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Application of the product net technique and Kadec–Klee property to study nonlinear ergodic theorems and weak convergence theorems in uniformly convex multi-Banach spaces

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Abstract

Let Y be a uniformly convex multi-Banach space which has not a Frechet differentiable norm. We use the technique of product net to obtain the nonlinear ergodic theorems in Y. Finally, let the dual of uniformly convex multi-Banach space have the Kadec–Klee property, we instate the weak convergence theorem in the case of reversible semi-group.

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1 Preliminaries

Dales and Polyakov in [1] introduced a multi–normed space by using the concept of operator sequence space, operator spaces, and Banach lattices; for more details and application, we refer to [1–3].

In this paper assume that $(Y, \|\cdot\|)$ is a complex normed space, and let $\ell \in \mathbb{N}$. We denote by Y^ℓ the vector space $Y \oplus \cdots \oplus Y$ consisting of ℓ-tuples $(y_1, \ldots, y_\ell)$, where $y_1, \ldots, y_\ell \in Y$. The linear operations on Y^ℓ are defined coordinate-wise. The zero element of either Y or Y^ℓ is denoted by 0. We denote by $\mathbb{N}_\ell$ the set $\{1, 2, \ldots, \ell\}$ and by Σ_ℓ the group of permutations on ℓ symbols.

Definition 1.1 Suppose that Y is a vector space, and take $\ell \in \mathbb{N}$. For $\sigma \in \Sigma_\ell$, define

$$B_\sigma(y) = (y_{\sigma(1)}, \ldots, y_{\sigma(\ell)}), \quad y = (y_1, \ldots, y_\ell) \in Y^\ell.$$

For $\beta = (\beta_j) \in \mathbb{C}^\ell$, define

$$K_\beta(y) = (\beta_j y_j), \quad y = (y_1, \ldots, y_\ell) \in Y^\ell.$$

Definition 1.2 Assume that $(Y, \|\cdot\|)$ is a complex (respectively, real) normed space, and take $m \in \mathbb{N}$. A multi-norm of level m on $\{Y^\ell : \ell \in \mathbb{N}_m\}$ is a sequence $(\|\cdot\|_\ell : \ell \in \mathbb{N}_m)$ such that $\|\cdot\|$ is a norm on Y^ℓ for each $\ell \in \mathbb{N}_m$, such that $\|y\|_1 = \|y\|$ for each $y \in Y$ (so that $\|\cdot\|_1$ is the initial norm), and such that the following Axioms (a1)–(a4) are satisfied for each $\ell \in \mathbb{N}_m$ with $k \geq 2$:

(a1) for each $\sigma \in \Sigma_\ell$ and $y \in Y^\ell$, we have

$$\|B_\sigma(y)\|_\ell = \|y\|_\ell;$$

(a2) for each $\beta_1, \dots, \beta_\ell \in \mathbb{C}$ (respectively, each $\beta_1, \dots, \beta_\ell \in \mathbb{R}$) and $y \in Y^\ell$, we have

$$\|K_\beta(y)\|_\ell \leq \Bigl(\max_{j \in \mathbb{N}_\ell} |\beta_j|\Bigr)\|y\|_\ell;$$

(a3) for each $y_1, \dots, y_{\ell-1}$, we have

$$\|(y_1, \dots, y_{\ell-1}, 0)\|_\ell = \|(y_1, \dots, y_{\ell-1})\|_{\ell-1};$$

(a4) for each $y_1, \dots, y_{\ell-1} \in Y$,

$$\|(y_1, \dots, y_{\ell-2}, y_{\ell-1}, y_{\ell-1})\|_\ell = \|(y_1, \dots, y_{\ell-1})\|_{\ell-1}.$$

In this case, $((Y^\ell, \|\cdot\|_\ell) : \ell \in \mathbb{N}_m)$ is a multi-normed space of level m.

A multi-norm on $\{Y^\ell : \ell \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_\ell) = (\|\cdot\|_\ell : \ell \in \mathbb{N})$$

such that $(\|\cdot\|_\ell : \ell \in \mathbb{N}_m)$ is a multi-norm of level m for each $m \in \mathbb{N}$. In this case, $((Y^m, \|\cdot\|_m) : m \in \mathbb{N})$ is a multi-normed space.

Lemma 1.3 ([3]) *Let* $((Y^\ell, \|\cdot\|_\ell) : \ell \in \mathbb{N})$ *be a multi-normed space, and take* $\ell \in \mathbb{N}_m$. *Then*

(a) $\|(y, \dots, y)\|_\ell = \|y\|$ $(y \in Y)$;

(b) $\max_{j \in \mathbb{N}_\ell} \|y_j\| \leq \|(y_1, \dots, y_\ell)\|_\ell \leq \sum_{j=1}^{\ell} \|y_j\| \leq \ell \max_{j \in \mathbb{N}_\ell} \|y_j\|$ $(y_1, \dots, y_\ell \in Y)$.

It follows from (b) that if $(Y, \|\cdot\|)$ is a Banach space, then $(Y^\ell, \|\cdot\|_\ell)$ is a Banach space for each $\ell \in \mathbb{N}$; in this case $((Y^\ell, \|\cdot\|_\ell) : \ell \in \mathbb{N})$ is a multi-Banach space.

Example 1.4 ([1]) The sequence $(\|\cdot\|_\ell : \ell \in \mathbb{N})$ on $\{Y^\ell : \ell \in \mathbb{N}\}$ defined by

$$\|(y_1, \dots, y_\ell)\|_\ell := \max_{j \in \mathbb{N}_\ell} \|y_j\| \quad (y_1, \dots, y_\ell \in Y)$$

is a multi-norm called the minimum multi-norm.

Example 1.5 ([1]) Assume that $\{(\|\cdot\|_\ell^\beta : \ell \in \mathbb{N}) : \beta \in B\}$ is the (non-empty) family of all multi-norms on $\{Y^\ell : \ell \in \mathbb{N}\}$. For $\ell \in \mathbb{N}$, set

$$\|(y_1, \dots, y_\ell)\|_k := \sup_{\beta \in B} \|(y_1, \dots y_\ell)\|_\ell^\beta \quad (y_1, \dots, y_\ell \in Y).$$

Then $(\|\cdot\|_\ell : \ell \in \mathbb{N})$ is a multi-norm on $\{Y^\ell : \ell \in \mathbb{N}\}$, called the maximum multi-norm.

By the property (b) of multi-norms and the triangle inequality for the norm $\|\cdot\|_k$, we can get the following properties. Suppose that $((Y^\ell, \|\cdot\|_\ell) : \ell \in \mathbb{N})$ is a multi-normed space. Let $\ell \in \mathbb{N}$ and $(y_1, \dots, y_\ell) \in Y^k$. For every $i \in \{1, \dots, \ell\}$, let $(y^i_m)_{m=1,2,\dots}$ be a sequence in Y such that $\lim_{m\to\infty} y^i_m = y_i$. Then for each $(z_1, \dots, z_\ell) \in Y^\ell$ we have

$$\lim_{m\to\infty} \left(y^1_m - z_1, \dots, y^\ell_m - z_\ell\right) = (y_1 - z_1, \dots, y_\ell - z_\ell).$$

A sequence (y_m) in Y is a *multi-null* sequence if, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\sup_{\ell\in\mathbb{N}} \left\|(y_n, \dots, y_{m+\ell-1})\right\|_\ell < \varepsilon \quad (m \ge m_0).$$

Let $y \in Y$. We say that the sequence (y_m) is *multi-convergent* to $y \in Y$ and write

$$\lim_{m\to\infty} y_m = y$$

when $(y_m - y)$ is a multi-null sequence.

Assume that G is a semi-topological semi-group. In this article, C is a nonempty bounded closed convex subset of a uniformly convex Banach space X. Let X^* be the dual of X, then the value of $u^* \in X^*$ at $u \in X$ will be denoted by $\langle u, u^*\rangle$, and we associate the set

$$J(u) = \left\{u^* \in X : \langle u, u^*\rangle = \|u\|^2 = \left\|u^*\right\|^2\right\}.$$

It is clear from the Hahn–Banach theorem that $J(u)$ is not empty for all $u \in X$. Then the multi-valued operator $J : X \to X^*$ is called the normalized duality mapping of X, also $\Im_k = \{J_k(t) : t \in G\}$ is a reversible semigroup of asymptotically nonexpansive functions acting on C. Let $F(\Im_k)$ denote the set of all fixed points of $\Im_k$, i.e., $F(\Im_k) = \{u \in C : J_k(t)u = u, \forall t \in G\}$. For each $\epsilon > 0$ and $p \in G$, we put

$$F_\epsilon\left(J_k(p)\right) = \left\{u \in C : \left\|\left(J_1(p)u - u, \dots, J_k(p)u - u\right)\right\|_k \le \epsilon\right\}.$$

Note that if, for any $\epsilon > 0$, there exists $p_\epsilon \in G$ such that for all $p > p_\epsilon$, $u \in F_\epsilon(J_k(p))$, then $\lim_{p\in G} J_k(p)u = u$; moreover, $u \in F(\Im_k)$ by the continuity of elements $\{J_k(p), p \in G\}$ (for more details, we refer to [4–9]).

We denote the set of all almost orbits of $\Im_k$ and the set $\{J_k(p)u_k(\cdot) : p \in G, u_k \in \mathrm{AO}(\Im_k)\}$ by $\mathrm{AO}(\Im_k)$ and $\mathrm{LAO}(\Im_k)$, respectively. Denote by $\omega_\omega(u_k)$ the set of all weak limit points of subnets of net $\{u_k(t)\}_{t\in G}$.

Lemma 1.6 ([10]) *Assume that X is a Banach space and J is the normalized duality function. Therefore*

$$\|u + v\|^2 \le \|u\|^2 + 2\left\langle v, j(u + v)\right\rangle$$

for all $j(u + v) \in J(u + v)$ and $u, v \in X$.

Lemma 1.7 ([11]) *Assume that* $\{(X^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ *is a uniformly convex multi-Banach space and* $\emptyset \neq C \subset X^k$ *is a bounded closed convex set. Then there exists a strictly increasing continuous convex function* $\xi : [0,+\infty) \to [0,+\infty)$ *with* $\xi(0) = 0$ *such that*

$$\xi\left(\left\|\left(J_1\left(\sum_{i=1}^n a_i u_i\right) - \sum_{i=1}^n a_i J_1 u_i, \dots, J_k\left(\sum_{i=1}^n a_i u_i\right) - \sum_{i=1}^n a_i J_k u_i\right)\right\|_k\right)$$
$$\leq \max_{1\leq i,j\leq n}\left\{\|u_i - u_j\| - \left\|(J_1 u_i - J_1 u_j, \dots, J_k u_i - J_k u_j)\right\|_k\right\}$$

for all integers $a_1, \dots, a_n \geq 0$, $n \geq 1$ *with* $\sum_{i=1}^n a_i = 1$, $u_1, \dots, u_n \in C$, *and every nonexpansive function* J_k *of* C *to* C.

Lemma 1.7 implies that, for all $a_1, \dots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$, $u_1, \dots, u_n \in C$,

$$\left\|\left(J_1(p)\left(\sum_{i=1}^n a_i u_i\right) - \sum_{i=1}^n a_i J_1(p) u_i, \dots, J_k(p)\left(\sum_{i=1}^n a_i u_i\right) - \sum_{i=1}^n a_i J_k(p) u_i\right)\right\|_k$$
$$\leq \left(1 + \alpha(p)\right)\xi^{-1}\left(\max_{1\leq i,j\leq n}\left\{\|u_i - u_j\|\right.\right.$$
$$\left.\left. - \frac{1}{1+\alpha(p)}\left\|\left(J_1(p)u_i - J_1(p)u_j, \dots, J_k(p)u_i - J_k(p)u_j\right)\right\|_k\right\}\right)$$
$$\leq \left(1 + \alpha(p)\right)\xi^{-1}\left(\max_{1\leq i,j\leq n}\left\{\|u_i - u_j\|\right.\right.$$
$$\left.\left. - \left\|\left(J_1(p)u_i - J_1(p)u_j, \dots, J_k(p)u_i - J_k(p)u_j\right)\right\|_k\right\} + d\cdot\alpha(p)\right)$$

in which $d = 4\sup\{\|u\| : u \in C\} + 1$.

For every $\epsilon \in (0,1]$, define

$$a(\epsilon) = \min\left\{\frac{\epsilon^2}{(d+2)^2}, \frac{\epsilon^3}{(3d+2)^2}\xi\left(\frac{\epsilon}{4}\right)\right\}$$

and

$$G_\epsilon = \left\{h \in G : \alpha(p) \leq \epsilon\right\},$$

in which $\xi(\cdot)$ is as Lemma 1.7. Then $G_\epsilon \neq \emptyset$ for $\epsilon > 0$, and if $p \in G_\epsilon$, then for all $t \geq p$, $t \in G_\epsilon$. Note that $G_{a(\epsilon)} \subset G_\epsilon$ for all $\epsilon \in (0,1]$.

2 Main result

For studies on ergodic theory and its history, we refer to [4–30]. The results of this paper are an extension and generalization of [31].

Lemma 2.1 *For all* $p \in G_{a(\epsilon)}$,

$$\overline{\mathrm{co}}\, F_{a(\epsilon)}\left(J_k(p)\right) \subset F_\epsilon\left(J_K(p)\right).$$

Proof Since $F_\epsilon(J_K(p))$ is closed, we only need to prove that, for all $p \in G_{a(\epsilon)}$,

$$\operatorname{co} F_{a(\epsilon)}\big(J_k(p)\big) \subset F_\epsilon\big(J_K(p)\big).$$

Let $v = \sum_{i=1}^n a_i v_i$, $v_i \in F_{a(\epsilon)}(J_k(p))$, $a_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n a_i = 1$. Then

$$\begin{aligned}
&\big\| \big(J_1(p)v - v, \ldots, J_k(p)v - v\big) \big\|_k \\
&\quad = \left\| \left(J_1(p) \sum_{i=1}^n a_i v_i - \sum_{i=1}^n a_i v_i, \ldots, J_k(p) \sum_{i=1}^n a_i v_i - \sum_{i=1}^n a_i v_i \right) \right\|_k \\
&\quad \leq \left\| \left(J_1(p) \sum_{i=1}^n a_i v_i - \sum_{i=1}^n a_i J_1(p) v_i, \ldots, J_k(p) \sum_{i=1}^n a_i v_i - \sum_{i=1}^n a_i J_k(p) v_i \right) \right\|_k \\
&\quad \leq 2\xi^{-1} \Big(\max_{1 \leq i,j \leq n} \big\{ \|v_i - v_j\| - \big\| \big(J_1(p)v_i - J_1(p)v_i, \ldots, J_k(p)v_i - J_k(p)v_i\big) \big\|_k \big\} + d \cdot \alpha(p) \Big) \\
&\qquad + a(\epsilon) \\
&\quad \leq 2\xi^{-1} \Big(\max_{1 \leq i,j \leq n} \big\{ \big\| \big(v_i - J_1(p)v_i, \ldots, v_i - J_k(p)v_i\big) \big\|_k + \big\| \big(v_j - J_1(p)v_j, \ldots, v_j - J_k(p)v_j\big) \big\|_k \big\} \\
&\qquad + d \cdot \alpha(p) \Big) + a(\epsilon) \\
&\quad \leq 2\xi^{-1} \big(2a(\epsilon) + d \cdot a(\epsilon) \big) + a(\epsilon) \\
&\quad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Lemma 2.2 *For every* $p \in G_{\frac{\epsilon}{4}}$,

$$F_{\frac{\epsilon}{4}}\big(J_k(p)\big) + B\left(0, \frac{\epsilon}{4}\right) \subset F_\epsilon\big(J_k(p)\big).$$

Proof Let $p \in G_{\frac{\epsilon}{4}}$ and $u = v + w \in F_{\frac{\epsilon}{4}}(J_k(p)) + B(0, \frac{\epsilon}{4})$, where $v \in F_{\frac{\epsilon}{4}}(J_k(p))$ and $w \in B(0, \frac{\epsilon}{4})$, then

$$\begin{aligned}
&\big\| \big(J_1(p)u - u, \ldots, J_k(p)u - u\big) \big\|_k \\
&\quad = \big\| \big(J_1(p)(v + w) - (v + w), \ldots, J_k(p)(v + w) - (v + w)\big) \big\|_k \\
&\quad \leq \big\| \big(J_1(p)(v + w) - J_1(p)v, \ldots, J_k(p)(v + w) - J_k(p)v\big) \big\|_k \\
&\qquad + \big\| \big(J_1(p)v - v, \ldots, J_k(p)v - v\big) \big\|_k + \|w\| \\
&\quad \leq 2\|w\| + \big\| \big(J_1(p)v - v, \ldots, J_k(p)v - v\big) \big\|_k + \|w\| \\
&\quad \leq 3\frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

□

Lemma 2.3 *Assume that* $\epsilon \in (0, 1]$ *and* $p \in G_{a(a(\frac{\epsilon}{4}))}$, *so we can find* $n_0 \in N$ *such that, for each* $n \geq n_0$ *and* $u \in C$,

$$\frac{1}{n} \sum_{i=1}^n J_k\big(p^i\big)u \in F_\epsilon\big(J_k(p)\big).$$

Proof Let $\epsilon \in (0,1]$ and $m = \frac{2d+1}{a(\frac{\epsilon}{4})}$. There is $n_0 \in N$ satisfying

$$n_0 \geq \max\left\{\frac{12md}{\epsilon}, 32m^2d(d+1)\left(\xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right)\epsilon\right)^{-1}\right\}.$$

For any $n \geq n_0$ and $p \in G_{a(a(\frac{\epsilon}{4}))}$, we can take a number

$$K = m^2d\big(1 + 2n\alpha(p)\big)\left(\xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right)\right)^{-1} \quad \left(k < \frac{n}{2}\right).$$

For every $i \in N$ and $u \in C$, we put

$$\begin{aligned} a_i(u) = \xi\Bigg(\frac{8}{9}\Bigg\|\Bigg(&\frac{1}{m}\sum_{j=1}^{m} J_1\big(p^{i+j+1}\big)u - J_1(p)\frac{1}{m}\sum_{j=1}^{m} J_1\big(p^{i+j}\big)u, \dots, \\ &\frac{1}{m}\sum_{j=1}^{m} J_k\big(p^{i+j+1}\big)u - J_k(p)\frac{1}{m}\sum_{j=1}^{m} J_k\big(p^{i+j}\big)u\Bigg)\Bigg\|_k\Bigg). \end{aligned}$$

By $\alpha(p) \leq \frac{1}{8}$ and

$$\begin{aligned} a_i(u) &\leq \max_{1\leq j,t\leq m}\big\{\big\|J_1\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u, \dots, J_k\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u\big\|_k \\ &\quad - \big\|J_1\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u, \dots, J_k\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u\big\|_k + d\cdot\alpha(p)\big\} \\ &\leq \sum_{1\leq j<t\leq m}\big(\big\|\big(J_1\big(p^{i+j}\big)u - J_1\big(p^{i+t}\big)u, \dots J_k\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u\big)\big\|_k \\ &\quad - \big\|\big(J_1\big(p^{i+j+1}\big)u - J_1\big(p^{i+t+1}\big)u, \dots J_k\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u\big)\big\|_k + d\alpha(p)\big), \end{aligned}$$

we get

$$\begin{aligned} &\sum_{i=1}^{n} a_i(u) \\ &\quad\leq \sum_{i=1}^{n}\sum_{1\leq j<t\leq m}\big(\big\|J_1\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u, \dots, J_k\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u\big\|_k \\ &\qquad - \big\|J_1\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u, \dots, J_k\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u\big\|_k + d\cdot\alpha(p)\big) \\ &\quad= \sum_{1\leq j<t\leq m}\sum_{i=1}^{n}\big(\big\|J_1\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u, \dots, J_k\big(p^{i+j}\big)u - J_k\big(p^{i+t}\big)u\big\|_k \\ &\qquad - \big\|J_1\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u, \dots, J_k\big(p^{i+j+1}\big)u - J_k\big(p^{i+t+1}\big)u\big\|_k + d\cdot\alpha(p)\big) \\ &\quad\leq \sum_{1\leq j<t\leq m}\big(d + nd\cdot\alpha(p)\big) \leq m^2d\big(1 + n\alpha(p)\big). \end{aligned}$$

Suppose that there is an element say t in $\{a_i(u) : i = 1, 2, \dots, 2n\}$ such that if $a_i(u) \geq \xi(\frac{a(\frac{\epsilon}{4})}{2})$, then

$$t\xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right) \leq m^2d\big(1 + 2n\alpha(p)\big).$$

Hence

$$t \le m^2 d\bigl(1 + 2n\alpha(p)\bigr)\left(\xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right)\right)^{-1} = K.$$

So, there are at most $N = [K]$ terms in $\{a_i(u) : i = 1, 2, \ldots, 2n\}$ with $a_i(u) \ge \xi(\frac{a(\frac{\epsilon}{4})}{2})$. Then, for every i in $\{1, 2, \ldots, n\}$, there exists at least one term $a_{i+j_0}(u)$ $(0 \le j_0 \le N)$ in $\{a_{i+j}(u) : j = 0, 1, \ldots, N\}$ hold $a_{i+j_0} < \xi(\frac{a(\frac{\epsilon}{4})}{2})$.

Put

$$\ell_i = \min\left\{j : a_{i+j}(u) < \xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right), 0 \le j \le N\right\},$$

$i = 1, 2, \ldots, n$. Now, there are at most N elements in $\{i = 1, 2, \ldots, n\}$ such that $\ell_i \ne 0$. Since

$$\begin{aligned}
&\Bigg\|\Bigg(J_1(p)\frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j}\bigr)u, \ldots, \\
&\quad J_k(p)\frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u\Bigg)\Bigg\|_k \\
&\le \Bigg\|\Bigg(J_1(p)\frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j+1}\bigr)u, \ldots, \\
&\quad J_k(p)\frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_k\bigl(h^{i+\ell_i+j+1}\bigr)u\Bigg)\Bigg\|_k \\
&\quad + \Bigg\|\Bigg(\frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_1\bigl(p^{i+\ell_i+j+1}\bigr)u, \ldots, \\
&\quad \frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u - \frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j+1}\bigr)u\Bigg)\Bigg\|_k \\
&\le \frac{9}{8}\xi^{-1}\bigl(a_{i+\ell_i}(u)\bigr) + \frac{d}{2m} \\
&\le \frac{9}{16}a\left(\frac{\epsilon}{4}\right) + \frac{1}{4}a\left(\frac{\epsilon}{4}\right) < a\left(\frac{\epsilon}{4}\right),
\end{aligned}$$

we can conclude that, for all $p \in G_{a(a(\frac{\epsilon}{4}))}$,

$$\frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u \in F_{a(\frac{\epsilon}{4})}\bigl(J_k(p)\bigr).$$

By Lemma 2.1, we get, for all $p \in G_{a(a(\frac{\epsilon}{4}))} \subset G_{a(\frac{\epsilon}{4})}$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} J_k\bigl(p^{i+\ell_i+j}\bigr)u \in \operatorname{co} F_{a(\frac{\epsilon}{4})}\bigl(J_k(p)\bigr) \subset F_{\frac{\epsilon}{4}}\bigl(J_k(p)\bigr).$$

Using Lemma 2.2 and

$$\begin{aligned}
&\left\| \left(\frac{1}{n}\sum_{i=1}^{n} J_1(p^i)u - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} J_1(p^{i+\ell_i+j})u, \ldots, \right.\right. \\
&\quad \left.\left. \frac{1}{n}\sum_{i=1}^{n} J_k(p^i)u - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} J_k(p^{i+\ell_i+j})u \right) \right\|_k \\
&\le \frac{1}{mn}\sum_{j=1}^{m} \left\| \left(\sum_{i=1}^{n} J_1(p^i)u - \sum_{i=1}^{n} J_1(p^{i+\ell_i+j})u, \ldots, \sum_{i=1}^{n} J_k(p^i)u - sum_{i=1}^{n} J_k(p^{i+\ell_i+j})u \right) \right\|_k \\
&\le \frac{1}{mn}\sum_{j=1}^{m} \left\| \left(\sum_{i=1}^{n} J_1(p^i)u - \sum_{i=1}^{n} J_1(p^{i+j})u, \ldots, \sum_{i=1}^{n} J_k(p^i)u - \sum_{i=1}^{n} J_k(p^{i+j})u \right) \right\|_k \\
&\quad + \frac{1}{mn}\sum_{j=1}^{m} \left\| \left(\sum_{i=1}^{n} L_1(p^{i+j})u - \sum_{i=1}^{n} J_1(p^{i+\ell_i+j})u, \ldots, \right.\right. \\
&\quad \left.\left. \sum_{i=1}^{n} J_k(p^{i+j})u - \sum_{i=1}^{n} J_k(p^{i+\ell_i+j})u \right) \right\|_k \\
&\le \frac{md}{n} + \frac{Nd}{n} \\
&\le \frac{\epsilon}{12} + \frac{m^2d^2(\xi(\frac{a(\frac{\epsilon}{4})}{2}))^{-1}}{n} + 2m^2d^2\alpha(p)\left(\xi\left(\frac{a(\frac{\epsilon}{4})}{2}\right)\right)^{-1} \\
&< \frac{\epsilon}{12} + \frac{\epsilon}{32} + \frac{\epsilon}{8} < \frac{\epsilon}{4},
\end{aligned}$$

we obtain

$$\frac{1}{n}\sum_{i=1}^{n} J_k(p^i)u \in F_{\frac{\epsilon}{4}}\big(J_k(p)\big) + B\left(0, \frac{\epsilon}{4}\right) \subset F_\epsilon\big(J_k(p)\big).$$

□

Lemma 2.4 *Suppose that $u_k(\cdot)$ is an almost orbit of $\Im_k$. So*

$$\lim_{t\in G} \big\| \big(\gamma u_1(t) + (1-\gamma)\varphi - g, \ldots, \gamma u_k(t) + (1-\gamma)\varphi - g\big) \big\|_k$$

exist for every $\gamma \in (0,1)$ and $\varphi, g \in F(\Im_k)$.

Proof To complete the proof, it is enough to prove that

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G} \big\| \big(\gamma u_1(ts) + (1-\gamma)\varphi - g, \ldots, \gamma u_k(ts) + (1-\gamma)\varphi - g\big) \big\|_k \\
&\quad \le \sup_{s\in G}\inf_{t\in G} \big\| \big(\gamma u_1(ts) + (1-\gamma)\varphi - g, \ldots, \gamma u_k(ts) + (1-\gamma)\varphi - g\big) \big\|_k.
\end{aligned}$$

We know, for every $\epsilon > 0$, there are t_0 and $s_0 \in G$ such that, for any $t \in G$, $\alpha(tt_0) < \frac{\epsilon}{1+d}$ and $\varphi(ts_0) < \epsilon$, where $\varphi(t) = \sup_{p\in G} \|(u_1(pt) - J_1(p)u_1(t), \ldots, u_k(pt) - J_k(p)u_k(t))\|_k$. So, for every

$a \in G$,

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\big\|\big(u_1(tss_0)-\varphi,\dots,u_k(tss_0)-\varphi\big)\big\|_k \\
&\quad\le \sup_{t\in G}\big\|\big(u_1(tt_0as_0)-\varphi,\dots,u_k(tt_0as_0)-\varphi\big)\big\|_k \\
&\quad\le \sup_{t\in G}\big\|\big(u_1(tt_0as_0)-J_1(tt_0)u_1(as_0),\dots,u_k(tt_0as_0)-J_k(tt_0)u_k(as_0)\big)\big\|_k \\
&\qquad+\sup_{t\in G}\big\|J_1(tt_0)u_1(as_0)-\varphi,\dots,J_k(tt_0)u_k(as_0)-\varphi\big\|_k \\
&\quad\le \varphi(as_0)+\sup_{t\in G}\big(1+\alpha(tt_0)\big)\cdot\big\|\big(u_1(as_0)-\varphi,\dots,u_k(as_0)-\varphi\big)\big\|_k \\
&\quad\le \big\|\big(u_1(as_0)-\varphi,\dots,u_k(as_0)-\varphi\big)\big\|_k+2\epsilon.
\end{aligned}$$

Hence

$$\inf_{s\in G}\sup_{t\in G}\big\|\big(u_1(tss_0)-\varphi,\dots,u_k(tss_0)-\varphi\big)\big\|_k\le\inf_{a\in G}\big\|\big(u_1(as_0)-\varphi,\dots,u_k(as_0)-\varphi\big)\big\|_k+2\epsilon.$$

Thus, there exists $s_1 \in G$ such that

$$\sup_{t\in G}\big\|\big(u_1(ts_1s_0)-\varphi,\dots,u_k(ts_1s_0)-\varphi\big)\big\|_k<\inf_{a\in G}\big\|\big(u_1(as_0)-\varphi,\dots,u_k(as_0)-\varphi\big)\big\|_k+3\epsilon.$$

Then, for every $a \in G$, we get

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\big\|\big(\gamma u_1(ts)+(1-\gamma)\varphi-g,\dots,\gamma u_k(ts)+(1-\gamma)\varphi-g\big)\big\|_k \\
&\quad\le \sup_{t\in G}\big\|\big(\gamma u_1(tt_0as_1s_0)+(1-\gamma)\varphi-g,\dots,\gamma u_k(tt_0as_1s_0)+(1-\gamma)\varphi-g\big)\big\|_k \\
&\quad\le \gamma\sup_{t\in G}\big\|\big(u_1(tt_0as_1s_0)-J_1(tt_0)u_1(as_1s_0),\dots,u_k(tt_0as_1s_0)-J_k(tt_0)u_k(as_1s_0)\big)\big\|_k \\
&\qquad+\sup_{t\in G}\big\|\big(\gamma J_1(tt_0)u_1(as_1s_0)+(1-\gamma)\varphi-g,\dots,\gamma J_k(tt_0)u_k(as_1s_0)+(1-\gamma)\varphi-g\big)\big\|_k \\
&\quad\le \varphi(as_1s_0)+\sup_{t\in G}\big\|\big(\gamma J_1(tt_0)u_1(as_1s_0)+(1-\gamma)\varphi-J_1(tt_0)\big(\gamma u_1(as_1s_0)+(1-\gamma)\varphi\big),\dots, \\
&\qquad\gamma J_k(tt_0)u_k(as_1s_0)+(1-\gamma)\varphi-J_k(tt_0)\big(\gamma u_k(as_1s_0)+(1-\gamma)\varphi\big)\big)\big\|_k \\
&\qquad+\sup_{t\in G}\big\|\big(J_1(tt_0)\big(\gamma u_1(as_1s_0)+(1-\gamma)\varphi\big)-g,\dots, \\
&\qquad J_k(tt_0)\big(\gamma u_k(as_1s_0)+(1-\gamma)\varphi\big)-g\big)\big\|_k \\
&\qquad+\epsilon\sup_{t\in G}\big(1+\alpha(tt_0)\big)\xi^{-1}\big(\big\|\big(u_1(as_1s_0)-\varphi,\dots,u_k(as_1s_0)-\varphi\big)\big\|_k \\
&\qquad-\big\|\big(J_1(tt_0)u_1(as_1s_0)-\varphi,\dots,J_k(tt_0)u_k(as_1s_0)-\varphi\big)\big\|_k+d\cdot\alpha(tt_0)\big) \\
&\qquad+\sup_{t\in G}\big(1+\alpha(tt_0)\big)\big\|\big(\gamma u_1(as_1s_0)+(1-\gamma)\varphi-g,\dots,\gamma u_k(as_1s_0)+(1-\gamma)\varphi-g\big)\big\|_k \\
&\quad\le \epsilon+(1-\epsilon)\sup_{t\in G}\xi^{-1}\big(\big\|\big(u_1(as_1s_0)-\varphi,\dots,u_k(as_1s_0)-\varphi\big)\big\|_k \\
&\qquad-\big\|\big(u_1(tt_0as_1s_0)-\varphi,\dots,u_k(tt_0as_1s_0)-\varphi\big)\big\|_k+\varphi(as_1s_0)+\epsilon\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1+\epsilon)\bigl\|\bigl(\gamma u_1(as_1s_0) + (1-\gamma)\varphi - g,\dots,\gamma u_k(as_1s_0) + (1-\gamma)\varphi - g\bigr)\bigr\|_k \\
&\le \epsilon + (1+\epsilon)\xi^{-1}(5\epsilon) \\
&\quad + (1+\epsilon)\bigl\|\bigl(\gamma u_1(as_1s_0) + (1-\gamma)\varphi - g,\dots,\gamma u_k(as_1s_0) + (1-\gamma)\varphi - g\bigr)\bigr\|_k.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\bigl\|\bigl(\gamma u_1(ts) + (1-\gamma)\varphi - g,\dots,\gamma u_k(ts) + (1-\gamma)\varphi - g\bigr)\bigr\|_k \\
&\quad\le \epsilon(1+\epsilon)\xi^{-1}(5\epsilon) \\
&\qquad + (1+\epsilon)\inf_{a\in G}\bigl\|\bigl(\gamma u_1(as_1s_0) + (1-\gamma)\varphi - g,\dots,\gamma u_k(as_1s_0) + (1-\gamma)\varphi - g\bigr)\bigr\|_k \\
&\quad\le \epsilon(1+\epsilon)\xi^{-1}(5\epsilon) \\
&\qquad + (1+\epsilon)\sup_{b\in G}\inf_{a\in G}\bigl\|\bigl(\gamma u_1(ab) + (1-\gamma)\varphi - g,\dots,\gamma u_k(ab) + (1-\gamma)\varphi - g\bigr)\bigr\|_k.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\bigl\|\bigl(\gamma u_1(ts) + (1-\gamma)\varphi - g,\dots,\gamma u_k(ts) + (1-\gamma)\varphi - g\bigr)\bigr\|_k \\
&\quad\le \sup_{s\in G}\inf_{t\in G}\bigl\|\bigl(\gamma u_1(ts) + (1-\gamma)\varphi - g,\dots,\gamma u_k(ts) + (1-\gamma)\varphi - g\bigr)\bigr\|_k
\end{aligned}
$$

because $\epsilon > 0$ is arbitrary. □

Theorem 2.5 *Assume that $\{(X^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ is a uniformly convex multi-Banach space, and suppose that $\emptyset \neq C \subset X$ is bounded and closed. Assume that $\Im_k = \{J_k(t) : t \in G\}$ for each $k \ge 1$ is a reversible semigroup of asymptotically nonexpansive functions on C. If D has a left invariant mean, then there exists a retraction P_k from $\mathrm{LAO}(\Im_k)$ onto $F(\Im_k)$ in which*:

(1) *P_k is nonexpansive in the sense*

$$
\begin{aligned}
&\bigl\|(P_1u_1 - P_1v_1,\dots,P_ku_k - P_kv_k)\bigr\|_k \\
&\quad\le \inf_{s\in G}\sup_{t\in G}\bigl\|\bigl(u_1(st) - v_1(st),\dots,u_k(st) - v_k(st)\bigr)\bigr\|_k,\quad \forall u_k, v_k \in \mathrm{LAO}(\Im_k);
\end{aligned}
$$

(2) *$P_kJ_k(p)u_k = J_k(p)P_ku_k = P_ku_k$ for all $u_k \in \mathrm{AO}(\Im_k)$ and $p \in G$;*

(3) *$P_ku_k \in \bigcap_{s\in G}\overline{\mathrm{conv}}\{u_k(t) : t \ge s\}$ for all $u_k \in \mathrm{LAO}(\Im_k)$.*

Proof We know D has a left invariant mean, so there is a net $\{\gamma_{k,\alpha} : \alpha \in A\}$ of finite means on G in which $\lim_{\alpha\in A}\|(\gamma_{1,\alpha} - \ell_s^*\gamma_{1,\alpha},\dots,\gamma_{k,\alpha} - \ell_s^*\gamma_{k,\alpha})\|_k = 0$ for every $s \in G$, in which A is a directed system. Putting $I = A \times G = \{\beta = (\alpha, t) : \alpha \in A, t \in G\}$. For $\beta_i = (\alpha_i, t_i) \in I$, $i = 1, 2$, define $\beta_1 \le \beta_2$ iff $\alpha_1 \le \alpha_2$, $t_1 \le t_2$. Then, I is also a directed system. For each $\beta = (\alpha, t) \in I$, define $P_{k,1}\beta = \alpha$, $P_{k,2}\beta = t$, and $\gamma_\beta = \gamma_\alpha$. So, for every $s \in G$,

$$
\lim_{\beta\in I}\bigl\|\bigl(\gamma_{1,\beta} - \ell^*\gamma_{1,\beta},\dots,\gamma_{k,\beta} - \ell^*\gamma_{k,\beta}\bigr)\bigr\|_k = 0. \tag{2.1}
$$

Assume that $\gamma = \{\{t_\beta\}_{\beta\in I}, t_\beta \ge P_{k,2}\beta, \forall\beta \in I\}$. Taking any $\{t_\beta, \beta \in I\} \in \gamma$, since $r_{t\beta}^*\gamma_{k,\beta}$ is bounded, without loss of generality, let $r_{t\beta}^*\gamma_{k,\beta}$ be *weak** convergent. Then, for all $u_k \in$

$\mathrm{LAO}(\Im_k)$, $\omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\langle u_k(tt_\beta)\rangle$ exist. We define

$$P_k u_k = \omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\big\langle u_k(tt_\beta)\big\rangle.$$

On the other hand, for every $u_k \in \mathrm{LAO}(\Im_k)$, $P_k u_k \in \bigcap_{s\in G}\overline{\mathrm{conv}}\{u(t) : t \geq s\}$. Next, we shall show that $P_k u_k \in F(\Im_k)$. Then, for every $\epsilon \in (0,1]$, there is $t_0 \in G$ such that, for each $t \geq t_0$, $\varphi(t) < \frac{a(\epsilon)}{4}$. Also, we can suppose that $P_{k2}\beta \geq t_0$ for every $\beta \in I$, so $t_\beta \geq t_0$, $\{t_\beta\} \in \gamma$. From Lemma 2.3, for every $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$, there is $n \in N$ such that, for each $t \in G$ and $\beta \in I$,

$$\frac{1}{n}\sum_{i=1}^{n} J_k\big(p^i\big)u_k(tt_\beta) \in F_{\frac{a(\epsilon)}{4}}\big(J_k(p)\big).$$

Since for every $t \in G$

$$\begin{aligned}&\left\|\left(\frac{1}{n}\sum_{i=1}^{n} J_1\big(p^i\big)u_1(tt_\beta) - \frac{1}{n}\sum_{i=1}^{n} u_1\big(p^i tt_\beta\big),\ldots,\frac{1}{n}\sum_{i=1}^{n} J_k\big(p^i\big)u_k(tt_\beta) - \frac{1}{n}\sum_{i=1}^{n} u_k\big(p^i tt_\beta\big)\right)\right\|_k\\ &\quad\leq \varphi(tt_\beta) < \frac{a(\epsilon)}{4},\end{aligned}$$

we have, for every $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$,

$$\frac{1}{n}\sum_{i=1}^{n} u_k\big(p^i tt_\beta\big) \in F_{\frac{a(\epsilon)}{4}}\big(J_k(p)\big) + B\left(0,\frac{a(\epsilon)}{4}\right) \subset F_{a(\epsilon)}\big(J_k(p)\big).$$

Equation (2.1) implies that

$$\begin{aligned}&\lim_{\beta\in I}\left\|\left(\gamma_{1,\beta}(t)\left\langle\frac{1}{n}\sum_{i=1}^{n} u_1\big(p^i tt_\beta\big)\right\rangle - \gamma_{1,\beta}\big\langle u_1(tt_\beta)\big\rangle,\ldots,\right.\right.\\ &\quad\left.\left.\gamma_{k,\beta}(t)\left\langle\frac{1}{n}\sum_{i=1}^{n} u_k\big(p^i tt_\beta\big)\right\rangle - \gamma_{k,\beta}\big\langle u_k(tt_\beta)\big\rangle\right)\right\|_k = 0.\end{aligned}$$

Combining it with the definition of $P_k u_k$, we get, for all $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$,

$$P_k u_k = \omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\left\langle\frac{1}{n}\sum_{i=1}^{n} u_k\big(p^i tt_\beta\big)\right\rangle \in \overline{\mathrm{co}}\, F_{a(\epsilon)}\big(J_k(p)\big).$$

Lemma 2.1 also implies that for every $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$, $P_k u_k \in F_\epsilon(J_k(p))$. Now, the continuity of $J_k(p)$ implies that $P_k u_k \in F(\Im_k)$. Obviously, for any $p \in G$,

$$\begin{aligned}P_k J_k(p)u_k &= \omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\big\langle J_k(p)u_k(tt_\beta)\big\rangle\\ &= \omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\big\langle u_k(htt_\beta)\big\rangle\\ &= \omega\text{-}\lim_{\beta\in I}\gamma_{k,\beta}(t)\big\langle u_k(tt_\beta)\big\rangle \quad \text{(using (2.1))}\\ &= P_k u_k\end{aligned}$$

and for every $v_k \in \mathrm{LAO}(\Im_k)$ and $s \in G$, we have

$$\begin{aligned}
&\big\|(P_1u_1 - P_1v_1, \dots, P_1u_1 - P_1v_1)\big\|_k \\
&\quad\le \liminf_{\beta\in I}\big\|\big(\gamma_{1,\beta}(t)\langle u_1(tt_\beta)\rangle - \gamma_{1,\beta}(t)\langle v_1(tt_\beta)\rangle, \dots, \gamma_{k,\beta}(t)\langle u_k(tt_\beta)\rangle - \gamma_{k,\beta}(t)\langle v_k(tt_\beta)\rangle\big)\big\|_k \\
&\quad= \liminf_{\beta\in I}\big\|\big(\gamma_{1,\beta}(t)\langle u_1(stt_\beta)\rangle - \gamma_{1,\beta}(t)\langle v_1(stt_\beta)\rangle, \dots, \\
&\qquad \gamma_{k,\beta}(t)\langle u_k(stt_\beta)\rangle - \gamma_{k,\beta}(t)\langle v_k(stt_\beta)\rangle\big)\big\|_k \quad (\text{by } (2.1)) \\
&\quad\le \liminf_{\beta\in I}\big\|\big(\gamma_{1,\beta}(t), \dots, \gamma_{1,\beta}(t)\big)\big\|_k \cdot \sup_{t\in G}\big\|\big(u_1(stt_\beta) - v_1(stt_\beta), \dots, u_k(stt_\beta) - v_k(stt_\beta)\big)\big\|_k \\
&\quad\le \sup_{t\in G}\big\|\big(u_1(st) - v_1(st), \dots, u_k(st) - v_k(st)\big)\big\|_k.
\end{aligned}$$

Thus,

$$\big\|(P_1u_1 - P_1v_1, \dots, P_ku_k - P_kv_k)\big\|_k \le \inf_{s\in G}\sup_{t\in G}\big\|\big(u_1(st) - v_1(st), \dots, u_k(st) - v_k(st)\big)\big\|_k. \qquad \square$$

Theorem 2.6 (Ergodic theorem [17]) *Assume that $\{(X^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ is a uniformly convex multi-Banach space, and suppose that $\emptyset \neq C \subset X$ is bounded and closed. Assume that $\Im_k = \{J_k(t) : t \in G\}$ is a reversible semigroup of asymptotically nonexpansive functions on C. If D has a left invariant mean and there is a unique retraction P_k from $\mathrm{LAO}(\Im_K)$ onto $F(\Im_k)$, which satisfies properties (1)–(3) in Theorem 2.5, then for every strongly net $\{v_{k,\alpha} : \alpha \in A\}$ on D and $u_k \in \mathrm{AO}(\Im_k)$,*

$$\omega\text{-}\lim_{\alpha\in A}\int u_k(tp)\,dv_{k,\alpha}(t) = P_k \in F(\Im_k) \quad \textit{uniformly in } p \in \gamma(G),$$

in which $\gamma(G) = \{s \in G : st = ts \text{ for all } t \in G\}$.

Theorem 2.7 *Assume that $\{(X^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ is a uniformly convex multi-Banach space, and suppose that $\emptyset \neq C \subset X$ is bounded and closed. Assume that $\Im_k = \{J_k(t) : t \in G\}$ of a reversible semigroup of asymptotically nonexpansive mappings on C, and let $u_k(\cdot)$ be an almost orbit of $\Im_k$. If*

$$\omega\text{-}\lim_{t\in G} u_k(pt) - u_k(t) = 0$$

for every $p \in G$, then

$$\omega_\omega(u_k) \subset F(\Im_k).$$

Proof Let $\epsilon \in (0,1]$, then there is $t_0 \in G$ such that, for $t \ge t_0$, $\varphi(t) < \frac{a(\epsilon)}{4}$. Suppose that $p_k \in \omega_\omega(u_k)$, so we can find a subnet $\{u_k(t_\alpha)\}_{\alpha\in A}$ of $\{u_k(t)\}_{t\in G}$ with $\omega\text{-}\lim_{\alpha\in A} u_k(t_\alpha) = p_k$ in which, for every $\alpha \in A$, $t_\alpha \ge t_0$, in which A is a directed system. Using Lemma 2.3, for every $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$, we can find $n \in \mathbb{N}$ such that, for every $\alpha \in A$,

$$\frac{1}{n}\sum_{i=1}^{n} J_k\big(p^i\big)u_k(t_\alpha) \in F_{\frac{a(\epsilon)}{4}}\big(J_k(p)\big).$$

Since for each $\alpha \in A$

$$\begin{aligned} &\left\| \left(\frac{1}{n}\sum_{i=1}^{n} J_1(p^i)u_1(t_\alpha) - \frac{1}{n}\sum_{i=1}^{n} u_1(p^i t_\alpha), \dots, \frac{1}{n}\sum_{i=1}^{n} J_k(p^i)u_k(t_\alpha) - \frac{1}{n}\sum_{i=1}^{n} u_k(p^i t_\alpha) \right) \right\|_k \\ &\quad \le \varphi(t_\alpha) < \frac{a(\epsilon)}{4}, \end{aligned}$$

we get

$$\frac{1}{n}\sum_{i=1}^{n} u_k(p^i t_\alpha) \in \frac{1}{n}\sum_{i=1}^{n} u_k(p^i t_\alpha) + B\left(0, \frac{a(\epsilon)}{4}\right) \subset F_{a(\epsilon)}\big(J_k(p)\big).$$

Since $u_k(pt) - u_k(t) \to 0$ for every $p \in G$, we have $u_k(p^i t_\alpha) \to p_k$, $i = 1, 2, \dots, n$. Then, for all $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$,

$$p_k = \omega\text{-}\lim_{\alpha \in A} \frac{1}{n}\sum_{i=1}^{n} u_k(p^i t_\alpha) \in \overline{\mathrm{co}}\, F_{a(\epsilon)}\big(J_k(p)\big).$$

So, Lemma 2.1, implies that for every $p \in G_{a(a(\frac{a(\epsilon)}{16}))}$, $p \in F_\epsilon(J(p))$, hence $p_k \in F(\Im_k)$. □

In three last theorems X has not a Frechet differentiable norm.

Theorem 2.8 *Assume that* $\{(X^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ *is a uniformly convex multi-Banach space with the Kadec–Klee property for its dual, and* $\emptyset \neq C \subset X$ *is bounded and closed. Suppose that* $\Im_k = \{J_k(t) : t \in G\}$ *of a reversible semigroup of asymptotically nonexpansive function on* C *and* $u_k(\cdot)$ *is an almost orbit of* $\Im_k$. *Then the following statements are equivalent*:

(1) $\omega_\omega(u_k) \subset F(\Im_k)$;

(2) $\omega\text{-}\lim_{t\in G} u_k(t) = p_k \in F(\Im_k)$;

(3) $\omega\text{-}\lim_{t\in G} u_k(pt) - u_k(t) = 0$ *for every* $p \in G$.

Proof (1) ⇒ (2). It is enough to prove that $\omega_\omega(u_k)$ is a singleton. The reflexivity of X implies that $X \neq \emptyset$. Suppose that φ_k and g_k are two elements in $\omega_\omega(u_k)$, then by (1) we get $\varphi, g \in F(\Im_k)$. For every $\gamma \in (0,1)$, using Lemma 2.4, we have $\lim_{t\in G} \|(\gamma u_1(t) + (1-\gamma)\varphi - g, \dots, \gamma u_k(t) + (1-\gamma)\varphi - g)\|_k$ exists. Put

$$h(\gamma) = \lim_{t\in G} \big\| \big(\gamma u_1(t) + (1-\gamma)\varphi - g, \dots, \gamma u_k(t) + (1-\gamma)\varphi - g\big) \big\|_k,$$

then for given $\epsilon > 0$, there is $t_1 \in G$ such that, for every $t > t_1$,

$$\big\| \big(\gamma u_1(t) + (1-\gamma)\varphi - g, \dots, \gamma u_k(t) + (1-\gamma)\varphi - g\big) \big\|_k \le h(\gamma) + \epsilon.$$

So, for every $t \ge t_1$,

$$\big\langle \gamma u_k(t) + (1-\gamma)\varphi - g, j(\varphi - g)\big\rangle \le \|\varphi - g\| \big(h(\gamma) + \epsilon\big),$$

in which $j(\varphi - g) \in J(\varphi - g)$. Let us note $\varphi \in \overline{\mathrm{co}}\{u_k(t) : t \ge t_1\}$, so

$$\big\langle \gamma \varphi + (1-\gamma)\varphi - g, j(\varphi - g)\big\rangle \le \|\varphi - g\| \big(h(\gamma) + \epsilon\big),$$

which means $\|\varphi - g\| \le h(\gamma) + \epsilon$. We know ϵ is arbitrary, then

$$\|\varphi - g\| \le h(\gamma).$$

$g \in \omega_\omega(u_k)$ implies that there is a subnet $\{u_k(t - \alpha)\}_{\alpha\in A}$ in $\{u_k(t)\}_{t\in G}$ such that $\omega\text{-}\lim_{\alpha\in A} u_k(t_\alpha) = g$, in which A is a directed system. Setting

$$I = A \times \mathbb{N} = \bigl\{\beta = (\alpha, n) : \alpha \in A, n \in \mathbb{N}\bigr\},$$

then for $\beta_i = (\alpha_i, n_i)$, $i \in I$, $i = 1, 2$, define $\beta_1 \le \beta_2$ iff $\alpha_1 \le \alpha_2$, $n_1 \le n_2$. For arbitrary $\beta = (\alpha, n) \in I$, define $P_{k,1}\beta = \alpha$, $P_{k,2}\beta = n$, $t_\beta = t_\alpha$, $\epsilon_\beta = \frac{1}{P_{k,2}\beta}$. Then $\omega\text{-}\lim_{\beta\in I} u_k(t_\beta) = g$ and $\lim_{\beta\in I} \epsilon_\beta = 0$. Using Lemma 1.6 implies that

$$\begin{aligned}
&\bigl\|\bigl(\gamma u_1(t_\beta) + (1-\gamma)\varphi - g, \ldots, \gamma u_k(t_\beta) + (1-\gamma)\varphi - g\bigr)\bigr\|_k \\
&\quad \le \|\varphi - g\|^2 + 2\gamma\bigl\langle u_k(t_\beta) - \varphi, j\bigl(\gamma u_k(t_\beta) + (1-\gamma)\varphi - g\bigr)\bigr\rangle.
\end{aligned}$$

Using Lemma 2.4 and the inequality $\|\varphi - g\| \le h(\gamma)$ implies that

$$\liminf_{\beta\in I}\bigl\langle u_k(t_\beta - \varphi, j\bigl(\gamma u_k(t_\beta) + (1-\gamma)\varphi - g\bigr)\bigr\rangle \ge 0.$$

So, for each $\xi \in I$, there is $\beta_\xi \in I$ such that $\beta_\xi \ge \gamma$ and

$$\bigl\langle u_k(t_{\beta_\xi}) - \varphi, j\bigl(\epsilon_\xi u_k(t_{\beta_\xi}) + (1 - \epsilon_\xi\varphi - g)\bigr)\bigr\rangle \ge -\epsilon_\xi. \tag{2.2}$$

It is well known that $\{\beta_\xi\}$ is also a subnet of I, then $\omega\text{-}\lim_{\xi\in I} u_k(t_{\beta_\xi}) = g$. Set

$$j_\xi = j\bigl(\epsilon_\xi u_k(t_{\beta_\xi}) + (1 - \epsilon_\xi\varphi - g)\bigr).$$

The reflexivity of X implies that X^* is also reflexive, and therefore the set of all weak limit points of $\{j_\xi, \xi \in I\}$ is nonempty. Then, without loss of generality, let $\omega\text{-}\lim_{\xi\in I} j_\xi = j \in X^*$. Then $\|j\| \le \liminf_{\xi\in I} \|j_\xi\| = \|\varphi - g\|$. Since

$$\langle \varphi - g, j_\xi\rangle = \bigl\|\epsilon_\xi u_k\bigl(t_{\beta_\xi} + (1-\epsilon_\xi)\varphi - g\bigr)\bigr\|^2 - \epsilon_\xi\bigl\langle u_k(t_{\beta_\xi} - \varphi, j_\xi\bigr\rangle.$$

Passing the limit for $\xi \in I$, we get $\langle \varphi - g, j\rangle = \|\varphi - g\|^2$, which implies $\|j\| \ge \|\varphi - g\|$. Then

$$\langle \varphi - g, j\rangle = \|\varphi - g\|^2 = \|j\|^2,$$

i.e., $j \in J(\varphi - g)$. Hence, $\omega\text{-}\lim_{\xi\in I} j_\xi = j$ and $\lim_{\xi\in I} \|j_\xi\| = \|j\|$. By the reflexivity of X^* and the Kadec–Klee property, we conclude that $\lim_{\xi\in I} j_\xi = j$. Take the limit for $\xi \in I$ in 2.2, we get $\langle g - \varphi, j\rangle \ge 0$, i.e., $\|\varphi - g\|^2 \le 0$, which implies $\varphi = g$.

(2) ⇒ (3). Obviously.

(3) ⇒ (1). See Theorem 2.7. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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