and the metric $\mathcal{G}_{\mu\nu}$ encodes both the information of the background spacetime metric and the background four-velocity of the fluid $v_t = -\dot{\theta}_0$, $v_i = \partial_i \theta_0$. The effective metric extracted from the Klein-Gordon equation is given by

$$\mathcal{G}_{\mu\nu} = \mathcal{H} \begin{pmatrix} g_{tt}(c_s^2 - v^2) & \vdots & -v_i v_t \\ \dots & \dots & \dots \\ -v_i v_t & \vdots & g_{ii}(c_s^2 - v_i v^i - v^2) \delta^{ij} + v_i v_j \end{pmatrix}, \quad (7)$$

where $\mathcal{H} = \frac{c_s}{\sqrt{c_s^2 - v_\mu v^\mu}}$ and $c_s^2 = \frac{b\rho_0}{2}$.

The background four-velocity obeys the relation $b\rho_0 = m^2 - v_t v^t - v_i v^i$. Now we switch to the spherical

coordinates (t, r, ϑ, ϕ) . Under the assumption $v_\vartheta = v_\phi = 0$, $v_t \neq 0$, $v_r \neq 0$ without any time dependence and $g_{rr}g_{tt} = -1$, with the coordinate transformation

$$dt = d\tilde{t} + \frac{v_t v_r}{g_{tt}(c_s^2 - v_r v^r)} dr, \quad (8)$$

we can write down the line element for static dumb black holes,

$$ds^2 = \frac{c_s^2}{\sqrt{c_s^2 - v_\mu v^\mu}} \left[(c_s^2 - v_r v^r) g_{tt} d\tilde{t}^2 + c_s^2 \frac{c_s^2 - v_\mu v^\mu}{c_s^2 - v_r v^r} g_{rr} dr^2 + (c_s^2 - v_\mu v^\mu) g_{\vartheta\vartheta} d\vartheta^2 + g_{\phi\phi} (c_s^2 - v_\mu v^\mu) d\phi^2 \right], \quad (9)$$

where $\mu = 0, 1, 2, 3$. Note that we can rescale parameters as $m^2 \rightarrow \frac{m^2}{2c_s^2}$ and $v_\mu v^\mu \rightarrow \frac{v_\mu v^\mu}{2c_s^2}$, so that the metric can be recast as

$$ds^2 = \frac{c_s^2}{\sqrt{1 - 2v_\mu v^\mu}} \left[(1 - 2v_r v^r) g_{tt} d\tilde{t}^2 + \frac{1 - 2v_\mu v^\mu}{1 - 2v_r v^r} g_{rr} dr^2 + (1 - 2v_\mu v^\mu) g_{\vartheta\vartheta} d\vartheta^2 + g_{\phi\phi} (1 - 2v_\mu v^\mu) d\phi^2 \right]. \quad (10)$$

Keeping in mind that $m^2 - v_\mu v^\mu = 1$ and working at the critical temperature $T = T_c$ (i.e., $m^2 = 0$), we can also put the metric in the form

$$ds^2 = \sqrt{3} c_s^2 \left[\frac{1}{3} (1 - 2v_r v^r) g_{tt} d\tilde{t}^2 + \frac{1}{1 - 2v_r v^r} g_{rr} dr^2 + g_{\vartheta\vartheta} d\vartheta^2 + g_{\phi\phi} d\phi^2 \right]. \quad (11)$$

Thus, in general, we can write the acoustic metric in curved spacetime as a Hadamard product $*$ of matrices with $g_{\mu\nu}^{\text{GR}}$

TABLE I. For an analogue metric embedded in the Schwarzschild spacetime, inside the optical event horizon $r < r_H$, the signs of \mathcal{G}_{tt} and \mathcal{G}_{rr} are different from the ordinary Schwarzschild black holes.

$r > r_{ac}$	$r_H < r < r_{ac}$	$r < r_H$
$\mathcal{G}_{tt} < 0, \mathcal{G}_{rr} > 0$	$\mathcal{G}_{tt} > 0, \mathcal{G}_{rr} < 0$	$\mathcal{G}_{tt} < 0, \mathcal{G}_{rr} > 0$
Both light and sound can escape	Sound cannot escape	Inside the black hole

TABLE II. For an analogue metric embedded in the Reissner-Nordström spacetime, inside the optical event horizon $r < r_H$, the signs of \mathcal{G}_{tt} and \mathcal{G}_{rr} are different from the charged black holes. Note that r_- denotes the inner event horizon.

$r > r_{ac}$	$r_H < r < r_{ac}$	$r_- < r < r_H$	$0 < r < r_-$
$\mathcal{G}_{tt} < 0, \mathcal{G}_{rr} > 0$	$\mathcal{G}_{tt} > 0, \mathcal{G}_{rr} < 0$	$\mathcal{G}_{tt} < 0, \mathcal{G}_{rr} > 0$	$\mathcal{G}_{tt} > 0, \mathcal{G}_{rr} < 0$
Both light and sound can escape	Sound cannot escape	Inside the black hole	Inside the black hole

denoting the spacetime metric in general relativity and $g_{\mu\nu}^{ac}$ the metric of analogue gravity as

$$ds^2 = (g^{\text{GR}} * g^{ac})_{\mu\nu} dx^\mu dx^\nu. \quad (12)$$

Although the background spacetime here is fixed to be static, the analogue metric can be a rotating one. A Kerr-like analogue black hole was derived for relativistic fluids in [23,24].

We thus obtain a metric consisting of an analogue metric multiplied by a real spacetime metric. The background real spacetime can be the Friedmann-Robertson-Walker (FRW) metric or a black hole metric. As $g_{\mu\nu}^{\text{GR}} \rightarrow \eta_{\mu\nu}$, Eq. (12) reduces to the acoustic black hole metric obtained in [23], while as $g_{\mu\nu}^{ac} \rightarrow \delta_{\mu\nu}$, Eq. (12) becomes the spacetime metric in general relativity. In principle, we are able to embed the analogue metric in asymptotically flat, de Sitter, or anti-de Sitter spacetimes. In the appendixes, we show how to embed the analogue black hole in a ‘‘wormhole’’ spacetime background. As an acoustic black hole is embedded in real black hole spacetimes, there could be two event horizons: the optical event horizon and acoustic horizon. One may notice interesting observations about the metric signature as follows:

- (i) For an acoustic black hole embedded in the Schwarzschild black hole background, two horizons sperate the spacetime into three regions as shown in Table I. It looks like the superfluid field makes the original black hole ‘‘charged’’ after condensation.
- (ii) Embedding in the Reissner-Nordström geometry with outside event horizon r_+ and inner event horizon r_- is shown in Table II. From the sign of \mathcal{G}_{tt} and \mathcal{G}_{rr} only, it seems that the causal structure of the original Reissner-Nordström black hole is modified. Note the fact that the acoustic metric obtained here works only in the probe limit and thus it cannot be extended to inside the optical event horizon region. Moreover, the analogue metric obtained here is not a solution of Einstein’s equation.
- (iii) Light-sound cones: A better understanding of the metric (12) as an acoustic black hole is the behavior of sound trajectories. Sound propagates along world lines for which $d\vartheta = d\phi = 0$ and $ds^2 = 0$. First we introduce the tortoise coordinate r_* defined by

$$dr_* = -\frac{\sqrt{3}}{(1 - 2v_r v^r) g_{tt}} dr. \quad (13)$$

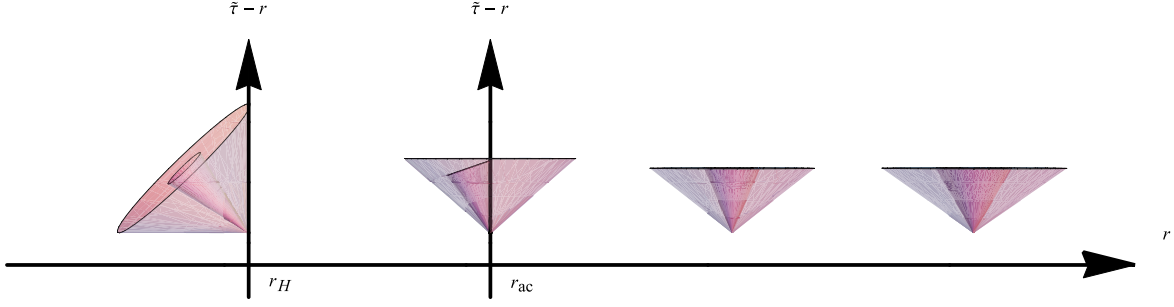


FIG. 1. Schematic view of light-sound cones for an acoustic black hole in curved spacetime. The orientation of the future light-sound cones at different radii is shown. Inside the acoustic horizon r_{ac} , the sound cones become tilted. Further into the optical event horizon r_H , both sound and light cones become tilted.

Note that as $v^r \rightarrow 0$, r_* reduces to the tortoise of optical black holes. Next, we adapt the Eddington-Finkelstein coordinate $\mathcal{V} = \tilde{t} + r_*$ and obtain

$$ds^2 = \sqrt{3}c_s^2 \left[\frac{1}{3}(1 - 2v_r v^r)g_{tt}d\mathcal{V}^2 + \frac{2}{\sqrt{3}}d\mathcal{V}dr \right]. \quad (14)$$

The optical event horizon is located at $g_{tt} = 0$ and the acoustic horizon satisfies $c_s^2 - v_r v^r = 0$. Especially, in the Schwarzschild geometry background, the optical event horizon should be at $r_H = 2M$, while the radial velocity satisfies the relation $v^r = (1 - 2M/r)^{1/2}c_s$ at the acoustic horizon. In the asymptotically flat region $r \rightarrow \infty$, we can recover the relation $v^r = c_s$, which characterizes the formation of the acoustic horizon given in [1]. For null geodesics $ds^2 = 0$, a simple solution is that some radial sound trajectories move along the curves³

$$\mathcal{V} = \text{const.} \quad (\text{infalling}). \quad (15)$$

The outgoing rays satisfy

$$\frac{d\mathcal{V}}{dr} = -\frac{2\sqrt{3}}{(1 - 2v_r v^r)g_{tt}}. \quad (16)$$

The conditions $1 - 2v_r v^r = 0$ and $g_{tt} = 0$ characterize the location of the acoustic and optical horizons r_{ac} and r_H , respectively, so that sound and light rays are neither ingoing nor outgoing there, respectively (see Fig. 1 for typical sound and light cones). A special condition is that as $g_{tt} = -1$, Eq. (16) only describes sound waves.

- (iv) Embedding in the FRW Universe: Let us consider a complex scalar field in the early Universe as given in (1). The flat FRW geometry is the simplest example of a homogeneous, isotropic cosmological metric $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$. We only

³Here we provide a degenerate description of light rays and sound trajectories as $ds^2 = 0$. Actually, light rays also satisfy the null condition $ds^2 = 0$.

consider the t -component fluctuations of the complex scalar field because at a sufficiently early time in the inflationary epoch, the expansion rate H is negligible compared to $\nabla\theta_1/a$ and the metric perturbation can be neglected [25]. In this case, the analogue metric can be expressed as

$$ds^2 = [(2c_s^2 + v_t^2)a^2(t) - v_z^2] \times \left\{ -\frac{2c_s^2 - v_z^2/a^2(t)}{(2c_s^2 + v_t^2)a^2(t) - v_z^2}d\tilde{t}^2 + \frac{2c_s^2}{2c_s^2 - v_z^2/a^2(t)}dz^2 + dx^2 + dy^2 \right\},$$

where we have used the transformation

$$d\tilde{t} = dt + \frac{v_t v_z a^2}{2c_s^2 a^2 - v_z^2} dz. \quad (17)$$

If we further set $v_z = 0$, then we have

$$ds^2 = -2c_s^2 dt^2 + (2c_s^2 + v_t^2)a^2(t)(dx^2 + dy^2 + dz^2). \quad (18)$$

See [26–29] for more references on the acoustic analogue of the expanding Universe produced in the Bose-Einstein condensate.

As $g_{rr}(r_H) \rightarrow \infty$ at the optical event horizon r_H , the acoustic horizon is located at $c_s^2 = g_{rr}(v^r)^2$, which implies that the location of the acoustic horizon should be outside the black hole optical event horizon. Furthermore, it seems that for a black hole surrounded by an acoustic horizon, a spacelike type singularity looks like a timelike singularity and vice versa. But this may not be true, because inside the black hole event horizon, the term $v_r v^r$ in Eq. (14) might decrease below c_s and then the spacelike singularity is still a spatial singularity for sound waves.⁴

⁴We would like to thank the anonymous referee for figuring out this point.

The Hawking temperature of the resultant acoustic solution encodes both the information of black holes and the acoustic metric. The Hawking temperature is given by

$$T = \frac{1}{4\pi\sqrt{(g^{\text{GR}} * g^{ac})_{rr}}} \left(\sqrt{\frac{g_{tt}^{ac}}{g_{tt}^{\text{GR}}} g_{tt}^{\text{GR}}} + \sqrt{\frac{g_{tt}^{\text{GR}}}{g_{tt}^{ac}} g_{tt}^{ac}} \right)_{\text{horizon}}, \quad (19)$$

where the prime ' denotes derivative with respect to r . Let us consider a four-dimensional black hole as an example: $g_{tt}^{\text{GR}} = -\frac{1}{g_{rr}^{\text{GR}}} = -f(r)$ and $g_{\theta\theta}^{\text{GR}} = r^2$. The acoustic Hawking temperature is then given by

$$T_a = \frac{1}{2\sqrt{3}\pi} \left| \frac{f}{\sqrt{c_s}} (2v_r v_r' f - v_r^2 f') \right|_{\text{horizon}}. \quad (20)$$

We notice that at the optical event horizon where $f(r_H) = 0$, the Hawking temperature of the acoustic black hole is vanishing. In general, v_r cannot be zero because it is the frequency related to the relativistic dispersion relation. In the $f(r) \rightarrow 1$ limit and after recovering the scaling $v_r \rightarrow \sqrt{2}v_r/c_s$, T_a can naturally reduce to the pure acoustic black hole Hawking temperature: $T_a = \frac{1}{2\pi} |c_s' - v_r'|_{c_s=v_r}$.

A. Energy released during “gravitational binding”

It is well known that gravitational binding is a more efficient mechanism for releasing rest energy than the thermonuclear mechanism [14,30]. This is why black holes and compact relativistic stars are at the heart of many energetic phenomena in the Universe. We are going to examine the energy release for a vortex moving from infinity to the stable orbit of an acoustic black hole in curved spacetime. Vortices can behave as relativistic particles with their dynamics governed by the fluid metric [31,32] and their stability ensured by a topological number. Vortices with mass m_0 given by the Einstein's relation $E = m_0 c_s^2$ [33,34] cannot propagate at velocities faster than the sound speed c_s . We can therefore utilize vortices to study the timelike geodesics of massive particles near an acoustic black hole in curved spacetime.

Since the metric is independent of coordinates t and ϕ , where \vec{u} is the four-velocity of the particle and $\vec{\xi}$ and $\vec{\eta}$ are Killing vectors, we can introduce energy per unit mass ϵ and angular momentum per unit mass ℓ [14]:

$$\epsilon = -\vec{\xi} \cdot \vec{u} = -\mathcal{G}_{tt} \frac{dt}{d\tau}, \quad (21)$$

$$\ell = \vec{\eta} \cdot \vec{u} = \mathcal{G}_{\phi\phi} \frac{d\phi}{d\tau}, \quad (22)$$

where $d\tau^2 = -ds^2$ denotes the proper time. We can normalize the four-velocity as $\vec{u} \cdot \vec{u} = \mathcal{G}_{\alpha\beta} u^\alpha u^\beta = -1$ and

this provides an integral for the geodesic equation in addition to the energy and angular momentum. Taking into account the equatorial plane condition $u^\theta = 0$ ($\theta = \frac{\pi}{2}$); writing $u^t = \frac{dt}{d\tau}$, $u^r = \frac{dr}{d\tau}$, $u^\phi = \frac{d\phi}{d\tau}$; and using the above relations to eliminate $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$, we can rewrite Eq. (10) as

$$\mathcal{G}_{tt}(u^t)^2 + \mathcal{G}_{rr}(u^r)^2 + \mathcal{G}_{\phi\phi}(u^\phi)^2 = -1. \quad (23)$$

Multiplying \mathcal{G}_{tt} on both sides and taking $\mathcal{G}_{\phi\phi} = r^2$, we obtain

$$\epsilon^2 + \mathcal{G}_{tt} \mathcal{G}_{rr} \left(\frac{dr}{d\tau} \right)^2 + \frac{\ell^2}{r^2} \mathcal{G}_{tt} = -\mathcal{G}_{tt}. \quad (24)$$

By further defining $\mathcal{E} \equiv \frac{(\epsilon^2 - 1)}{2}$ and the effective potential

$$V_{\text{eff}} = -\frac{1}{2} \left[\mathcal{G}_{tt} \left(1 + \frac{\ell^2}{r^2} \right) + 1 \right], \quad (25)$$

we obtain

$$\mathcal{E} = -\frac{1}{2} \mathcal{G}_{tt} \mathcal{G}_{rr} \left(\frac{dr}{d\tau} \right)^2 + V_{\text{eff}}. \quad (26)$$

Suppose there are some stable circular orbits outside the horizon. The angular velocity of a particle or a vortex in a circular orbit with respect to the acoustic time is the rate measured with respect to a stationary clock at infinity. For any equatorial orbit, the angular velocity is given by

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = -\frac{\ell \mathcal{G}_{tt}}{\epsilon r^2}. \quad (27)$$

The effective potential has its minimum $\partial_r V_{\text{eff}} = 0$ at the radius of the orbit. The value of the total energy \mathcal{E} equals the value of the effective potential at the minimum. These two requirements yield the ratio ℓ/ϵ of circular orbits

$$\frac{\ell}{\epsilon} = \left(-\frac{r^3 \partial_r \mathcal{G}_{tt}}{2\mathcal{G}_{tt}^2} \right)^{1/2}. \quad (28)$$

The components of the four-velocity of a particle or a vortex in a circular orbit are then $u^\alpha = u^t(1, 0, 0, \Omega)$. The component u^t is determined by the normalization condition $\vec{u} \cdot \vec{u} = -1$. Now there is a contribution from the angular velocity, so

TABLE III. Orbits in the equatorial plane: the radius r_{ISCO} of the innermost stable circular orbit, the angular momentum per unit mass, and the energy release ratio as a function of the parameter γ . M is the mass of the Schwarzschild black hole.

	$\gamma = 0$	$\gamma = 5$	$\gamma = 7$	$\gamma = 9$	$\gamma = 11$
r_{ISCO}	$6M$	$23.59M$	$35.78M$	$47.86M$	$59.89M$
ℓ	$2\sqrt{3}M$	$16.70M$	$23.71M$	$30.68M$	$37.63M$
$1 - \epsilon$	0.057	0.083	0.074	0.071	0.068

$$u^t = \left(-\mathcal{G}_{tt} + \frac{r}{2} \partial_r \mathcal{G}_{tt} \right)^{-1/2}. \quad (29)$$

The energy difference between a free particle/vortex and one bound in a circular orbit of radius r that is available for release is

$$\frac{\text{released energy}}{\text{rest energy}} = 1 + \mathcal{G}_{tt} \left(-\mathcal{G}_{tt} + \frac{r}{2} \partial_r \mathcal{G}_{tt} \right)^{-1/2} \Big|_{r=r_{\text{ISCO}}}. \quad (30)$$

For the Schwarzschild metric $\mathcal{G}_{tt} = -(1 - 2M/r)$ with the innermost stable circular orbit (ISCO) at $r = 6M$, the energy release ratio is 5.7%. The gravitational binding is a more efficient mechanism for releasing rest energy than thermonuclear fusion, which is approximately 1%. As an example, let us consider an orbit of a vortex that executes radial free fall from infinity starting from rest outside the Schwarzschild black hole. The radial component of the four-velocity v^r is taken to be the escape velocity of an observer maintaining a stationary position at the Schwarzschild coordinate radius r , that is to say, $v^r = (\gamma \frac{2M}{r})^{1/2}$, and thus

$$\mathcal{G}_{tt} = - \left(1 - \frac{2M}{r} \right) [1 - 2\gamma M/r(1 - 2M/r)] \quad (31)$$

by further setting $c_s^2 = \frac{1}{\sqrt{3}}$, where γ is a tuning parameter. γ is introduced by hand to guarantee that the acoustic horizon is outside the optical event horizon. In this case, there are horizons with the radius $r_H = 2M$ and $r_{ac} = (\gamma \pm \sqrt{\gamma^2 - 4\gamma})M$. The acoustic horizon r_{ac} is located outside the real black hole and requires $\gamma \geq 4$. The energy release ratio is summarized in Table III. The energy release ratio obtained here is slightly higher than that of a Schwarzschild black hole at 5.7%.

III. ANALOGUE GRAVITY FROM EINSTEIN-YANG-MILLS THEORY

In this section, we show how to produce a (2 + 1)-dimensional acoustic metric from the (3 + 1)-dimensional

Yang-Mills theory. To compare our results with the gauge/gravity duality theory, we demonstrate that a (2 + 1)-dimensional Minkowski metric can emerge from the SU(2) Yang-Mills theory.

The action for the Einstein-Yang-Mills theory in (3 + 1)-dimensional spacetime is

$$S = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa_4^2} \left(R + \frac{6}{l^2} \right) - \frac{1}{2g_{\text{YM}}^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \right], \quad (32)$$

where κ_4 is the gravitational coupling, R is the Ricci scalar curvature, l is the radius of the AdS space, and g_{YM} is the gauge coupling constant. The gauge field strength is given by $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c$ and $A = A_\mu dx^\mu = \tau^a A_\mu^a dx^\mu$. Here τ^a are the generators of SU(2), which obey the relation $[\tau^a, \tau^b] = \epsilon^{abc} \tau^c$ and are related to the Pauli matrices by $\tau^a = \sigma^a/2i$. ϵ^{abc} is the totally antisymmetric tensor with $\epsilon^{123} = 1$. The Yang-Mills Lagrangian becomes $\text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -F_{\mu\nu}^a F^{a\mu\nu}/2$. In the probe limit, we focus on the gauge fields only.

The equations of motion for the gauge fields are

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{a\mu\nu}) + \epsilon^{abc} A_\mu^b F^{c\mu\nu} = 0. \quad (33)$$

We consider the symmetric metric as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{tt} dt^2 + g_{xx} (dx^2 + dy^2) + g_{rr} dr^2, \end{aligned} \quad (34)$$

$$A = \tau^3 \phi(t, x, y, r) dt + w(t, x, y, r) (\tau^1 dx + \tau^2 dy), \quad (35)$$

where $g_{\mu\nu}$ are functions of the radial coordinate r only, but ϕ and w are considered as functions of (t, x, y, r) . Such a coordinate is usually used for planar AdS black holes. The U(1) subgroup of SU(2) generated by τ^3 is identified with the electromagnetic gauge group and ϕ is the electromagnetic scalar potential. There are no analytic solutions of the Einstein-Yang-Mills theory, but numerical calculations have been widely studied (see [35–37] for reviews). In case $\phi \neq 0$, but $w = 0$, the solution is simply the Reissner–Nordström–anti–de Sitter (RNAdS) metric.

The (real) field w is charged under the U(1) group and represents the amplitudes of the $p_x + ip_y$ components of the superfluid order parameter [38]. In the presence of vortices, w is a complex variable and can be considered as a composite of two real fields $w = w_R + iw_I$. We choose $A_x^1 = A_y^2 = w_R$ and $A_x^2 = -A_y^1 = w_I$. The y -components of the Yang-Mills equation for $a = 1$ and $a = 2$ are given by

$$\begin{aligned}
0 &= -\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}g^{yy}\partial_r w_I) \\
&\quad + \frac{1}{\sqrt{-g}}\partial_x[\sqrt{-g}g^{xx}g^{yy}(-\partial_x w_I - \partial_y w_R)] \\
&\quad + \frac{1}{\sqrt{-g}}\partial_t[\sqrt{-g}g^{tt}g^{yy}(-\partial_t w_I - \phi w_R)] \\
&\quad + g^{xx}g^{yy}w_I(w_R^2 + w_I^2) - g^{tt}g^{yy}\phi(\partial_t w_R - \phi w_I), \\
0 &= \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}g^{yy}\partial_r w_R) \\
&\quad + \frac{1}{\sqrt{-g}}\partial_x[\sqrt{-g}g^{xx}g^{yy}(\partial_x w_R - \partial_y w_I)] \\
&\quad + \frac{1}{\sqrt{-g}}\partial_t[\sqrt{-g}g^{tt}g^{yy}(\partial_t w_R - \phi w_I)] \\
&\quad - g^{xx}g^{yy}w_R(w_R^2 + w_I^2) + g^{tt}g^{yy}\phi(-\partial_t w_I - \phi w_R).
\end{aligned}$$

The t component for $a = 3$ is given by

$$\begin{aligned}
&\partial_r(\sqrt{-g}g^{rr}g^{tt}\partial_r\phi) + \partial_t(\sqrt{-g}g^{tt}g^{ij}g^{ij}\partial_j\phi) \\
&\quad - 2\sqrt{-g}g^{xx}g^{tt}[\phi(w_R^2 + w_I^2) + w_R\partial_t w_I - w_I\partial_t w_R] = 0.
\end{aligned}$$

In what follows, the electromagnetic-scalar potential ϕ is only regarded as a background field and fluctuations of ϕ will not be considered [23]. For consistency and the purpose of deriving the acoustic metric, we assume w_I and w_R to be y -independent, so that there will be no x - and y -dependent mixing terms. As we can see later, this will lead to an acoustic black hole metric, where the space dimension is reduced by 1 and this in turn is consistent with the holographic principle, which states that the description of a volume of space can be thought of as encoded on a lower-dimensional boundary. Under this assumption, the equations of motion for w_R and w_I can be combined into a single equation,

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}g^{yy}\partial_r w) + \frac{1}{\sqrt{-g}}\partial_x(\sqrt{-g}g^{xx}g^{yy}\partial_x w) \\
&\quad + \frac{1}{\sqrt{-g}}\partial_t[\sqrt{-g}g^{tt}g^{yy}(\partial_t w + i\phi w)] \\
&\quad - g^{xx}g^{yy}|w|^2 w + g^{tt}g^{yy}\phi(i\partial_t w - \phi w), \quad (36)
\end{aligned}$$

where $w = w_R + iw_I$. Notice that this is an equation in $(2 + 1)$ -dimensional spacetime. In the condensed phases, the quantized field describing the microscopic system can be replaced by a classical mean field with a macroscopic wave function. Thus, with the assumption $w = \sqrt{\rho(\vec{x}, t)}e^{i\theta(\vec{x}, t)}$,

the resulting equation of motion for the complex scalar field w reduces to real and imaginary parts, respectively,

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}g^{yy}\rho\partial_r\theta) + \frac{1}{\sqrt{-g}}\partial_x(\sqrt{-g}g^{xx}g^{yy}\rho\partial_x\theta) \\
&\quad + \frac{1}{\sqrt{-g}}\partial_t[\sqrt{-g}g^{tt}g^{yy}(\rho\partial_t\theta + \phi\rho)] + g^{tt}g^{yy}\frac{\partial_t\rho}{2}, \\
0 &= \frac{1}{\sqrt{-g}\sqrt{\rho}}\left(g^{tt}g^{yy}\partial_t^2 + g^{rr}g^{yy}\partial_r^2 + g^{xx}g^{yy}\frac{\partial_x^2}{2}\right)\sqrt{\rho} \\
&\quad - g^{tt}g^{yy}(\partial_t\theta)^2 - g^{xx}g^{yy}(\partial_x\theta)^2 - g^{rr}g^{yy}(\partial_r\theta)^2 - g^{xx}g^{yy}\rho. \quad (37)
\end{aligned}$$

When the metric $g_{\mu\nu}$ reduces to the Minkowski metric $\eta_{\mu\nu}$, the above equations are completely equivalent to those of an irrotational and inviscid fluid apart from the quantum potential and the θ -dependent terms. In what follows, we neglect the first term in the second equation of (37) from the observation that the quantum potential term contains the second derivative of slowly varying ρ , which is small in the hydrodynamic regime where the related frequency and momentum are small.

Now we linearize the fields around the background (ρ_0, θ_0) , $\rho = \rho_0 + \varepsilon\rho_1$ and $\theta = \theta_0 + \varepsilon\theta_1$, and write the ε terms as

$$\begin{aligned}
0 &= \partial_t[\sqrt{-g}g^{tt}g^{yy}(\rho_0\partial_t\theta_1 + \rho_1\partial_t\theta_0)] \\
&\quad + \partial_x[\sqrt{-g}g^{xx}g^{yy}(\rho_0\partial_x\theta_1 + \rho_1\partial_x\theta_0)] \\
&\quad + \partial_r[\sqrt{-g}g^{rr}g^{yy}(\rho_0\partial_r\theta_1 + \rho_1\partial_r\theta_0)], \quad (38)
\end{aligned}$$

$$0 = 2g^{tt}\partial_t\theta_0\partial\theta_1 + 2g^{xx}\partial_x\theta_0\partial_x\theta_1 + 2g^{rr}\partial_r\theta_0\partial_r\theta_1 + \rho_1g^{xx}. \quad (39)$$

Defining the velocity field by

$$v_t = -\partial_t\theta_0, \quad v_i = \partial_i\theta_0, \quad (40)$$

where v_t corresponds to the frequency obeying a ‘‘relativistic dispersion relation,’’ $g^{xx}\rho_0 = -v_t v^t - v_r v^r - v_x v^x$, we can also define three-velocity as

$$v^\mu = g^{\mu\nu}v_\nu. \quad (41)$$

Equations (38) and (39) can be written as a single equation for θ_1 :

$$\begin{aligned}
0 &= \partial_t\left\{\sqrt{-g}g^{tt}\left[\frac{1}{2}g^{yy}\rho_0\partial_t\theta_1 + (v^t\partial_t\theta_1 - v^x\partial_x\theta_1 - v^r\partial_r\theta_1)v_t\right]\right\} + \partial_x\left\{\sqrt{-g}g^{xx}\left[\frac{1}{2}g^{yy}\rho_0\partial_x\theta_1 + (v^t\partial_t\theta_1 - v^x\partial_x\theta_1 - v^r\partial_r\theta_1)v_x\right]\right\} \\
&\quad + \partial_r\left\{\sqrt{-g}g^{rr}\left[\frac{1}{2}g^{yy}\rho_0\partial_r\theta_1 + (v^t\partial_t\theta_1 - v^x\partial_x\theta_1 - v^r\partial_r\theta_1)v_r\right]\right\}. \quad (42)
\end{aligned}$$

Comparing this with the massless Klein-Gordon equation

$$\frac{1}{\sqrt{-\mathcal{G}}} \partial_\mu (\sqrt{-\mathcal{G}} g^{\mu\nu} \partial_\nu \theta_1) = 0, \quad (43)$$

we can extract a metric from (42) for the sound modes:

$$\sqrt{-\mathcal{G}} g^{\mu\nu} \equiv \sqrt{-g} \begin{pmatrix} g^{tt} \left(\frac{1}{2} g^{xx} \rho_0 - v_t v^t \right) & \vdots & v^i v^t \\ \dots\dots\dots & \cdot & \dots\dots\dots \\ v^i v^t & \vdots & g^{ii} \left(\frac{1}{2} g^{xx} \rho_0 \delta^{ij} - v_i v^j \right) \end{pmatrix}. \quad (44)$$

By further defining the local speed of sound as

$$c_s^2 = \frac{1}{2} g^{xx} \rho_0, \quad (45)$$

we can determine the acoustic metric simply by inverting this 3×3 matrix. The acoustic metric is obtained as

$$\mathcal{G}_{\mu\nu} = \mathcal{H} \begin{pmatrix} g_{tt}(c_s^2 - v^2) & \vdots & -v_t v_i \\ \dots\dots\dots & \cdot & \dots\dots\dots \\ -v_t v_i & \vdots & g_{ii}(c_s^2 - v_t v^t - v^2) \delta^{ij} + v_i v_j \end{pmatrix},$$

where

$$\mathcal{H} = \frac{(-g)^{1/4}}{\sqrt{c_s^2 - (v_t v^t + v^2)}}, \quad (46)$$

$v^2 = v_i v^i$, and $g \equiv \det(g_{\mu\nu})$ is the determinant of the background spacetime metric.

In the case $v_i = 0$ and $g_{rr} = \frac{-1}{g_{tt}}$, but $v_r \neq 0$, we can simplify the metric as

$$\begin{aligned} ds^2 &= (g^{\text{GR}} * g^{ac})_{\mu\nu} dx^\mu dx^\nu \\ &= \mathcal{H} \left[(c_s^2 - v_r v^r) g_{tt} d\tau^2 + c_s^2 \frac{3c_s^2}{c_s^2 - v_r v^r} g_{rr} dr^2 \right. \\ &\quad \left. + 3c_s^2 g_{xx} dx^2 \right], \end{aligned} \quad (47)$$

where the Hadamard product of matrices was used with $g_{\mu\nu}^{\text{GR}}$ representing the spacetime metric in general relativity and $g_{\mu\nu}^{ac}$ the metric of analogue gravity. Note that the coordinate transformation has been used,

$$dt = d\tau + \frac{v_r v_t}{g_{tt}(c_s^2 - v_r v^r)} dr, \quad (48)$$

and the assumption that v^μ is independent of time has been made.

(i) *Emergent asymptotic Minkowski spacetime.* The obtained metric can reduce to a pure black hole case with an overall factor c_s^2 , when each component of the fluid velocity vanishes (i.e., $v_t = 0$, $v_i = 0$). The acoustic metric in this case becomes

$$ds^2 = c_s^2 (g_{tt} dt^2 + g_{rr} dr^2 + g_{xx} dx^2). \quad (49)$$

To see how the Minkowski metric is emergent, let us first assume that ρ_0 is constant so that the sound velocity c_s^2 mainly depends on g^{xx} . Further setting the background spacetime to be the pure AdS space in the Poincaré coordinate, one can easily find that the resultant acoustic metric becomes

$$\begin{aligned} ds^2 &= c_s^2 r^2 (-dt^2 + dr^2 + dx^2) \\ &= \frac{\rho_0}{2} (-dt^2 + dr^2 + dx^2). \end{aligned} \quad (50)$$

One observation is that one can establish a quantum field theory even in this analogue spacetime. In case the background spacetime has a black hole inside, the acoustic metric has its form

$$ds^2 = \frac{\rho_0}{2} \left(-f(r) dt^2 + dx^2 + \frac{1}{f(r)} dr^2 \right), \quad (51)$$

which is also asymptotically flat as $f(r \rightarrow \infty) = 1$.

- (ii) *FRW version.* An interesting example is the case in which the analogue metric is embedded in the flat FRW metric. The line element of the acoustic metric is listed as

$$ds^2 \sim a^{-\frac{1}{2}}(-dt^2 + dx^2 + dz^2). \quad (52)$$

This implies that as the spacetime expands the analogue metric produced by the Yang-Mills field shrinks.

IV. CONCLUSION AND DISCUSSION

In summary, we have shown that acoustic black holes can also be produced in curved spacetime and we use the Gross-Pitaevskii and the Yang-Mills equations for a concrete demonstration. The analogue metric can be written as the Hadamard product of a real metric matrix and an analogue metric matrix. Working at the critical temperature can simplify the form of the metric. For acoustic black holes produced in the Gross-Pitaevskii fluid, the light-sound cone of the analogue geometry in the background of the Schwarzschild spacetime can be summarized as follows. (i) Outside the acoustic horizon, both light and sound can escape to infinity. (ii) In the region inside the acoustic horizon but outside the optical event horizon, light can escape although sound cannot. (iii) Inside the optical event horizon of the Schwarzschild black hole, both light and sound cannot escape. Intriguingly, the optical event horizon also plays the role of the acoustic horizon.⁵ The physical picture can be understood through a thought experiment as follows: consider a large enough black hole with small enough tidal forces so that a box full of relativistic condensate matter falling into the black hole is not to be destroyed by the tidal force. Suppose there is an observer who freely falls into the acoustic black hole with the box. For the box partially inside the acoustic horizon and partially outside, sound waves cannot propagate out when the observer perturbs it inside the acoustic horizon. However, a faraway observer can still receive the light sent by the observer who is located between the acoustic horizon and the optical event horizon. When the box is located half outside and half inside the optical event horizon, when the observer perturbs it again from inside, not only sound but also light cannot propagate across the optical event horizon. The Hawking temperature of the acoustic black hole comes from the contributions from the fluid and the background spacetime. Regarding quantum vortices as “test particles,” we calculated their energy-release ratio during “gravitational binding.”

A $(2+1)$ -dimensional acoustic black hole can be produced by considering the combination of two $(3+1)$ -dimensional Yang-Mills fields. Surprisingly, this is consistent

⁵Acoustic horizons naturally are not optical event horizons, but optical event horizons are naturally acoustic horizons.

with the holographic principle. What is more, we show how the flat Minkowski metric is emergent from the bulk anti-de Sitter space by considering perturbations of the Yang-Mills field. The shortcoming of analogue gravity is that such models do not have a dynamical description but only reflect kinetic aspects of black holes, though there are some papers trying to resolve this problem [39–42]. It is also worth investigating analogue gravity from the holographic duality viewpoint. In a previous paper [43], some of us have managed to construct an acoustic black hole from the d -dimensional fluid located at the timelike cutoff surface of a neutral black brane in asymptotically AdS _{$d+1$} spacetime. We also have shown that the phonon field, which comes from the normal mode excitation of the fluid at the cutoff surface and scatters in the acoustic black hole geometry, is dual to the scalar field—the sound channel of quasinormal modes propagating in the bulk perturbed AdS black brane (see also [44]).

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APPENDIX A: ACOUSTIC NONTRAVERSABLE WORMHOLES

In [1], Unruh was the first to obtain an acoustic black hole metric by considering a spherically symmetric, stationary, and convergent flow,

$$ds^2 = \frac{\rho_0}{c_s} \left[-(c_s^2 - v_0^2) d\tau^2 + \frac{c_s^2}{c_s^2 - v_0^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (A1)$$

where ρ_0 is the background fluid density, c_s is the local velocity of sound, and v_0^r is the radial velocity of the background fluid. Assuming at some value of $r = R$, v_0^r exceeds the velocity of sound $v_0^r = -c + \alpha(r - R) + \dots$, we obtain the Schwarzschild-like metric

$$ds^2 = \frac{\rho_0}{c_s} \left[-2\alpha c_s(r - R) d\tau^2 + \frac{dr^2}{2\alpha(r - R)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (A2)$$

We consider a coordinate transformation $u^2 = r - R$ and obtain

$$ds^2 = \frac{\rho_0}{c_s} \left[-2\alpha c_s u^2 d\tau^2 + \frac{2c_s du^2}{\alpha} + (u^2 + R)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (\text{A3})$$

This metric is asymptotically flat as $u \rightarrow -\infty$ and $u \rightarrow \infty$, corresponding to $r \rightarrow \infty$. This is an acoustic version of the Einstein-Rosen bridge which is nontraversable for sound. The acoustic black hole is expected to be useful in studying the relationship between spacetime geometry and quantum entanglement.

APPENDIX B: ACOUSTIC TRAVERSABLE WORMHOLES

It would be interesting to consider an acoustic black hole embedded in a traversable wormhole background.

A two-dimensional wormhole metric can be achieved by setting $\theta = \pi/2$ in an Ellis wormhole [45]:

$$ds_{\text{GR}}^2 = -d\tau^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2). \quad (\text{B1})$$

Under coordinate transformation $l^2 = r^2 - b_0^2$, the two-dimensional wormhole metric is precisely equivalent to the line element of a catenoid:

$$ds_{\text{GR}}^2 = -d\tau^2 + \frac{r^2}{r^2 - b_0^2} dr^2 + r^2 d\phi^2. \quad (\text{B2})$$

An acoustic black hole embedded in a traversable wormhole is then given by

$$ds^2 = \mathcal{H} \left[-(c_s^2 - v_r v^r) d\tau^2 + c_s^2 \frac{c_s^2 - v_t v^t - v_r v^r}{c_s^2 - v_r v^r} \times \frac{r^2}{r^2 - b_0^2} dr^2 + (c_s^2 - v_t v^t - v_r v^r) r^2 d\phi^2 \right], \quad (\text{B3})$$

where \mathcal{H} is given by Eq. (7).

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