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STABILITY OF TRIGINTIC FUNCTIONAL EQUATION IN MULTI-BANACH SPACES: FIXED POINT APPROACH

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Abstract. In this paper, we introduce the pioneering trigintic functional equation. Moreover, we establish the general solution of the trigintic functional equation and prove the Hyers-Ulam sum and product stabilities of the same equation in multi-Banach spaces by employing the fixed point approach.

1. Introduction

The stability problem for functional equations starts from the famous talk of Ulam and Hyers gave a partial solution to the Ulam's problem see ([6, 13]). Thereafter, Rassias [12] attempted to solve the stability problem of the Cauchy additive functional equation in a more general setting. The concept introduced by Rassias' theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations (see [2, 3, 7, 14, 16, 17]).

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In 2013, Moradlou [9] proved the Hyers-Ulam stability of the Euler-Lagrange-Jensen type additive mapping in multi-Banach spaces. In 2015, Yang, Chang and Liu [15] estabilished the orthogonal stability of mixed additive-quadratic Jensen type functional equation in multi-Banach spaces. In 2015, Brzdęk, Fechner, Moslehian and Sikorska [4] discussed recent developments of the conditional stability of the homomorphism equation. In 2016, Alizadeh and Moradlou [1] proved the Hyers-Ulam stability of the quadratic mapping in multi-Banach spaces. Recently, Rassias, Murali, Rassias and Raj [11] introduced the general solution, the stability and the non-stability of quattuorvigintic functional equation in multi-Banach spaces.

Now, let us recall some concepts concerning multi-Banach spaces. Let $(\wp, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\wp^k$ the linear space $\wp \oplus \wp \oplus \wp \oplus \cdots \oplus \wp$ consisting of k-tuples $(x_1, ..., x_k)$ where $x_1, ..., x_k \in \wp$. The linear operations on $\wp^k$ are defined coordinate wise. The zero element of either $\wp$ or $\wp^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 1.1. [5] A multi-norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

1. $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1...x_k)\|_k$, for $\sigma \in \Psi_k, x_1, ..., x_k \in \wp$;
2. $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1...x_k)\|_k$ for $\alpha_1...\alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;
3. $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, for $x_1, ..., x_{k-1} \in \wp$;
4. $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ for $x_1, ..., x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space. Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space, and take $k \in \mathbb{N}$. We need the following two properties of multi-norms. They can be found in [5].

$$(a)\; \|(x, ...x)\|_k = \|x\|, \forall x \in \wp,$$

$$(b)\; \max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|, \forall x_1, ..., x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$;
In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-Banach space.

THEOREM 1.2. *[10] Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that (i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
(ii) The sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point y^* of $\mathcal{J}$;
(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ for all $y \in Y$.

Let X and Y be real vector spaces. For convenience, we use the following abbreviation for a mapping $f : X \to Y$

$$\begin{aligned}
Df(x, y) &= f(x + 15y) - 30f(x + 14y) + 435f(x + 13y) - 4060f(x + 12y) \\
&+27405f(x+11y) - 142506f(x+10y) + 593775f(x+9y) - 2035800f(x+8y) \\
&+5852925f(x + 7y) - 14307150f(x + 6y) + 30045015f(x + 5y) \\
&-54627300f(x + 4y) + 86493225f(x + 3y) - 119759850f(x + 2y) \\
&+145422675f(x + y) - 155117520f(x) + 145422675f(x - y) \\
&-119759850f(x - 2y) + 86493225f(x - 3y) - 54627300f(x - 4y) \\
&+30045015f(x - 5y) - 14307150f(x - 6y) + 5852925f(x - 7y) \\
&-2035800f(x-8y) + 593775f(x-9y) - 142506f(x-10y) + 27405f(x-11y) \\
&-4060f(x - 12y) + 435f(x - 13y) - 30f(x - 14y) + f(x - 15y) - 30!f(y)
\end{aligned}$$

for all $x, y \in X$, where $30! = 2.652528598 \times 10^{32}$.

In this paper, we introduce the trigintic functional equation:

$$Df(x, y) = 0. \tag{1}$$

for all $x, y \in X$. Moreover, we prove the stability of the trigintic functional equation (1) in multi-Banach spaces by using fixed point method.

2. General soultion of trigintic functional equation in (1)

In this section, we solve the trigintic functional equation in (1) in vector spaces.

THEOREM 2.1. *Let X and Y be vector spaces. If $f : X \to B$ satisfies the function equation (1) for all $x, y \in X$, then f is a trigintic mapping, i.e., $f(2x) = 2^{30} f(x)$ for all $x \in X$.*

Proof. Substituting $x = 0$ and $y = 0$ in (1), we obtain that $f(0) = 0$. Substituting (x, y) with (x, x) and $(x, -x)$ in (1), respectively, and subtracting two resulting equations, we can arrive at $f(-x) = f(x)$, that is to say, f is an even function.
Letting (x, y) by $(15x, x)$ and $(0, 2x)$ respectively in (1), and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&30f(29x) - 465f(28x) + 4060f(27x) - 26970f(26x) + 142506f(25x)\\
&-597835f(24x) + 2035800f(23x) - 5825520f(22x) + 14307150f(21x)\\
&-30187611f(20x) + 54627300f(19x) - 85899450f(18x)\\
&+119759850f(17x) - 147458475f(16x) + 155117520f(15x)\\
&-139569750f(14x) + 119759850f(13x) - 100800375f(12x)\\
&+54627300f(11x) + 14307150f(9x) - 60480225f(8x) + 2035800f(7x)\\
&+85899450f(6x) + 142506f(5x) - 119787255f(4x) + 4060f(3x)\\
&- 1.326264299 \times 10^{32} f(2x) + 30! f(x) = 0 \qquad (2)
\end{aligned}$$

for all $x \in X$. Taking $x = 14x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 30, and subtracting the obtained result from (2), we arrive at

$$\begin{aligned}
&435f(28x) - 8990f(27x) + 94830f(26x) - 679644f(25x) + 3677345f(24x)\\
&-15777450f(23x) + 55248480f(22x) - 161280600f(21x)\\
&+399026889f(20x) - 846723150f(19x) + 1552919550f(18x)\\
&-2475036900f(17x) + 3445337025f(16x) - 4207562730f(15x)\\
&+4513955850f(14x) - 4242920400f(13x) + 3491995125f(12x)\\
&-2540169450f(11x) + 1638819000f(10x) - 887043300f(9x)\\
&+368734275f(8x) - 173551950f(7x) + 146973450f(6x) - 17670744f(5x)\\
&-115512075f(4x) - 818090f(3x)\\
&- 1.326264299 \times 10^{32} f(2x) + 30!(31) f(x) = 0 \qquad (3)
\end{aligned}$$

for all $x \in X$. Taking $x = 13x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 435, and subtracting the obtained result from (3), we arrive at

$$\begin{aligned}
&4060f(27x) - 94395f(26x) + 1086456f(25x) - 8243830f(24x)\\
&+46212660f(23x) - 203043645f(22x) + 724292400f(21x)\\
&-2146995486f(20x) + 5376887100f(19x) - 11516661980f(18x)\\
&+2.12878386 \times 10^{10} f(17x) - 3.417921585 \times 10^{10} f(16x)\\
&+4.788797202 \times 10^{10} f(15x) - 5.874490778 \times 10^{10} f(14x)\\
&+6.32332008 \times 10^{10} f(13x) - 5.97668685 \times 10^{10} f(12x)\\
&+4.95553653 \times 10^{10} f(11x) - 3.598573388 \times 10^{10} f(10x)\\
&+2.28758322 \times 10^{10} f(9x) - 1.270084726 \times 10^{10} f(8x) + 6050058300f(7x)
\end{aligned}$$

$$-2399048925f(6x)+867902256f(5x)-373804200f(4x)+61172020f(3x)$$

$$-1.326264299\times 10^{32}f(2x)+30!(466)f(x)=0 \tag{4}$$

for all $x\in X$. Taking $x=12x$ and replacing $y=x$ in (1), further multiplying the resulting equation by 4060, and subtracting the obtained result from (4), we get

$$\begin{aligned}
&27405f(26x)-679644f(25x)+8239770f(24x)-65051640f(23x)\\
&+375530715f(22x)-1686434100f(21x)+6118352514f(20x)\\
&-18385988400f(19x)+46570367030f(18x)-1.006949223\times 10^{11}f(17x)\\
&+1.876076222\times 10^{11}f(16x)-3.032745215\times 10^{11}f(15x)\\
&+4.274800832\times 10^{11}f(14x)-5.271828597\times 10^{11}f(13x)\\
&+5.700102627\times 10^{11}f(12x)-5.408606952\times 10^{11}f(11x)\\
&+4.502392571\times 10^{11}f(10x)-3.282866613\times 10^{11}f(9x)\\
&+2.090859907\times 10^{11}f(8x)-1.159327026\times 10^{11}f(7x)\\
&+5.568798008\times 10^{10}f(6x)-2.289497324\times 10^{10}f(5x)\\
&+7891543800f(4x)-2349558540f(3x)
\end{aligned}$$

$$-1.326264299\times 10^{32}f(2x)+30!(4526)f(x)=0 \tag{5}$$

for all $x\in X$. Taking $x=11x$ and replacing $y=x$ in (1), further multiplying the resulting equation by 27405, and subtracting the obtained result from (5), we arrive at

$$\begin{aligned}
&142506f(25x)-3681405f(24x)+46212660f(23x)-375503310f(22x)\\
&+2218942830f(21x)-10154051370f(20x)+1037405110600f(19x)\\
&-113829042600f(18x)+2.913925235\times 10^{11}f(17x)\\
&-6.357760139\times 10^{11}f(16x)+1.193786635\times 10^{12}f(15x)\\
&-1.942866748\times 10^{12}f(14x)+2.75483583\times 10^{12}f(13x)\\
&-3.415298146\times 10^{12}f(12x)+3.710134941\times 10^{12}f(11x)\\
&-3.535069151\times 10^{12}f(10x)+2.953732028\times 10^{12}f(9x)\\
&-2.16126084\times 10^{12}f(8x)+1.381128454\times 10^{12}f(7x)\\
&-7.67695656\times 10^{11}f(6x)+3.691924726\times 10^{11}f(5x)\\
&-1.525078932\times 10^{11}f(4x)+5.344236261\times 10^{10}f(3x)
\end{aligned}$$

$$-1.326264299\times 10^{32}f(2x)+30!(31932)f(x)=0 \tag{6}$$

for all $x\in X$. Taking $x=10x$ and replacing $y=x$ in (1), further multiplying the resulting equation by 142506, and subtracting the obtained result from (6), we arrive at

$$\begin{aligned}
&593775f(24x)-15777450f(23x)+203071050f(22x)\\
&-1686434100f(21x)+10153908670f(20x)-47211389550f(19x)\\
&+17628467220f(18x)-5.426844066\times 10^{11}f(17x)
\end{aligned}$$

$+1.403078704 \times 10^{12} f(16x) - 3.087808273 \times 10^{12} f(15x)$
$+5.841851266 \times 10^{12} f(14x) - 9.570967692 \times 10^{12} f(13x)$
$+1.36511990 \times 10^{13} f(12x) - 1.701346878 \times 10^{13} f(11x)$
$+1.857010816 \times 10^{13} f(10x) - 1.776987169 \times 10^{13} f(9x)$
$+1.490523634 \times 10^{13} f(8x) - 1.094467507 \times 10^{13} f(7x)$
$+7.017022358 \times 10^{12} f(6x) - 3.912402577 \times 10^{12} f(5x)$
$+1.8863511 \times 10^{12} f(4x) - 7.806965576 \times 10^{11} f(3x)$

$$- 1.326264299 \times 10^{32} f(2x) + 30!(174437) f(x) = 0 \tag{7}$$

for all $x \in X$. Taking $x = 9x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 593775, and subtracting the obtained result from (7), we arrive at
$2035800 f(23x) - 55221075 f(22x) + 724292400 f(21x)$
$-6118495206 f(20x) + 37405110600 f(19x)$
$-176284078400 f(18x) + 6.661227384 \times 10^{11} f(17x)$
$-2.072241838 \times 10^{12} f(16x) + 5.407419719 \times 10^{12} f(15x)$
$-1.199812752 \times 10^{13} f(14x) + 2.286535737 \times 10^{13} f(13x)$
$-3.770631564 \times 10^{13} f(12x) + 5.409693615 \times 10^{13} f(11x)$
$-6.777824069 \times 10^{13} f(10x) + 7.433503375 \times 10^{13} f(9x)$
$-7.144311251 \times 10^{13} f(8x) + 6.016572986 \times 10^{13} f(7x)$
$-4.434049291 \times 10^{13} f(6x) + 2.852394029 \times 10^{13} f(5x)$
$-1.595388597 \times 10^{13} f(4x) + 7.71694216 \times 10^{12} f(3x)$

$$- 1.326264299 \times 10^{32} f(2x) + 30!(768212) f(x) = 0 \tag{8}$$

for all $x \in X$. Taking $x = 8x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 2035800, and subtracting the obtained result from (8), we arrive at
$5852925 f(22x) - 161280600 f(21x) + 2146852794 f(20x) - 18385988400 f(19x)$
$+113829636400 f(18x) - 5.426844066 \times 10^{11} f(17x)$
$+2.072239802 \times 10^{12} f(16x) - 6.507964996 \times 10^{12} f(15x)$
$+1.712836845 \times 10^{13} f(14x) - 3.830028417 \times 10^{13} f(13x)$
$+7.35039417 \times 10^{13} f(12x) - 1.219859713 \times 10^{14} f(11x)$
$+1.760288619 \times 10^{14} f(10x) - 2.217164481 \times 10^{14} f(9x)$
$+2.443451347 \times 10^{14} f(8x) - 2.358857539 \times 10^{14} f(7x)$
$+1.994666708 \times 10^{14} f(6x) - 1.475598527 \times 10^{14} f(5x)$
$+9.526463673 \times 10^{13} f(4x) - 5.350449048 \times 10^{13} f(3x)$

$$- 1.326264299 \times 10^{32} f(2x) + 30!(2804012) f(x) = 0 \tag{9}$$

for all $x \in X$. Taking $x = 7x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 5852925, and subtracting the obtained result from (9), we arrive at

$$\begin{aligned}
&14307150f(21x) - 399169581f(20x) + 65376887100f(19x)\\
&-46569773250f(18x) + 2.913925235 \times 10^{11}f(17x)\\
&-1.40308074 \times 10^{12}f(16x) + 5.407419719 \times 10^{12}f(15x)\\
&-1.71283626 \times 10^{13}f(14x) + 4.543839174 \times 10^{13}f(13x)\\
&-1.023472777 \times 10^{14}f(12x) + 1.977435186 \times 10^{14}f(11x)\\
&-3.30209497 \times 10^{14}f(10x) + 4.79228972 \times 10^{14}f(9x)\\
&-6.068028812 \times 10^{14}f(8x) + 6.720056324 \times 10^{14}f(7x)\\
&-6.516838853 \times 10^{14}f(6x) + 5.534093293 \times 10^{14}f(5x)\\
&-4.111341216 \times 10^{14}f(4x) + 2.670590763 \times 10^{14}f(3x)\\
&- 1.326264299 \times 10^{32}f(2x) + 30!(8656937)f(x) = 0 \qquad (10)
\end{aligned}$$

for all $x \in X$. Taking $x = 6x$ and replacing $y = x$ in (1),further multiplying the resulting equation 14307150, and subtracting the obtained result from (10), we arrive at

$$\begin{aligned}
&30045015f(20x) - 846723150f(19x) + 11517255750f(18x)\\
&-1.006949223 \times 10^{11}f(17x) + 6.357739781 \times 10^{11}f(16x)\\
&-3.087808273 \times 10^{12}f(15x) + 1.199813337 \times 10^{13}f(14x)\\
&-3.830028417 \times 10^{13}f(13x) + 1.023472634 \times 10^{14}f(12x)\\
&-2.321150178 \times 10^{14}f(11x) + 4.513514782 \times 10^{14}f(10x)\\
&-7.58242586 \times 10^{14}f(9x) + 1.106619686 \times 10^{15}f(8x)\\
&-1.408584616 \times 10^{15}f(7x) + 1.567663828 \times 10^{15}f(6x)\\
&-1.527566783 \times 10^{15}f(5x) + 1.304326871 \times 10^{15}f(4x)\\
&-9.789076957 \times 10^{14}f(3x) - 1.326264299 \times 10^{32}f(2x)\\
&+ 30!(22964087)f(x) = 0 \qquad (11)
\end{aligned}$$

for all $x \in X$. Taking $x = 5x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 30045015, and subtracting the obtained result from (11), we arrive at

$$\begin{aligned}
&54627300f(19x) - 1552325775f(18x) + 2.12878386 \times 10^{10}f(17x)\\
&-1.87609658 \times 10^{11}f(16x) + 1.193786635 \times 10^{12}f(15x)\\
&-5.841845413 \times 10^{12}f(14x) + 2.286535737 \times 10^{13}f(13x)\\
&-7.350395601 \times 10^{13}f(12x) + 1.977435186 \times 10^{14}f(11x)\\
&-4.513514782 \times 10^{14}f(10x) + 8.83036363 \times 10^{14}f(9x)\\
&-1.492083626 \times 10^{15}f(8x) + 2.189723856 \times 10^{15}f(7x)\\
&-2.802386007 \times 10^{15}f(6x) + 3.137223027 \times 10^{15}f(5x)
\end{aligned}$$

$-3.08273956 \times 10^{15} f(4x) + 2.680444435 \times 10^{15} f(3x)$

$$(12) \qquad - 1.326264299 \times 10^{32} f(2x) + 30!(53009102) f(x) = 0$$

for all $x \in X$. Taking $x = 4x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 54627300, and subtracting the obtained result from (12), we arrive at

$86493225 f(18x) - 2475036900 f(17x) + 3.417718005 \times 10^{10} f(16x)$
$-3.032745215 \times 10^{11} f(15x) + 1.942872601 \times 10^{12} f(14x)$
$-9.570967692 \times 10^{12} f(13x) + 3.770630133 \times 10^{13} f(12x)$
$-1.219860259 \times 10^{14} f(11x) + 3.302111358 \times 10^{14} f(10x)$
$-7.58265448 \times 10^{14} f(9x) + 1.492280066 \times 10^{15} f(8x)$
$-2.536664555 \times 10^{15} f(7x) + 3.747555965 \times 10^{15} f(6x)$
$-4.839261392 \times 10^{15} f(5x) + 5.502121998 \times 10^{15} f(4x)$
$-5.583333149 \times 10^{15} f(3x)$

$$(13) \qquad - 1.326264299 \times 10^{32} f(2x) + 30!(107636402) f(x) = 0$$

for all $x \in X$. Taking $x = 3x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 86493225, and subtracting the obtained result from (13), we arrive at

$119759850 f(17x) - 3447372833 f(16x) + 4.788797202 \times 10^{10} f(15x)$
$-4.274742302 \times 10^{10} f(14x) + 2.75483583 \times 10^{12} f(13x)$
$-1.365129984 \times 10^{13} f(12x) + 5.409947635 \times 10^{13} f(11x)$
$-1.760648477 \times 10^{14} f(10x) + 4.79557259 \times 10^{14} f(9x)$
$-1.108780523 \times 10^{15} f(8x) + 2.200552599 \times 10^{15} f(7x)$
$-3.784879521 \times 10^{15} f(6x) + 5.695237168 \times 10^{15} f(5x)$
$-7.582192512 \times 10^{15} f(4x) + 9.070752951 \times 10^{15} f(3x)$

$$(14) \qquad - 1.326264299 \times 10^{32} f(2x) + 30!(194129627) f(x) = 0$$

for all $x \in X$. Taking $x = 2x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 119759850, and subtracting the obtained result from (14), we arrive at

$145422675 f(16x) - 4207562730 f(15x)$
$+5.875076090 \times 10^{10} f(14x) - 5.273026195 \times 10^{11} f(13x)$
$+3.41879014 \times 10^{12} f(12x) - 1.706302412 \times 10^{13} f(11x)$
$+6.822847991 \times 10^{13} f(10x) - 2.246701798 \times 10^{14} f(9x)$
$+6.21708118 \times 10^{14} f(8x) - 1.468744296 \times 10^{15} f(7x)$
$+3.001084836 \times 10^{15} f(6x) - 5.364123902 \times 10^{15} f(5x)$
$+8.473651298 \times 10^{15} f(4x) - 1.194323128 \times 10^{16} f(3x)$

$$(15) \qquad - 1.326264299 \times 10^{32} f(2x) + 30!(313889477) f(x) = 0$$

for all $x \in X$. Taking $x = x$ and replacing $y = x$ in (1), further multiplying the resulting equation by 145422675, and subtracting the obtained result from (15), we arrive at

$$\begin{aligned}
&155117520f(15x) - 4653525530f(14x) + 6.74612125 \times 10^{10}f(13x)\\
&-6.29777131 \times 10^{11}f(12x) + 4.250995636 \times 10^{12}f(11x)\\
&-2.210517735 \times 10^{13}f(10x) + 9.210490544 \times 10^{13}f(9x)\\
&-3.157882471 \times 10^{14}f(8x) + 9.078912107 \times 10^{14}f(7x)\\
&-2.219289626 \times 10^{15}f(6x) + 4.660508218 \times 10^{15}f(5x)\\
&-8.473651302 \times 10^{15}f(4x) + 1.341661456 \times 10^{16}f(3x)\\
&- 1.326264299 \times 10^{32}f(2x) + 30!(459312152)f(x) = 0 \qquad (16)
\end{aligned}$$

for all $x \in X$. Taking $x = 0$ and replacing $y = x$ in (1), further multiplying the resulting equation by 155117520, and subtracting the obtained result from (16), we arrive at

$$- 1.326264299 \times 10^{32}f(2x) + 30!(536870912)f(x) = 0 \qquad (17)$$

for all $x \in X$. From (17), we get

$$f(2x) = 2^{30}f(x) \qquad (18)$$

for all $x \in X$. □

3. Hyers-Ulam stability of the trigintic functional equation (1) in multi-Banach spaces

In this section, we prove the Hyers-Ulam stability of the trigintic functional equation in (1) in multi-Banach spaces.

THEOREM 3.1. *Let X be a vector space and let $\big((Y^k, \|.\|_k) : k \in \mathbb{N}\big)$ be a multi-Banach space. Suppose that δ is a nonnegative real number and $f : X \to Y$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(Df(x_1, y_1), \cdots, Df(x_k, y_k))\|_k \leq \delta \qquad (19)$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in X$. Then there exists a unique trigintic mapping $\mathcal{T} : X \to Y$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{T}(x_1), \cdots, f(x_k) - \mathcal{T}(x_k))\|_k \leq \frac{1073741825}{30!(1073741824)}\delta \qquad (20)$$

for all $x_i \in X$, where $i = 1, 2, \cdots, k$.

Proof. Letting (x_i, y_i) by $(15x_i, x_i)$ and $(0, 2x_i)$ in (19), respectively, and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(30f(29x_1) - 465f(28x_1) + 4060f(27x_1) - 26970f(26x_1)\\
&+142506f(25x_1) - 597835f(24x_1) + 2035800f(23x_1) - 5825520f(22x_1)\\
&+14307150f(21x_1) - 30187611f(20x_1) + 54627300f(19x_1)\\
&-85899450f(18x_1) + 119759850f(17x_1) - 147458475f(16x_1)\\
&+155117520f(15x_1) - 139569750f(14x_1) + 119759850f(13x_1)\\
&-100800375f(12x_1) + 54627300f(11x_1) + 14307150f(9x_1)\\
&-60480225f(8x_1) + 2035800f(7x_1) + 85899450f(6x_1) + 142506f(5x_1)\\
&-119787255f(4x_1) + 4060f(3x_1) - 1.326264299 \times 10^{32} f(2x_1) + 30!f(x_1),\\
&\cdots, 30f(29x_k) - 465f(28x_k) + 4060f(27x_k) - 26970f(26x_k)\\
&+142506f(25x_k) - 597835f(24x_k) + 2035800f(23x_k) - 5825520f(22x_k)\\
&+14307150f(21x_k) - 30187611f(20x_k) + 54627300f(19x_k)\\
&-85899450f(18x_k) + 119759850f(17x_k) - 147458475f(16x_k)\\
&+155117520f(15x_k) - 139569750f(14x_k) + 119759850f(13x_k)\\
&-100800375f(12x_k) + 54627300f(11x_k) + 14307150f(9x_k)\\
&-60480225f(8x_k) + 2035800f(7x_k) + 85899450f(6x_k)\\
&+142506f(5x_k) - 119787255f(4x_k) + 4060f(3x_k)
\end{aligned}$$

$$-1.326264299 \times 10^{32} f(2x_k) + 30!f(x_k))\|_k \leq \frac{3}{2}\delta \tag{21}$$

for all $x_i \in X$, where $i = 1, 2, \cdots, k$. Taking $x_i = 14x_i$ and replacing $y_i = x_i$ in (19), further multiplying the resulting equation by 30, and subtracting the obtained result from (21), we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(435f(28x_1) - 8990f(27x_1) + 94830f(26x_1) - 679644f(25x_1)\\
&+3677345f(24x_1) - 15777450f(23x_1) + 55248480f(22x_1)\\
&-161280600f(21x_1) + 399026889f(20x_1) - 846723150f(19x_1)\\
&+1552919550f(18x_1) - 2475036900f(17x_1) + 3445337025f(16x_1)\\
&-4207562730f(15x_1) + 4513955850f(14x_1) - 4242920400f(13x_1)\\
&+3491995125f(12x_1) - 2540169450f(11x_1) + 1638819000f(10x_1)\\
&-887043300f(9x_1) + 368734275f(8x_1) - 173551950f(7x_1)\\
&+146973450f(6x_1) - 17670744f(5x_1) - 115512075f(4x_1) - 818090f(3x_1)\\
&-1.326264299 \times 10^{32} f(2x_1) + 30!(31)f(x_1), \cdots, 435f(28x_k)\\
&-8990f(27x_k) + 94830f(26x_k) - 679644f(25x_k) + 3677345f(24x_k)\\
&-15777450f(23x_k) + 55248480f(22x_k) \quad -161280600f(21x_k)\\
&+399026889f(20x_k) - 846723150f(19x_k) + 1552919550f(18x_k)\\
&-2475036900f(17x_k) + 3445337025f(16x_k) - 4207562730f(15x_k)\\
&+4513955850f(14x_k) - 4242920400f(13x_k) + 3491995125f(12x_k)
\end{aligned}$$

$$-2540169450f(11x_k) + 1638819000f(10x_k) - 887043300f(9x_k)$$
$$+368734275f(8x_k) - 173551950f(7x_k) + 146973450f(6x_k)$$
$$-17670744f(5x_k) - 115512075f(4x_k) - 818090f(3x_k)$$

$$-1.326264299 \times 10^{32}f(2x_k) + 30!(31)f(x_k))\big\|_k \leq \frac{63}{2}\delta \tag{22}$$

for all $x_i \in X$, where $i = 1, 2, \cdots, k$. Applying the same procedure of Theorem 2.1 and using (17), we get

$$\begin{aligned}\sup_{k\in\mathbb{N}} &\left\| \left(f(x_1) - \frac{1}{2^{30}}f(2x_1), \cdots, f(x_k) - \frac{1}{2^{30}}f(2x_k) \right) \right\|_k \\ &\leq \frac{1073741825}{(30!)(1073741824)}\delta \end{aligned} \tag{23}$$

for all $x_i \in X$, where $i = 1, 2, \cdots, k$.
Let $\Lambda = \{g : X \to Y | g(0) = 0\}$ and introduce the generalized metric d defined on Λ by

$$\begin{aligned} d(u, v) = \inf\{\lambda \in [0, \infty] | &\sup_{k\in\mathbb{N}} \|(u(x_1) - v(x_1), \cdots, u(x_k) - v(x_k))\|_k \\ &\leq \lambda, \quad \forall x_1, \cdots, x_k \in X\}. \end{aligned}$$

Then it is easy to show that (Λ, d) is a generalized complete metric space. See [8].
We define an operator $\mathcal{J} : \Lambda \to \Lambda$ by

$$\mathcal{J}u(x) = \frac{1}{2^{30}}u(2x) \ \forall x \in X.$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $u, v \in \Lambda$, let $\lambda \in (0, \infty)$ be an arbitary constant with $d(u, v) \leq \lambda$. From the definition of d, it follows that

$$\sup_{k\in\mathbb{N}} \|(u(x_1) - v(x_1), \cdots, u(x_k) - v(x_k))\|_k \leq \lambda,$$

for all $x_1, \cdots, x_k \in X$. Therefore,

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(\mathcal{J}u(x_1) - \mathcal{J}v(x_1), \cdots, \mathcal{J}u(x_k) - \mathcal{J}v(x_k))\|_k \\ &\leq \sup_{k\in\mathbb{N}} \left\| \left(\frac{1}{2^{30}}u(2x_1) - \frac{1}{2^{30}}v(2x_1), ..., \frac{1}{2^{30}}u(2x_k) - \frac{1}{2^{30}}v(2x_k) \right) \right\|_k \\ &\leq \frac{1}{2^{30}}\lambda \end{aligned}$$

for all $x_1, \cdots, x_k \in X$. It holds that $d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2^{30}}\lambda$, *i.e.*, $d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2^{30}} d(u, v)$ for all $u, v \in \Lambda$. This means that $\mathcal{J}$ is a strictly contractive operator on Λ with the Lipschitz constant $\mathcal{L} = \frac{1}{2^{30}}$.
By (23), we have $d(\mathcal{J}h, h) \leq \frac{1073741825}{(30!)(1073741824)}\delta$. According to Theorem 1.2, we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $\mathcal{T} : X \to Y$ such that

$$\mathcal{T}(2x) = 2^{30}\mathcal{T}(x) \quad \forall x \in X.$$

Moreover, we have $d\left(\mathcal{J}^n h, \mathcal{T}\right) \to 0$, which implies

$$\mathcal{T}(x) = \lim_{n\to\infty} \mathcal{J}^n h(x) = \lim_{n\to\infty} \frac{h(2^n x)}{2^{30n}}$$

for all $x \in X$.
Also, $d(h, \mathcal{T}) \leq \frac{1}{1-\mathcal{L}} d(\mathcal{J}h, h)$ implies the inequality

$$d(h, \mathcal{T}) \leq \frac{1}{1 - \frac{1}{2^{30}}} d(\mathcal{J}h, h) \leq \frac{1073741825}{(30!)(1073741824)}\delta.$$

Setting $x_1 = \cdots = x_k = 2^n x, y_1 = \cdots = y_k = 2^n y$ in (19) and dividing both sides by 2^{30n}. Then, using property (a) of multi-norms, we obtain

$$\|D\mathcal{T}(x, y)\| = \lim_{n\to\infty} \frac{1}{2^{30n}} \left\|Dh\left(2^n x, 2^n y\right)\right\| = 0$$

for all $x, y \in X$. Hence $\mathcal{T}$ is a trigintic mapping.
The uniqueness of $\mathcal{T}$ follows from the fact that $\mathcal{T}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{T}(x_1), ..., f(x_k) - \mathcal{T}(x_k))\|_k \leq \ell$$

for all $x_1, \cdots, x_k \in X$.
This completes the proof of the theorem. □

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