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THE STABILITY OF AN ADDITIVE (ρ_1,ρ_2)–FUNCTIONAL INEQUALITY IN BANACH SPACES

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Abstract. In this paper, we introduce and solve the following additive (ρ_1,ρ_2)-functional inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \|\rho_1(f(x+y)+f(x-y)-2f(x))\| + \left\|\rho_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|, \tag{1}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\sqrt{2}|\rho_1|+|\rho_2|<1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (ρ_1,ρ_2)-functional inequality (1) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [25] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

Park [21, 22] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 8, 12, 13, 14, 15, 16, 20]).

We recall a fundamental result in fixed point theory.

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THEOREM 1.1 [3, 7] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 18, 19, 23]).

In Section 2, we solve the additive (ρ_1, ρ_2)-functional inequality (1) and prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (1) in Banach spaces by using the fixed point method. In Section 3, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (1) in Banach spaces by using the direct method.

Throughout this paper, let X be a real or complex normed space with norm $\|\cdot\|$ and Y a complex Banach space with norm $\|\cdot\|$. Assume that ρ_1 and ρ_2 are fixed nonzero complex numbers with $\sqrt{2}|\rho_1| + |\rho_2| < 1$.

2. Additive (ρ_1, ρ_2)-functional inequality (1): a fixed point method

In this section, we solve and investigate the additive (ρ_1, ρ_2)-functional inequality (1) in complex Banach spaces.

LEMMA 2.1 *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \|\rho_1(f(x+y) + f(x-y) - 2f(x))\| + \left\|\rho_2\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \tag{2}$$

for all $x, y \in X$, then f is additive.

Proof. Assume that f satisfies the inequality (2).

Letting $y = x$ in (2), we get $\|f(2x) - 2f(x)\| \leqslant |\rho_1| \cdot \|f(2x) - 2f(x)\|$ and so $f(2x) = 2f(x)$ for all $x \in X$, since $|\rho_1| < 1$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3}$$

for all $x \in X$.

It follows from (2) and (3) that

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &\leqslant \|\rho_1(f(x+y)+f(x-y)-2f(x))\| \\ &\quad + \left\|\rho_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= \|\rho_1(f(x+y)+f(x-y)-2f(x))\| \\ &\quad + \|\rho_2(f(x+y)-f(x)-f(y))\|\end{aligned}$$

and so

$$(1-|\rho_2|)\|f(x+y)-f(x)-f(y)\| \leqslant |\rho_1|\cdot\|f(x+y)+f(x-y)-2f(x)\| \tag{4}$$

for all $x,y\in X$.

Letting $z=x+y$ and $w=x-y$ in (4), we get

$$(1-|\rho_2|)\left\|f(z)-f\left(\frac{z+w}{2}\right)-f\left(\frac{z-w}{2}\right)\right\| \leqslant |\rho_1|\cdot\left\|f(z)+f(w)-2f\left(\frac{z+w}{2}\right)\right\|$$

and so

$$\frac{1}{2}(1-|\rho_2|)\|f(z+w)+f(z-w)-2f(z)\| \leqslant |\rho_1|\cdot\|f(z+w)-f(z)-f(w)\|$$

for all $z,w\in X$.

It follows from (4) and (5) that

$$\frac{1}{2}(1-|\rho_2|)^2\|f(x+y)-f(x)-f(y)\| \leqslant |\rho_1|^2\|f(x+y)-f(x)-f(y)\|$$

for all $x,y\in X$. Since $\sqrt{2}|\rho_1|+|\rho_2|<1$, $f(x+y)=f(x)+f(y)$ for all $x,y\in X$. Thus f is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (ρ_1,ρ_2)-functional inequality (2) in complex Banach spaces.

THEOREM 2.2 *Let $\varphi: X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant \frac{L}{2}\varphi(x,y) \tag{5}$$

for all $x,y\in X$. Let $f: X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &\leqslant \|\rho_1(f(x+y)+f(x-y)-2f(x))\| \\ &\quad + \left\|\rho_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| + \varphi(x,y)\end{aligned} \tag{6}$$

for all $x,y\in X$. Then there exists a unique additive mapping $A: X\to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{L}{2(1-L)(1-|\rho_1|)}\varphi(x,x)$$

for all $x\in X$.

Proof. Letting $y = x$ in (6), we get

$$(1-|\rho_1|)\|f(2x)-2f(x)\| \leqslant \varphi(x,x) \tag{7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \ \ h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x)-h(x)\| \leqslant \mu\varphi(x,x), \ \ \forall x \in X\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [17]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x)-h(x)\| \leqslant \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x)-Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)\right\| \leqslant 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2}\right)\\ &\leqslant 2\varepsilon\frac{L}{2}\varphi(x,x) = L\varepsilon\varphi(x,x)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h)$$

for all $g,h \in S$.

It follows from (7) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leqslant \frac{1}{1-|\rho_1|}\varphi\left(\frac{x}{2},\frac{x}{2}\right) \leqslant \frac{L}{2(1-|\rho_1|)}\varphi(x,x)$$

for all $x \in X$. So $d(f,Jf) \leqslant \frac{L}{2(1-|\rho_1|)}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x)-A(x)\| \leqslant \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leqslant \frac{1}{1-L} d(f,Jf)$, which implies

$$\|f(x)-A(x)\| \leqslant \frac{L}{2(1-L)(1-|\rho_1|)}\varphi(x,x)$$

for all $x \in X$.

It follows from (5) and (6) that

$$\begin{aligned}
&\|A(x+y)-A(x)-A(y)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| \\
&\leqslant \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^n |\rho_2| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \|\rho_1(A(x+y)+A(x-y)-2A(x))\| + \left\|\rho_2\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
\|A(x+y)-A(x)-A(y)\| \leqslant\ & \|\rho_1(A(x+y)+A(x-y)-2A(x))\| \\
&+ \left\|\rho_2\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A: X \to Y$ is additive. □

COROLLARY 2.3 *Let $r > 1$ and θ be nonnegative real numbers, and let $f: X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
\|f(x+y)-f(x)-f(y)\| \leqslant\ & \|\rho_1(f(x+y)+f(x-y)-2f(x))\| \\
&+ \left\|\rho_2\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{9}$$

for all $x,y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{(2^r - 2)(1 - |\rho_1|)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

THEOREM 2.4 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leqslant 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x,y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (6). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{1}{2(1-L)(1 - |\rho_1|)} \varphi(x,x)$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leqslant \frac{1}{2(1 - |\rho_1|)} \varphi(x,x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

COROLLARY 2.5 *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{(2 - 2^r)(1 - |\rho_1|)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

REMARK 2.6. If ρ is a real number such that $\sqrt{2}|\rho_1| + |\rho_2| < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive (ρ_1,ρ_2)-functional inequality (1): a direct method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1,ρ_2)-functional inequality (2) in complex Banach spaces by using the direct method.

THEOREM 3.1 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{10}$$

for all $x,y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (6). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{1}{2(1-|\rho_1|)}\Psi(x,x) \tag{11}$$

for all $x \in X$.

Proof. Letting $y = x$ in (6), we get

$$(1-|\rho_1|)\|f(2x) - 2f(x)\| \leqslant \varphi(x,x) \tag{12}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leqslant \frac{1}{1-|\rho_1|}\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leqslant \sum_{j=l}^{m-1} \frac{2^j}{2(1-|\rho_1|)}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\end{aligned} \tag{13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (13) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (13), we get (11).

It follows from (6) and (10) that

$$\begin{aligned}
&\|A(x+y)-A(x)-A(y)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| \\
&\leqslant \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right)\right\| \\
&\quad + \lim_{n\to\infty} 2^n |\rho_2| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \|\rho_1(A(x+y)+A(x-y)-2A(x))\| + \left\|\rho_2\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
\|A(x+y)-A(x)-A(y)\| \leqslant \|\rho_1(A(x+y)+A(x-y)-2A(x))\| \\
+ \left\|\rho_2\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (11). Then we have

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leqslant \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\| 2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leqslant \frac{2^q}{1-|\rho_1|} \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

COROLLARY 3.2 *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leqslant \frac{2\theta}{(2^r-2)(1-|\rho_1|)} \|x\|^r$$

for all $x \in X$.

THEOREM 3.3 *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (6) and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x,y\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leqslant\frac{1}{2(1-|\rho_1|)}\Psi(x,x)\tag{14}$$

for all $x\in X$.

Proof. It follows from (12) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|\leqslant\frac{1}{2(1-|\rho_1|)}\varphi(x,x)$$

for all $x\in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\|&\leqslant\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|\\&\leqslant\sum_{j=l}^{m-1}\frac{1}{2^{j+1}(1-|\rho_1|)}\varphi(2^jx,2^jx)\end{aligned}\tag{15}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (15) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is Cauchy for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A:X\to Y$ by

$$A(x):=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (15), we get (14).

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.4 *Let $r<1$ and θ be positive real numbers, and let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (9). Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\|f(x)-A(x)\|\leqslant\frac{2\theta}{(2-2^r)(1-|\rho_1|)}\|x\|^r$$

for all $x\in X$.

REFERENCES

[1] M. ADAM, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.

[2] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] L. CĂDARIU, V. RADU, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[4] L. CĂDARIU, V. RADU, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **3**46 (2004), 43–52.

[5] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2**008, Art. ID 749392 (2008).
[6] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **2**7 (1984), 76–86.
[7] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **7**4 (1968), 305–309.
[8] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces*, J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[9] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **1**84 (1994), 431–436.
[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **2**7 (1941), 222–224.
[11] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **1**9 (1996), 219–228.
[12] B. Khosravi, M. B. Moghimi, A. Najati, *Generalized Hyers-Ulam stability of a functional equation of Hosszu type*, Nonlinear Funct. Anal. Appl. **2**3 (2018), 157–166.
[13] G. Kim, H. Shin, *Approximately quadratic mappings in non-Archimedean fuzzy normed spaces*, Nonlinear Funct. Anal. Appl. **2**3 (2018), 369–380.
[14] Y. Lee, S. Jung, *A general theorem on the fuzzy stability of a class of functional equations including quadratic-additive functional equations*, Nonlinear Funct. Anal. Appl. **2**3 (2018), 353–368.
[15] Y. Manar, E. Elqorachi, Th. M. Rassias, *Hyers-Ulam stability of the Jensen functional equation in quasi-Banach spaces*, Nonlinear Funct. Anal. Appl. **1**5 (2010), 581–603.
[16] Y. Manar, E. Elqorachi, Th. M. Rassias, *On the Hyers-Ulam stability of the quadratic and Jensen functional equations on a restricted domain*, Nonlinear Funct. Anal. Appl. **1**5 (2010), 647–655.
[17] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **3**43 (2008), 567–572.
[18] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2**007, Art. ID 50175 (2007).
[19] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2**008, Art. ID 493751 (2008).
[20] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[21] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.
[22] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.
[23] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[24] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **7**2 (1978), 297–300.
[25] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **5**3 (1983), 113–129.
[26] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

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