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Approximation of derivations and the superstability in random Banach ∗-algebras

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Abstract

We prove that approximations of derivations on random Banach ∗-algebras are exactly derivations by using a fixed point method. Furthermore, we show that approximations of quadratic ∗-derivations on random Banach ∗-algebras are exactly quadratic ∗-derivations. We, moreover, prove that approximations of derivations on random C^*-ternary algebras are exactly derivations by using a fixed point method.

MSC: 46S50; 47H10; 26E60

Keywords: Derivation; Quadratic derivation; Superstability; Fixed point method; Random Banach ∗-algebra; Random C^*-ternary algebra

1 Introduction

Ulam [1] presented an effective lecture at the University of Wisconsin in which he stated a number of essential unsolved problems, in the fall of 1940. The next question concerning the stability of homomorphisms was among those:

Assume that Ω_1 is a group and suppose that Ω_2 is a metric group with a metric $\Delta(\cdot,\cdot)$. Let $\xi > 0$, is there $\eta > 0$ such that if a function $\varphi : \Omega_1 \to \Omega_2$ satisfies the inequality $\Delta(\varphi(uv), \varphi(u)\varphi(v)) < \eta$ for all $u, v \in \Omega_1$ then there is a homomorphism $\Phi : \Omega_1 \to \Omega_2$ with $\Delta(\varphi(u), \Phi(u)) < \xi$ for all $u \in \Omega_1$?

When the answer is established, the functional equation for homomorphisms is stable.

The first mathematician who presented the result concerning the stability of functional equations was Hyers [2]. He intelligently answered Ulam's question when Ω_1 and Ω_2 are Banach spaces. Recently, Rassias [3] and others have obtained important results on stability and applied them to the investigations in the nonlinear sciences.

2 Preliminaries

Assume that Δ^+ is the family of distribution functions, i.e., the family of all left-continuous functions $G : [-\infty, \infty] \to [0, 1]$ such that G is increasing on $[-\infty, \infty]$, $G(0) = 0$ and $G(+\infty) = 1$. $D^+ \subseteq \Delta^+$ contains each function $G \in \Delta^+$ for which $\ell^- G(+\infty) = 1$ and $\ell^- g(x)$ is the left limit of the map g at x, i.e., $\ell^- g(x) = \lim_{t\to x^-} g(t)$. In Δ^+, we have $H \le F$ if and only if $H(s) \le F(s)$ for all s in $\mathbb{R}$ (partially ordered). Note that the function ε_u defined by

$$\varepsilon_u(s) = \begin{cases} 0, & \text{if } s \le u, \\ 1, & \text{if } s > u, \end{cases}$$

is an element of Δ^+ and ε_0 is the maximal element in this space. For more details see [4–6].

Definition 2.1 ([6]) Let $I = [0,1]$. A continuous triangular norm (briefly, *ct*-norm) is a function T from I to I with continuity property such that:

(a) $T(\theta,\vartheta) = T(\vartheta,\theta)$ and $T(\theta,T(\vartheta,\iota)) = T(T(\theta,\vartheta),\iota)$ for all $\theta,\vartheta,\iota \in I$;
(b) $T(\theta,1) = \theta$ for $0 \le \theta \le 1$;
(c) $T(\theta,\vartheta) \le T(\iota,\kappa)$ whenever $\theta \le \iota$ and $\vartheta \le \kappa$ for each $\theta,\vartheta,\iota,\kappa \in I$.

$T_P(\theta,\vartheta) = \theta\vartheta$, $T_M(\theta,\vartheta) = \min(\theta,\vartheta)$ and $T_L(\theta,\vartheta) = \max(\theta + \vartheta - 1, 0)$ (the Lukasiewicz t-norm) are some examples of t-norms. Also, we define $\prod_{j=1}^n \theta_j = T^{n-1}(\theta_1,\dots,\theta_n)$.

Definition 2.2 ([6]) Suppose that T is a *ct*-norm, V is a vector space and let μ be a map from V to D^+. In this case, the ordered triple (V,μ,T) with the properties

(RN1) $\mu_\nu(\theta) = \varepsilon_0(\theta)$ for all $\theta > 0$ if and only if $\nu = 0$;
(RN2) $\mu_{\alpha\nu}(\theta) = \mu_\nu(\frac{\theta}{|\alpha|})$ for all $\nu \in V$, $\alpha \neq 0$;
(RN3) $\mu_{u+\nu}(\theta + \vartheta) \ge T(\mu_u(\theta), \mu_\nu(\vartheta))$ for all $u,\nu \in V$ and all $\theta,\vartheta \ge 0$,

is said to be a *random normed space* (in short, RN-space).

Let $(V, \|\cdot\|)$ be a linear normed space. Then

$$\mu_\nu(\vartheta) = \frac{\vartheta}{\vartheta + \|\nu\|}$$

for all $\vartheta > 0$, defines a random norm, and the ordered triple (V,μ,T_M) is an RN-space.

Definition 2.3 Assume that the following algebraic structure on an RN-space (V,μ,T) holds:

(RN-4) $\mu_{u\nu}(\theta\vartheta) \ge T'(\mu_u(\theta), \mu_\nu(\vartheta))$ for each $u,\nu \in V$ and all $\theta,\vartheta > 0$, where T' is a *ct*-norm.

Then (V,μ,T,T') is called a *random normed algebra*.

Suppose that $(V,\|\cdot\|)$ is a normed algebra. Then (V,μ,T_M,T_P) is a random normed algebra, where

$$\mu_\nu(\vartheta) = \frac{\vartheta}{\vartheta + \|\nu\|}$$

for all $\vartheta > 0$ if and only if

$$\|u\nu\| \le \|\nu\|\|u\| + \theta\|u\| + \vartheta\|\nu\| \quad (\nu,u \in V; \theta,\vartheta > 0).$$

For more details, see [7–22].

Definition 2.4 A random Banach $*$-algebra $\mathcal{B}$ is a random complex Banach algebra $(\mathcal{B},\mu,T,T')$, together with an involution on $\mathcal{B}$ which is a mapping $g \mapsto g^*$ from $\mathcal{B}$ into $\mathcal{B}$ that satisfies

(i) $g^{**} = g$ for $g \in \mathcal{B}$;

(ii) $(ag + bh)^* = \overline{a}g^* + \overline{b}h^*$;

(iii) $(gh)^* = h^*g^*$ for $g, h \in \mathcal{B}$.

If, in addition, $\mu_{g^*g}(\theta\vartheta) = T'(\mu_g(\theta), \mu_g(\vartheta))$ for $g \in \mathcal{B}$ and $\theta, \vartheta > 0$, then $\mathcal{B}$ is called a random C^*-algebra.

Assume that $\mathcal{B}$ is a random Banach $*$-algebra. A *derivation* on $\mathcal{B}$ is a mapping δ from $\mathcal{B}$ to $\mathcal{B}$ such that:

$$\delta(\lambda g + h) = \lambda\delta(g) + \delta(h), \tag{2.1}$$

$$\delta(gh) = \delta(g)h + g\delta(h) \tag{2.2}$$

for all $g, h \in \mathcal{B}$ and all $\lambda \in \mathbb{C}$. A derivation δ is called a $*$-derivation on $\mathcal{B}$ if $\delta(g^*) = \delta(g)^*$ for all $g \in \mathcal{B}$ (see [23]).

Recall that

$$\omega(u + v) = \omega(u) + \omega(v), \tag{2.3}$$

$$\omega(u + v) + \omega(u - v) = 2\omega(u) + 2\omega(v), \tag{2.4}$$

respectively, are Cauchy additive and Cauchy quadratic functional equations.

Firstly, Baker, Lawrence and Zorzitto [24] defined the concept of superstability. Let $(\mathcal{B}, \mu, T, T')$ be an RN algebra. The random norm is multiplicative if $\mu_{uv}(\theta\vartheta) = T'(\mu_u(\theta), \mu_v(\vartheta))$ for all $u, v \in \mathcal{B}$ and all $\theta, \vartheta > 0$.

Suppose that $\Gamma \neq \emptyset$. A function $\Delta : \Gamma \times \Gamma \to [0, \infty]$ is a *generalized metric* (GM) on Γ if

(1) $\Delta(\rho, \varrho) = 0$ if and only if $\rho = \varrho$;

(2) $\Delta(\rho, \varrho) = \Delta(\varrho, \rho)$ for all $\rho, \varrho \in \Gamma$;

(3) $\Delta(\rho, \varrho) \leq \Delta(\rho, \sigma) + \Delta(\sigma, \varrho)$ for all $\rho, \varrho, \sigma \in \Gamma$.

Theorem 2.1 ([25, 26]) *Suppose that (Γ, Δ) is a complete GM space and assume that the selfmapping Υ on Γ with Lipschitz constant $0 < L < 1$ is strictly contractive. Then, for $\varrho \in \Gamma$, either*

$$\Delta\left(\Upsilon^n\varrho, \Upsilon^{n+1}\varrho\right) = \infty$$

for each $0 \leq n \in \mathcal{Z}$, or there exists $n_0 \in \mathbb{N}$ such that

(1) $\Delta(\Upsilon^n\varrho, \Upsilon^{n+1}\varrho) < \infty, \forall n \geq n_0$;

(2) *the sequence $\{\Upsilon^n\varrho\}$ tends to σ^* in Γ*;

(3) $\Upsilon(\sigma^*) = \sigma^*$;

(4) $\Upsilon(\sigma^*) = \sigma^*$ *and is unique in* $\mathbb{E} = \{\sigma \in \Gamma | \Delta(\Upsilon^{n_0}\varrho, \sigma) < \infty\}$

(5) $(1 - L)\Delta(\sigma, \sigma^*) \leq \Delta(\sigma, \Upsilon\sigma)$ *for all* $\sigma \in \Gamma$.

3 Approximation of derivations on random Banach $*$-algebras

Assume that a random $*$-Banach algebra $\mathcal{B}$ has unit e. Our results improve and expand the result presented by Jang [27].

Theorem 3.1 *Let $\psi_1 : \mathcal{B} \times \mathcal{B} \to D^+$ and $\psi_2 : \mathcal{B} \to D^+$ be distribution functions. Assume that $f : \mathcal{B} \to \mathcal{B}$ is a mapping such that*

$$\mu_{f(\xi p+q)-\xi f(p)-f(q)}(t) \geq \psi_1(p,q,t), \tag{3.1}$$

$$\mu_{f(pq)-pf(q)-f(p)q}(t) \geq \psi_1(p,q,t), \tag{3.2}$$

$$\mu_{f(p^*)-f(p)^*}(t) \geq \psi_2(p,t), \tag{3.3}$$

for all $\xi \in \mathbb{T}$, $p,q \in \mathcal{B}$ and $t > 0$. If there exist $n \in \mathbb{N}$ and $0 < L < 1$ such that $\psi_1(sp,sq,Lst) > \psi_1(p,q,t)$, $\psi_1(sp,q,Lst) > \psi_1(p,q,t)$, $\psi_1(p,sq,Lst) > \psi_1(p,q,t)$ and $\psi_2(sp,Lst) > \psi_2(p,t)$ for all $p,q \in \mathcal{B}$ and $t > 0$. Then f on $\mathcal{B}$ is a $*$-derivation.

Proof Putting $p = q$ and $\xi = 1$ in (3.1), we get

$$\mu_{f(2p)-2f(p)}(t) \geq \psi_1(p,p,t) \tag{3.4}$$

for all $p \in \mathcal{B}$ and $t > 0$. By induction, we can prove that

$$\mu_{f(np)-nf(p)}(t) \geq \prod_{j=1}^{n-1} \psi_1(jp,p,t_j) \tag{3.5}$$

for all $p,q \in \mathcal{B}$, $t > 0$ and $n \geq 2$ where $\sum_{j=1}^{n-1} t_j = t$.

Define

$$\Psi(p,t) = \prod_{j=1}^{s-1} \psi_1(jp,p,t_j)$$

for $p \in \mathcal{B}$, $t > 0$ and $s \geq 2$ where $\sum_{j=1}^{s-1} t_j = t$. So

$$\mu_{f(sp)-sf(p)}(t) \geq \Psi(p,t). \tag{3.6}$$

Put $\Gamma = \{g; g : \mathcal{B} \to \mathcal{B}\}$. Define a function $\Delta : \Gamma \times \Gamma \to [0,\infty]$ such that

$$\Delta(\vartheta,\upsilon) = \inf\bigl\{\nu > 0 : \mu_{\vartheta(p)-\upsilon(p)}(\nu t) \geq \Psi(p,t), \forall p \in \mathcal{B}, t > 0\bigr\},$$

where $\vartheta,\upsilon \in \Gamma$. Miheț and Radu [28] proved that (Γ,Δ) is a complete GM space. Define a mapping $H : \Gamma \to \Gamma$ by $H(\vartheta)(p) = s^{-1}\upsilon(sp)$. Put

$$\Delta(\vartheta,\upsilon) = \nu,$$

where $\vartheta,\upsilon \in \Gamma$. Then

$$\mu_{H(\vartheta)(p)-H(\upsilon)(p)}(t) = \mu_{\vartheta(sp)-\upsilon(sp)}(st) \geq \Psi\left(sp, \frac{s}{\alpha}t\right) \geq \Psi\left(p, \frac{t}{L\alpha}\right).$$

So, for $\vartheta,\upsilon \in S$, we have

$$\Delta\bigl(H(\vartheta),H(\upsilon)\bigr) \leq L\Delta(\vartheta,\upsilon). \tag{3.7}$$

Then the mapping H on Γ with Lipschitz constant L is strictly contractive. From (3.6), we have

$$\mu_{(Hf)(p)-f(p)}(t) = \mu_{f(sp)-f(p)}(st) = \mu_{f(sp)-sf(p)}(st) \geq \Psi(p,st),$$

which implies that $\Delta(H(f),f) \leq 1/|s|$. Theorem 2.1 implies that, in the set

$$U = \{\vartheta \in \Gamma : \Delta(\vartheta, H(f)) < \infty\},$$

$h : \mathcal{B} \to \mathcal{B}$ is a unique fixed point of H. Also for every $p \in \mathcal{A}$

$$h(p) = \lim_{m\to\infty} H^m(f(p)) = \lim_{m\to\infty} s^{-m} f(s^m p). \tag{3.8}$$

Using (3.6), we get

$$\begin{aligned} \mu_{h(\xi p+q)-\xi h(p)-h(q)}(t) &= \lim_{n\to\infty} \mu_{f(s^n(\xi p+q))-\xi f(s^n p)-f(s^n q)}(s^n t) \\ &\geq \lim_{n\to\infty} \psi_1(s^n p, s^n q, s^n t) \\ &\geq \lim_{n\to\infty} \psi_1\left(p,q,\frac{t}{L^n}\right) = 1 \end{aligned}$$

for all $p,q \in \mathcal{B}$, $\xi \in T$ and $t > 0$. Let $\xi = \xi_1 + i\xi_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{R}$ and let $\mu_1 = \xi_1 - [\xi_1]$ and $\mu_2 = \xi_2 - [\xi_2]$ where $[\xi]$ denotes the integer part of ξ. So $0 \leq \mu_i < 1$ $(1 \leq i \leq 2)$. Now, we represent μ_i as $\mu_i = \frac{\xi_{i,1}+\xi_{i,2}}{2}$ such that $\xi_{i,j} \in \mathbb{T}$ $(1 \leq i, j \leq 2)$. Since $h(\xi p + q) = \lambda h(p) + h(q)$ for $\xi \in T$, we conclude that

$$\begin{aligned} h(\xi p) &= h(\xi_1 p) + ih(\xi_2 p) \\ &= ([\xi_1]h(p) + \delta(\mu_1 p)) + i([\xi_2]h(p) + h(\mu_2 p)) \\ &= \left([\xi_1]h(p) + \frac{1}{2}h(\xi_{1,1}p + \xi_{1,2}p)\right) + i\left([\xi_2]h(p) + \frac{1}{2}h(\xi_{2,1}p + \xi_{2,2}p)\right) \\ &= \left([\xi_1]h(p) + \frac{1}{2}\xi_{1,1}h(p) + \frac{1}{2}\xi_{1,2}h(p)\right) + i\left([\xi_2]h(p) + \frac{1}{2}\xi_{2,1}h(p) + \frac{1}{2}\xi_{2,2}h(p)\right) \\ &= \xi_1 h(p) + i\xi_2 h(p) \\ &= h(p) \end{aligned}$$

for all $p \in \mathcal{B}$ and $\xi \in \mathbb{C}$. So, on $\mathcal{B}$, h is a $\mathbb{C}$-linear mapping. For the involution of h, we have

$$\begin{aligned} \mu_{h(p^*)-h(p)^*}(t) &= \lim_{n\to\infty} \mu_{f(s^n p^*)-f(s^n p)^*}(s^n t) \\ &\geq \lim_{n\to\infty} \psi_2(s^n p, s^n t) \\ &\geq \lim_{n\to\infty} \psi_2\left(p, \frac{t}{L^n}\right) \\ &= 1. \end{aligned}$$

Now, we prove the derivation property of h. In (3.2), we replace p by $s^n p$, q by $s^n q$, divide by s^{2n} and get

$$\mu_{\frac{f(s^n p s^n q)}{s^{2n}} - p\frac{f(s^n q)}{s^n} - \frac{f(s^n p)}{s^n} p}(t) \geq \psi_1\bigl(s^n p, s^n q, s^{2n} t\bigr) \geq \psi_1\biggl(p, q, \frac{t}{L^{2n}}\biggr). \tag{3.9}$$

In (3.9), letting $n \to \infty$, we get

$$h(pq) = ph(q) + h(p)q \tag{3.10}$$

for all $p, q \in \mathcal{B}$. So h is a $*$-derivation on $\mathcal{B}$. Now, in (3.2), replacing p by $s^n p$ and dividing by s^n, we get

$$\mu_{\frac{f(s^n pq)}{s^n} - pf(q) - \frac{f(s^n p)}{s^n} q}(t) \geq \psi_1\bigl(s^n p, q, s^n t\bigr) \geq \psi_1\biggl(p, q, \frac{t}{L^n}\biggr)$$

for all $p, q \in \mathcal{B}$, $n \in \mathbb{N}$ and $t > 0$. Letting $n \to \infty$, we get

$$h(pq) = pf(q) + h(p)q \tag{3.11}$$

for all $p, q \in \mathcal{B}$. Fix $m \in \mathbb{N}$. From

$$\begin{aligned} pf\bigl(s^m q\bigr) &= h\bigl(s^m pq\bigr) - h(p)s^m q \\ &= s^m pf(q) \end{aligned} \tag{3.12}$$

for all $p, q \in \mathcal{B}$, we have $pf(q) = p\frac{f(s^m q)}{s^m}$ for all $p, q \in \mathcal{B}$ and $m \in \mathbb{N}$. Letting $m \to \infty$, we get $pf(q) = ph(q)$. Putting $p = e$, we get $h(q) = f(q)$ for all $q \in \mathcal{B}$. Hence f is a $*$-derivation on $\mathcal{B}$. □

4 Approximation of quadratic $*$-derivations on random Banach $*$-algebras

Definition 4.1 Assume that a mapping $\delta : \mathcal{B} \to \mathcal{B}$ satisfies

(1) $\delta(\eta + \kappa) + \delta(\eta - \kappa) - 2\delta(\eta) - 2\delta(\kappa) = 0$;
(2) δ is quadratic homogeneous, that is, $\delta(\lambda\eta) = \lambda^2\delta(\eta)$;
(3) $\delta(\eta\kappa) = \delta(\eta)\kappa^2 + \eta^2\delta(\kappa)$;
(4) $\delta(\eta^*) = \delta(\eta)^*$;

for all $\eta, \kappa \in \mathcal{B}$ and $\lambda \in \mathbb{C}$. Then it is called a $*$-quadratic derivation on $\mathcal{B}$.

Theorem 4.2 *Assume that $\psi_1 : \mathcal{B} \times \mathcal{B} \to D^+$ and $\psi_2 : \mathcal{B} \to D^+$ are distribution functions. Let $f : \mathcal{B} \to \mathcal{B}$ be a function such that*

$$\mu_{f(p+q)+f(p-q)-2f(p)-2f(q)}(t) \geq \psi_1(p, q, t), \tag{4.1}$$

$$\mu_{f(pq)-p^2 f(q)-f(p)q^2}(t) \geq \psi_1(p, q, t), \tag{4.2}$$

$$\mu_{f(\xi p)-\lambda^2 f(p)}(t) \geq \psi_2(p, t), \tag{4.3}$$

$$\mu_{f(p^*)-f(p)^*}(t) \geq \psi_2(p, t), \tag{4.4}$$

for all $\xi \in \mathbb{C}$, $p,q \in \mathcal{B}$ and $t > 0$. If there exist $s \in \mathbb{N}$ and $0 < L < 1$ such that $\psi_1(2^s p, 2^s q, 2^{2s}Lt) > \psi_1(p,q,t)$, $\psi_1(2^s p, q, 2^{2s}Lt) > \psi_1(p,q,t)$, $\psi_1(p, 2^s q, 2^{2s}Lt) > \psi_1(p,q,t)$ and $\psi_2(2^s p, 2^{2s}Lt) > \psi_2(p,t)$ for all $p,q \in \mathcal{B}$ and $t > 0$. Then, on $\mathcal{B}$, f is a $*$-quadratic derivation.

Proof Putting $p = q$ and $\xi = 1$ in (4.1), we get

$$\mu_{f(2p)-4f(p)}(t) \geq \psi_1(p,p,t)$$

for all $p \in \mathcal{B}$ and $t > 0$. Induction on n yields

$$\mu_{f(2^n p)-2^{2n}f(p)}(t) \geq \prod_{i=0}^{n-1} \psi_1\left(2^i p, 2^i p, \frac{t_i}{2^{2(n-i)}}\right) \tag{4.5}$$

for all $p,q \in \mathcal{B}$, $n \geq 2$ and $t > 0$ where $\sum_{i=0}^{n-1} t_i = t$. Define

$$\Psi(p,t) = \prod_{i=0}^{s-1} 2^{2(s-i)}\psi_1\left(2^i p, 2^i p, \frac{t_i}{2^{2(n-i)}}\right). \tag{4.6}$$

Then we have

$$\mu_{f(2^s p)-2^{2s}f(p)}(t) \geq \Psi(p,t).$$

The set of all mappings $\zeta : \mathcal{B} \to \mathcal{B}$ is denoted by Γ. Define a function $\Delta : \Gamma \times \Gamma \to [0,\infty]$ by

$$\Delta(\zeta,\eta) = \inf\left\{\nu > 0 : \mu_{\zeta(p)-\eta(p)}(t) \geq \Psi\left(p, \frac{t}{\nu}\right), \forall p \in \mathcal{B}\right\}.$$

Miheț and Radu [28] proved that (Γ,Δ) is a complete GM space. Now, define a mapping $H : \Gamma \to \Gamma$ by $H(\zeta)(p) = 2^{-2s}\zeta(2^s p)$. Putting

$$\Delta(\zeta,\eta) = \nu \quad (\zeta,\eta \in \Gamma),$$

we obtain

$$\mu_{H(\zeta)(p)-H(\eta)(p)}(t) = \mu_{\zeta(2^s p)-\eta(2^s p)}\left(\frac{t}{2^{2s}}\right) \geq \Psi\left(2^s p, \frac{t}{\nu 2^{2s}}\right) \geq \Psi\left(p, \frac{t}{L\alpha}\right).$$

Then, for $\zeta,\eta \in S$, we have

$$\Delta\big(H(\zeta),H(\eta)\big) \leq L\Delta(\zeta,\eta), \tag{4.7}$$

which means that H on Γ, with Lipschitz constant L is a strictly contractive mapping. Also, for $p \in \mathcal{B}$, we have

$$\mu_{(Hf)(p)-f(p)}(t) = \mu_{2^{-2s}f(2^s p)-f(p)}(t) = \mu_{f(2^s)2^{2s}f(p)}\big(2^{2s}t\big) \geq \Psi\big(p,2^{2s}t\big),$$

which implies that $\Delta(H(f),f) \leq 1/2^{2s}$. Using Theorem 2.1, we conclude that, in the set

$$U = \{\zeta \in \Gamma : \Delta(\zeta, H(f)) < \infty\} \tag{4.8}$$

and for each $p \in \mathcal{B}$, $h : \mathcal{B} \to \mathcal{B}$ is a unique fixed point of H and

$$h(p) = \lim_{m\to\infty} H^m(f(p)) = \lim 2^{-2sm} f(2^{sm}p). \tag{4.9}$$

By (4.9), we have

$$\begin{aligned}
&\mu_{h(p+q)+h(p-q)-2h(p)-2h(q)}(t) \\
&\quad = \lim_{n\to\infty} \mu_{f(2^{sn}(p+q)+f(2^{sn}(p-q))-2f(2^{sn}p)-2f(2^{sn}q)}(2^{2sn}t) \\
&\quad \geq \lim_{n\to\infty} \psi_1(2^{ns}p, 2^{ns}q, 2^{2ns}t) \geq \lim_{n\to\infty} \psi_1\left(p, q, \frac{t}{L^n}\right) = 1
\end{aligned}$$

for all $p, q \in \mathcal{B}$ and $t > 0$. Then h is a quadratic mapping on $\mathcal{B}$. Also, we have

$$\begin{aligned}
\mu_{h(\xi p)-\lambda^2 h(p)}(t) &= \lim_{n\to\infty} \mu_{f(2^{ns}(\xi p)-\lambda^2 f(2^{ns}p)}(2^{2ns}t) \\
&\geq \lim_{n\to\infty} \psi_2(2^{ns}p, 2^{2ns}t) \\
&\geq \lim_{n\to\infty} \psi_2\left(p, \frac{t}{L^n}\right) \\
&= 1,
\end{aligned}$$

which implies that h is quadratic homogeneous.

Now, replacing p by $2^{ns}p$ in (4.2) and dividing by 2^{-2sn}, we get

$$\mu_{\frac{f(2^{ns}pq)}{2^{2ns}} - p^2 f(q) - \frac{f(2^{ns}p)}{2^{2ns}} q^2}(t) \geq \psi_1(2^{ns}p, q, 2^{2ns}t) \geq \psi_1\left(p, q, \frac{t}{L^n}\right) \tag{4.10}$$

for all $p, q \in \mathcal{B}$, $n \in \mathbb{N}$ and $t > 0$. Letting $n \to \infty$, we get

$$h(pq) = p^2 f(q) + h(p)q^2, \tag{4.11}$$

for all $p, q \in \mathcal{B}$. Let $m \in \mathbb{N}$. We have

$$\begin{aligned}
p^2 f(2^{ms}q) &= h(2^{ms}pq) - h(2^{ms}p)q^2 \\
&= 2^{2ms}p^2 f(q) + h(2^{ms}p)q^2 - h(2^{ms}p)q^2 \\
&= 2^{2ms}p^2 f(q)
\end{aligned} \tag{4.12}$$

for all $p, q \in \mathcal{B}$, and so $p^2 f(q) = p^2 \frac{f(2^{ms}q)}{2^{2ms}}$ for all $p, q \in \mathcal{B}$ and $m \in \mathbb{N}$. Letting $m \to \infty$ yields $p^2 f(q) = p^2 h(q)$. Putting $p = e$, we get $h(q) = f(q)$ for all $q \in \mathcal{B}$. Hence, on $\mathcal{B}$, f is a $*$-quadratic derivation. □

5 Derivations on random C^*-ternary algebras

A complex random Banach space $(\mathcal{B},\mu,T,T')$, which has a ternary product $(f,g,h)\longmapsto [f,g,h]$ of $\mathcal{B}^3$ into $\mathcal{B}$, is a random C^*-ternary algebra if (see [29]):

(1) $[\xi f+v,g,h]=\xi[f,g,h]+[v,g,h]$ for all $\xi\in\mathbb{C}$;
(2) $[f,\xi g+v,h]=\xi[f,g,h]+[f,v,h]$ for all $\xi\in\mathbb{C}$;
(3) $[f,g,\xi h+v]=\xi[f,g,h]+[f,g,v]$ for all $\xi\in\mathbb{C}$;
(4) $[f,g,[h,k,j]]=[f,[k,h,g],j]=[[f,g,h],k,j]$;
(5) $\|[f,g,h]\|\leq\|f\|\cdot\|g\|\cdot\|h\|$;
(6) $\|[f,f,f]\|=\|f\|^3$;

for $f,g,h,v,k,j\in\mathcal{B}$.

If $(\mathcal{B},\mu,T,T')$ has the unit e satisfying $f=[f,e,e]=[e,e,f]$ for all $f\in\mathcal{B}$, then the random C^*-ternary algebra has unit e. If for $f\in\mathcal{B}$, we have $[e,f,e]=f^*$, then $*$ is an involution on the C^*-ternary algebra. A C^*-ternary derivation is a mapping $\delta:\mathcal{B}\longrightarrow\mathcal{B}$ such that

$$\delta\bigl([f,g,h]\bigr)=\bigl[\delta(f),g,h\bigr]+\bigl[f,\delta(g),h\bigr]+\bigl[f,g,\delta(h)\bigr],$$
$$\delta(\xi f+g)=\xi\delta(f)+\delta(g)$$

for all $f,g,h\in\mathcal{B}$ and $\xi\in\mathbb{C}$. Recall that $\delta([e,f,e])=[e,\delta(f),e]$ implies that δ is an involution.

Theorem 5.1 *Assume that $\mathcal{B}$ is a random C^*-ternary algebra which has the unit e. Suppose that $\psi_1:\mathcal{B}^2\longrightarrow[0,\infty)$ and $\psi_2:\mathcal{B}^3\longrightarrow[0,\infty)$ are functions. Let $f:\mathcal{B}\longrightarrow\mathcal{B}$ be a mapping such that*

$$\mu_{f(\xi p+q)-\lambda f(p)-f(q)}(t)\geq\psi_1(p,q,t),\tag{5.1}$$
$$\mu_{f([p,q,r])-[f(p),q,r]-[p,f(q),r][p,q,f(r)]}(t)\geq\psi_2(p,q,r,t),\tag{5.2}$$
$$\mu_{f([e,q,e])-[e,f(q),e]}(t)\geq\psi_2(e,q,e,t)\tag{5.3}$$

for all $\lambda\in\mathbb{C}$, $p,q,r\in\mathcal{B}$ and $t>0$. Assume there exist $s\in\mathbb{N}$ and $0<L<1$ such that $\psi_1(s^ip,s^jq,s^{(i+j)}L^{(i+j)}t)>\psi_1(p,q,t)$, $\psi_2(s^ip,s^jq,s^kr,s^{(i+j+k)}L^{(i+j+k)}t)>\psi_2(p,q,r,t)$ for all $p,q,r\in\mathcal{B}$ and $i,j,k=0,1$. Then on $\mathcal{B}$, f is a $*$-derivation.

Proof Put

$$\Psi(p,t)=\prod_{j=1}^{s-1}\psi_1(jp,p,t_j)$$

for $p\in\mathcal{B}$ and $t>0$ where $\sum_{j=1}^{s-1}t_j=t$. Then we have

$$\mu_{f(sp)-sf(p)}(t)\geq\Psi(p,t).\tag{5.4}$$

We use similar method presented in the proof of Theorem 3.1. Let Γ be the set of all mappings $r:\mathcal{B}\longrightarrow\mathcal{B}$. Define a function $\Delta:\Gamma\times\Gamma\longrightarrow[0,\infty]$ by

$$\Delta(\zeta,\eta)=\inf\bigl\{\nu>0:\mu_{\zeta(z)-\eta(z)}(\nu s)\geq\Psi(z,s)\bigr\}$$

for $\zeta,\eta\in\Gamma$, $z\in\mathcal{B}$ and $t>0$. Miheţ and Radu [28] proved that (Γ,Δ) is a complete GM space. Define a mapping $H:\Gamma\longrightarrow\Gamma$ by $H(\zeta)(z)=s^{-1}\zeta(sz)$. Now

$$\Delta(\zeta,\eta)=\nu(\zeta,\eta\in\Gamma)$$

implies that

$$\mu_{H(\zeta)(z)-H(\eta)(z)}(t)=\mu_{\zeta(sz)-\eta(sz)}(\nu st)\geq\Psi(sz,st)\geq\Psi\left(z,\frac{t}{L\nu}\right)$$

and for $\zeta,\eta\in\Gamma$

$$\Delta\big(H(\zeta),H(\eta)\big)\leq L\Delta(\zeta,\eta). \tag{5.5}$$

Therefore H on Γ with Lipschitz constant L is a strictly contractive function. From (5.4), we have

$$\mu(Hf)(z)-f(z)(t)=\mu_{s^{-1}f(sz)-f(z)}(t)=\mu_{f(sz)-sf(z)}(st)\geq\Psi(z,st).$$

So $\Delta(H(f),f)\leq 1/|s|$. Using Theorem 2.1, we conclude that, in the set

$$U=\big\{\zeta\in\Gamma:\Delta\big(\zeta,H(f)\big)<\infty\big\},$$

$h:\mathcal{B}\longrightarrow\mathcal{B}$ is a unique fixed point of H.

Now, for every $z\in\mathcal{B}$, we have

$$h(z)=\lim_{m\to\infty}H^m\big(f(z)\big)=\lim_{m\to\infty}s^{-m}f\big(s^m z\big) \tag{5.6}$$

which implies that h is a $\mathbb{C}$-linear mapping on $\mathcal{B}$. Also, we can show that h has the C^*-ternary derivation property,

$$\begin{aligned}&\mu_{h([p,q,r])[h(p),q,r][p,h(q),r][p,q,h(r)]}(t)\\&\quad=\lim_{n\to\infty}\mu_{f(s^{3n}[p,q,r])-s^{2n}[f(s^np),q,r]-s^{2n}[p,f(s^nq),r]-s^{2n}[p,q,f(s^nr)]}\big(s^{3n}t\big)\\&\quad\geq\lim_{n\to\infty}\psi_1\big(s^np,s^nq,s^nr,s^{3n}t\big)\geq\lim_{n\to\infty}\psi_1\left(p,q,r,\frac{t}{L^{3n}}\right)=1.\end{aligned}$$

So

$$h\big([p,q,r]\big)=\big[h(p),q,r\big]+\big[p,h(q),r\big]+\big[p,q,h(r)\big] \tag{5.7}$$

for all $p,q,r\in\mathcal{B}$. Also,

$$\begin{aligned}\mu_{h([e,p,e])-[e,h(p),e]}(t)&=\lim_{n\to\infty}\mu_{f(s^{3n}[e,p,e])-s^{2n}[e,f(s^np),e]}\big(s^{3n}t\big)\\&\geq\lim_{n\to\infty}\psi_1\big(s^ne,s^np,s^ne,s^{3n}t\big)\\&\geq\lim_{n\to\infty}L^{3n}\psi_1\left(e,p,e,\frac{t}{L^{3n}}\right)\\&=1,\end{aligned}$$

which implies that, on $\mathcal{B}$, h is a $*$-derivation.

Now, in (5.2), we replace q by $s^n q$, r by $s^n r$ and divide by s^{2n}. Letting $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} \mu_{s^{-2n}(f([p,s^nq,s^nr])-[f(p),s^nq,s^nr]-s^n[p,f(s^nq),r]-s^n[p,q,f(s^nr)])}(t) \\
&\quad = \lim_{n\to\infty} \mu_{f(s^{2n}[p,q,r])-s^{2n}[f(p),q,r]-s^n[p,f(s^nq),r]-s^n[p,q,f(s^nr)]}\left(s^{2n}t\right) \\
&\quad \geq \lim_{n\to\infty} \psi_1\left(p,s^nq,s^nr,s^{2n}\right) \geq \lim_{n\to\infty} \psi_1\left(p,q,r\frac{t}{L^{2n}}\right) = 1,
\end{aligned}$$

which implies that

$$h\big([p,q,r]\big) = \big[f(p),q,r\big] + \big[p,h(q),r\big] + \big[p,q,h(r)\big] \tag{5.8}$$

for all $p,q,r \in \mathcal{B}$. Putting $f(p) - h(p)$ instead of q and r in (5.7) and (5.8), we obtain $\mu_{h(p)-f(p)}(t) = 1$. Hence, on $\mathcal{B}, f$ is a $*$-derivation. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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