information

MDPI

Article

On Homomorphism Theorem for Perfect Neutrosophic Extended Triplet Groups

Xiaohong Zhang [1,2,*], Xiaoyan Mao [2,3], Florentin Smarandache [4] and Choonkil Park [5]

1. Department of Mathematics, Shaanxi University of Science &Technology, Xi'an 710021, China
2. Department of Mathematics, Shanghai Maritime University, Shanghai 201306, China; maoxiaoyan@nbu.edu.cn
3. College of Science and Technology, Ningbo University, Ningbo 315212, China
4. Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu
5. Department of Mathematics, Hanyang University, Seoul 04763, Korea; baak@hanyang.ac.kr

* Correspondence: zhangxiaohong@sust.edu.cn or zhangxh@shmtu.edu.cn

Received: 7 September 2018; Accepted: 13 September 2018; Published: 18 September 2018

Abstract: Some homomorphism theorems of neutrosophic extended triplet group (NETG) are proved in the paper [Fundamental homomorphism theorems for neutrosophic extended triplet groups, Symmetry 2018, 10(8), 321; doi:10.3390/sym10080321]. These results are revised in this paper. First, several counterexamples are given to show that some results in the above paper are not true. Second, two new notions of normal NT-subgroup and complete normal NT-subgroup in neutrosophic extended triplet groups are introduced, and their properties are investigated. Third, a new concept of perfect neutrosophic extended triplet group is proposed, and the basic homomorphism theorem of perfect neutrosophic extended triplet groups is established.

Keywords: fuzzy set; neutrosophic extended triplet group (NETG); complete NT-subgroup; homomorphism theorem; perfect neutrosophic extended triplet group

1. Introduction

As an extension of fuzzy sets and intuitionistic fuzzy sets, F. Smarandache proposed the new concept of neutrosophic sets [1]. Because the existence of intermediate states (neutral) is allowed, neutrosophic sets have more flexibility in expressing uncertainty, which has attracted much research interest. At present, neutrosophic sets have been applied to many fields, for examples, logical algebraic systems, decision making, medical diagnosis and data analysis [2–11], and more in-depth theoretical studies have also made new progress [12–14].

As an application of the basic idea of neutrosophic sets (more general, neutosophy), the new notion of neutrosophic triplet group (NTG) is proposed by F. Smarandache and M. Ali in [15,16]. As a new algebraic structure, NTG is a generalization of classical group, but it has different properties from the classical group. For NTG, the neutral element is relative and local, that is, for a neutrosophic triplet group (N, *), every element a in N has its own neutral element (denote by $neut(a)$) satisfying condition $a*neut(a) = neut(a)*a = a$, and there exists at least one opposite element (denote by $anti(a)$) in N relative to $neut(a)$ such condition $a*anti(a) = anti(a)*a = neut(a)$. In the original definition of NTG by the authors of [16], $neut(a)$ is different from the traditional unit element. Later, the concept of neutrosophic extended triplet group (NETG) was introduced (see [15]), in which the neutral element may be a traditional unit element, it is just a special case.

It should be noted that from the point of view of Neutrosophy, the neutrosophic set and neutrosophic extended triplet group are related: for a neutrosophic set, the membership of each element x is divided into three independent parts, $T(x)$, $I(x)$, $F(x)$; for a neutrosophic extended triplet group, every element a and its neutral element $neut(a)$, opposition element $anti(a)$ constitute a triple (a, $neut(a)$, $anti(a)$). In other words, the concepts of the neutrosophic set and the neutrosophic extended triplet group reflect the thought "*Trinity*". Of course, neutrosophic set and neutrosophic extended triplet group are two different mathematical concepts, one is expressed by function, the other is expressed by algebraic structure, and their deeper connection needs further study.

For the nature and structure of NTG, recently, some new results have been published: cancellable NTGs are discussed in [17]; several homomorphism theorems of commutative NTGs are proved in [18]; some new properties of NTGs are obtained in [19]; the relationships between generalized NTGs and logical algebraic systems are investigated in [20]. In these papers, the name "neutrosophic triplet group" essentially refers to "neutrosophic extended triplet group", which is illustrated by the authors of [17–20].

As we know, for an algebraic system, homomorphism basic theorems are important, similar to the classical group. In [21], the authors studied the homomorphism basic theorems of NETGs, and obtained some useful results. Unfortunately, we found that some of the results need to be corrected. In this paper, we first give some counter examples to show that several theorems in [21] are wrong, and then we prove a quotient structure theorem of weak commutative NETGs. Moreover, we introduce a new concept of perfect NETG and establish basic homomorphism theorem of perfect NETGs, which will play a positive role in the further study of neutrosophic extended triplet groups.

2. Preliminaries

Definition 1 ([15,16])**.** *Let N be a non-empty set together with a binary operation*. Then, N is called a neutrosophic extended triplet set if for any* $a \in N$*, there exist a neutral of "a" (denote by neut(a)), and an opposite of "a" (denote by anti(a)), such that* $neut(a) \in N$, $anti(a) \in N$ *and:*

$$a*neut(a) = neut(a)*a = a;$$

$$a* anti(a) = anti(a)*a = neut(a).$$

The triplet (a, neut(a), anti(a)) is called a neutrosophic extended triplet.

Note that, for a neutrosophic triplet set $(N, *)$, $a \in N$, $neut(a)$ and $anti(a)$ may not be unique. In order not to cause ambiguity, we use the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;
$\{neut(a)\}$: denote the set of all neutral of a.
$anti(a)$: denote any certain one of opposite of a;
$\{anti(a)\}$: denote the set of all opposite of a.

Definition 2 ([15,16])**.** *Let (N, *) be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:*

(1) *(N, *) is well-defined, i.e., for any* $a, b \in N$*, one has* $a*b \in N$.
(2) *(N, *) is associative, i.e.,* $(a*b)*c = a*(b*c)$ *for all* $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if for all $a, b \in N$, $a*b = b*a$.

Proposition 1 ([19])**.** *Let (N, *) be a NETG. Then*

(1) *neut(a) is unique for any a in N.*
(2) $neut(a)*neut(a) = neut(a)$ *for any a in N.*

Proposition 2 ([19])**.** *Let (N, *) be a NETG. Then* $\forall a \in N$, $\forall anti(a) \in \{anti(a)\}$,

(1) $neut(a)*p = q*neut(a)$*, for any* $p, q \in \{anti(a)\}$;

(2) *neut(neut(a)) = neut(a);*
(3) *anti(neut(a))*anti(a)∈{anti(a)};*
(4) *neut(a*a)*a = a*neut(a*a) = a; neut(a*a)*neut(a) = neut(a)*neut(a*a) = neut(a);*
(5) *neut(anti(a))*a = a*neut(anti(a)) = a; neut(anti(a))*neut(a) = neut(a)*neut(anti(a)) = neut(a);*
(6) *anti(neut(a))*a = a*anti(neut(a)) = a, for any anti(neut(a))∈{anti(neut(a))};*
(7) *a∈{anti(neut(a)*anti(a))};*
(8) *neut(a)*anti(a)∈{anti(a)}; anti(a)*neut(a)∈{anti(a)};*
(9) *a∈{anti(anti(a))}, that is, there exists p∈{anti(a)} such that a∈{anti(p)};*
(10) *neut(a)*anti(anti(a)) = a.*

Definition 3 ([19])**.** *Let (N, *) be a NETG. Then N is called a weal commutative neutrosophic extended triplet group (briefly, WCNETG) if a*neut(b) = neut(b)*a for all a, b∈N.*

Proposition 3 ([19])**.** *Let (N, *) be a NETG. Then (N, *) is weak commutative if and only if N satisfies the following conditions:*

(1) *neut(a)*neut(b) = neut(b)*neut(a) for all a, b∈N.*
(2) *neut(a)*neut(b)*a = a*neut(b) for all a, b∈N.*

Definition 4 ([19])**.** *Let (N, *) be a NETG and H be a subset of N. Then H is called a NT-subgroup of N if*

(1) *a*b∈H for all a, b∈H;*
(2) *there exists anti(a)∈{anti(a)} such that anti(a)∈H for all a∈H, where {anti(a)} is the set of opposite element of a in (N, *).*

Proposition 4 ([19])**.** *Let (N, *) be a weak commutative NETG. Then (∀a, b∈N)*

(1) *neut(a)*neut(b) = neut(b*a);*
(2) *anti(a)*anti(b)∈{anti(b*a)}.*

3. Some Counterexamples

The following examples show that Theorem 12 in [21] is not true.

Example 1. *Let N = {1, 2, 3, 4}. Define operation * on N as following Table 1. Then, (N, *) is a commutative NETG.*

Table 1. Commutative neutrosophic extended triplet group (NETG).

*	1	2	3	4
1	1	1	4	4
2	1	2	4	4
3	4	4	3	4
4	4	4	4	4

Denote H_1 = {2}, H_2 = {3}, then H_1, H_2 are neutrosophic extended triplet subgroup of N (according to Definition 3 in [21]), and H_1H_2 = H_2H_1 = {4}; but H_1H_2 is not a neutrosophic extended triplet subgroup of N (according to Definition 3 in [21]), since 4∈H_1H_2, 1∈{anti(4)}, 1∉H_1H_2.

Remark 1. *According to Definition 3 in [21], H_1H_2 is not a neutrosophic extended triplet subgroup of N. However, according to Definition 4 in this paper, H_1H_2 is a NT-subgroup of N.*

Example 2. *Let N = {1, 2, 3, 4}. Define operation * on N as following Table 2. Then, (N, *) is a non-commutative NETG.*

Table 2. Non-commutative NETG.

*	1	2	3	4
1	1	1	3	4
2	1	2	3	4
3	1	4	3	4
4	1	4	3	4

Denote H_1 = {2}, H_2 = {3}, then H_1, H_2, H_1H_2 = {3} are neutrosophic extended triplet subgroup of N (according to Definition 3 in [21]); but H_1H_2 = {3} $\neq$ {4} = H_2H_1.

The following example shows that Theorem 13 in [21] is not true.

Example 3. *Let N_1 = {a, b, c, d}, N_2 = {1, 2, 3, 4, 5, 6}. Define operations $*_1$ and $*_2$ on N_1 and N_2 as following Tables 3 and 4. Then, (N_1, $*_1$) and (N_2, $*_2$) are commutative NETGs.*

Table 3. The operation $*_1$ on N_1.

$*_1$	*a*	*b*	*c*	*d*
a	*c*	*d*	*a*	*b*
b	*a*	*b*	*c*	*d*
c	*a*	*b*	*c*	*d*
d	*c*	*d*	*a*	*b*

Table 4. The operation $*_2$ on N_2.

$*_1$	1	2	3	4	5	6
1	3	4	1	2	1	1
2	1	2	3	4	3	3
3	1	2	3	4	3	3
4	3	4	1	2	1	1
5	1	2	3	4	5	3
6	1	2	3	4	3	6

Define mapping ϕ: $N_1 \to N_2$; $a \mapsto 1$, $b \mapsto 2$, $c \mapsto 3$, $d \mapsto 4$. Then ϕ is a neutro-homomorphism (according to Definition 13 in [21]), this can be verified by software MATLAB (the program is omitted). However, according to Definition 8 in [21], ker(ϕ) = {b, c}, and

$$a\ ker(\phi) = d\ ker(\phi) = \{a, d\},\ b\ ker(\phi) = c\ ker(\phi) = \{b, c\} = ker(\phi).$$

However, a ker(ϕ) = {a, d} $\neq$ {a} = ker(ϕ)a, N_1/ker(ϕ) is not isomorphic to im(ϕ) = {1, 2, 3, 4}.

Remark 2. *It should be pointed out that many of the results in [22] are cited in [21], but in fact, some results in [22] are not true, please see [19].*

The following example shows that Theorem 14 (a) and (c) in [21] are not true.

Example 4. *Let N = {1, 2, 3, 4, 5}. Define operation * on N as following Table 5. Then, (N, *) is a commutative NETG.*

Denote H = {5}, K = {1, 3, 4, 5}, then H, K is two neutrosophic extended triplet subgroup of N (according to Definition 3 in [21]), and H is a neutrosophic triplet normal subgroup of K (according to Definition 11 in [21]). However, HK = {1, 5} is not a neutrosophic triplet subgroup of N, since

$$1 \in HK,\ 3, 4 \in \{anti(1)\} \text{ and } 3, 4 \notin HK.$$

Hence, Theorem 14 (a) and (c) in [21] are not true.

Table 5. The operation * on *N*.

*	1	2	3	4	5
1	1	2	1	1	1
2	2	1	2	2	2
3	1	2	4	3	1
4	1	2	3	4	1
5	1	2	1	1	5

4. On Complete Normal NT-Subgroups of NETGs and Homomorphism Theorem of WCNETGs

For the omissions in the literature mentioned above, one of the main reasons is that there is no careful analysis of the various definitions of subgroups of neutrosophic extended triplet group (NETG). In this section, we propose new concepts of normal NT-subgroups and complete normal NT-subgroups of NETGs and discuss their basic properties. Moreover, based on complete normal NT-subgroups, we establish homomorphism theorem of weak commutative neutrosophic extended triplet groups (WCNETGs).

Definition 5. *Let (N, *) be a NETG and H be a NT-subgroup of N. Then H is called a normal NT-subgroup of N if for all a∈N and every anti(a)∈{anti(a)}, aH(anti(a)) ⊆ H.*

Obviously, for any commutative NETG (*N*, *), a NT-subgroup *H* of *N* is normal if and only if for all *a*∈*N*, *H*(*neut*(*a*)) ⊆ *H*. The following examples show that there exists some NT-subgroups which are not normal, for some commutative NETGs.

Example 5. *Let (N, *) be the commutative NETG in Example 1. Denote H = {1}, then H is a NT-subgroup of N. But,*

$$3H(anti(3)) = \{4\} \subseteq \{1\} = H.$$

Example 6. *Let (N, *) be the commutative NETG in Example 4. Denote H = {5}, then H is a NT-subgroup of N. But, 2H(anti(2)) = {2} ⊆ {5} = H.*

Definition 6. *Let (N, *) be a NETG and H be a normal NT-subgroup of N. Then H is called to be complete normal if it satisfies:*

(1) *for all a∈N, neut(a)∈H.*
(2) *for all h∈H, anti(h)∈H.*

The following examples show that a normal NT-subgroup may be not a complete normal.

Example 7. *Let (N, *) be the commutative NETG in Example 1. Denote H = {4}, then H is a normal NT-subgroup of N. But, H is not a complete NT-subgroup of N, since 1 = neut(1)∉H. Moreover, 1, 2, 3∈{anti(4)}, but 1, 2, 3∉H.*

Example 8. *Let N = {1, 2, 3, 4, 5, 6}. Define operation * on N as following Table 6. Then, (N, *) is a non-commutative NETG.*

Denote H = {2, 3}, then H is a normal NT-subgroup of N. But, H is not a complete normal NT-subgroup of N, since neut(5) = 5∉H. Moreover, 2∈H, 5∈{anti(2)}, but 5∉H.

It is easy to verify that the following proposition is true (the proof is omitted).

Table 6. Non-commutative NETG.

*	1	2	3	4	5	6
1	3	1	1	3	1	3
2	4	2	2	4	2	4
3	1	3	3	1	3	1
4	2	4	4	2	4	2
5	1	2	3	4	5	6
6	3	4	1	2	6	5

Proposition 5. *Let (N, *) be a commutative NETG and H be a non-empty subset of N. Then H is complete normal NT-subgroup of N if and only if it satisfies:*

(1) *for all a, b∈H, a*b∈H.*
(2) *for all a∈N, neut(a)∈H.*
(3) *for all h∈H, anti(h)∈H.*

Proposition 6. *Let (N, *) be a weak commutative NETG and H be a complete normal NT-subgroup of N, a, b∈N. Then the following conditions are equivalent:*

(1) *there exists anti(a)∈{anti(a)} and p N such that anti(a)*b*neut(p)∈H;*
(2) *for any anti(b)∈{anti(b)}, there exists p∈N such that anti(b)*a*neut(p)∈H;*
(3) *for any anti(a)∈{anti(a)}, there exists p∈N such that anti(a)*b*neut(p)∈H.*

Proof. (1) ⇒ (2): Assume that *anti*(*a*)**b***neut*(*p*)∈*H*, *p*∈*N*. By Defition 6 (2), *anti*(*anti*(*a*)**b***neut*(*p*))∈*H*, for any *anti*(*anti*(*a*)**b***neut*(*p*))∈{*anti*(*anti*(*a*)**b***neut*(*p*))}. On the other hand, using Proposition 4 (2),

$$anti(neut(p))*anti(b)*anti(anti(a))\in\{anti(anti(a)*b*neut(p))\}.$$

It follows that *anti*(*neut*(*p*))**anti*(*b*)**anti*(*anti*(*a*))∈*H*, for any *anti*(*b*)∈{*anti*(*b*)}. Then, by Definition 4 (1), Definition 6 (1), Definition 3, Proposition 2 (10) and (2) we get

$$(anti(neut(p))*anti(b)*anti(anti(a)))*neut(a)\in H,$$

$$(anti(neut(p))*anti(b))*(neut(a)*anti(anti(a)))\in H,$$

$$anti(neut(p))*anti(b)*a\in H,$$

$$neut(p)*(anti(neut(p))*anti(b)*a)\in H,$$

$$(neut(p)*\ anti(neut(p)))*anti(b)*a\in H,$$

$$neut(neut(p))*anti(b)*a\in H,$$

$$neut(p)*anti(b)*a\in H,$$

$$anti(b)*a*neut(p)\in H.$$

(2) ⇒ (3): Assume that *anti*(*b*)**a***neut*(*p*)∈*H*, *p*∈*N*, for any *anti*(*b*)∈{*anti*(*b*)}. Using the proof process similar to the previous one, we can get that *anti*(*a*)**b***neut*(*p*)∈*H*, *p*∈*N*, for any *anti*(*a*)∈{*anti*(*a*)}.

(3) ⇒ (1): Obviously. □

Theorem 1. *Let (N, *) be a weak commutative neutrosophic triplet group and H be a complete normal NT-subgroup of N. Define binary relation* $\approx_H$ *on N as follows: ∀a, b∈N,*

A $\approx_H$ *b if and only if there exists p∈N such that anti(a)*b*neut(p)∈H, where anti(a)∈{anti(a)}.*

Then

(1) $\approx_H$ *is an equivalent relation on N;*
(2) $\forall\, c \in N,\ a \approx_H b \Rightarrow c{*}a \approx_H c{*}b;$
(3) $\forall\, c \in N,\ a \approx_H b \Rightarrow a{*}c \approx_H b{*}c;$
(4) *define binary operation * on* $N/H=\{[a]_H \mid a \in N\}$ *as follows:* $[a]_H {*} [b]_H = [a{*}b]_H,\ \forall a, b \in N$. *We can obtained a homomorphism from (N, *) to (N/H, *), that is,* $f\colon N \to N/H;\ f(a)= [a]_H$ *for all* $a \in N$.

Proof. (1) Suppose $a \in N$, then $anti(a){*}a{*}neut(a) = neut(a){*}neut(a) = neut(a) \in H$, applying Proposition 2 (2) and Definition 6 (1). Hence, $a \approx_H a$.

Assume $a \approx_H b$, then there exists $p \in N$ such that $anti(a){*}b{*}neut(p) \in H$, where $anti(a) \in \{anti(a)\}$. By Proposition 6 (2) and (3), $anti(b){*}a{*}neut(p) \in H,\ \forall\, anti(b) \in \{anti(b)\}$. Thus, $b \approx_H a$.

If $a \approx_H b$ and $b \approx_H c$, then there exists $p \in N$ and $q \in N$ such that $anti(a){*}b{*}neut(p) \in H$, $anti(b){*}c{*}neut(q) \in H$, *where* $anti(a) \in \{anti(a)\}$, $anti(b) \in \{anti(b)\}$. Using Definition 4 (1), Definition 3 and Proposition 4 (1) we have

$$(anti(a){*}b{*}neut(p)){*}(anti(b){*}c{*}neut(q)) \in H,$$

$$anti(a){*}(b{*}anti(b)){*}c{*}(neut(p){*}neut(q)) \in H,$$

$$anti(a){*}neut(b){*}c{*}(neut(p){*}neut(q)) \in H,$$

$$anti(a){*}c{*}(neut(b){*}neut(p){*}neut(q)) \in H,$$

$$anti(a){*}c{*}(neut(q{*}p{*}b) \in H.$$

It follows that $a \approx_H c$.

Combining the results above, $\approx_H$ is an equivalent relation on *N*.

(2) Suppose $a \approx_H b$. Then there exists $p \in N$ such that $anti(a){*}b{*}neut(p) \in H$, where $anti(a) \in \{anti(a)\}$. By Definition 3, Definition 6 (1) and Definition 4 (1),

$$(anti(a){*}anti(c)){*}(c{*}b){*}neut(p)$$

$$= anti(a){*}(anti(c){*}c){*}b{*}neut(p)$$

$$= anti(a){*}neut(c){*}b{*}neut(p)$$

$$= anti(a){*}b{*}neut(c)$$

$$= (anti(a){*}b{*}neut(p)){*}neut(c) \in H.$$

By Proposition 4 (2), $anti(a){*}anti(c) \in \{anti(c{*}a)\}$. Hence, there exists $anti(c{*}a) \in \{anti(c{*}a)\}$ and $p \in N$ such that $anti(c{*}a){*}(c{*}b){*}neut(p) \in H$. Applying Proposition 6, for any $anti(c{*}a) \in \{anti(c{*}a)\}$, there exists $p \in N$ such that $anti(c{*}a){*}(c{*}b){*}neut(p) \in H$. That is, $(c{*}a) \approx_H (c{*}b)$.

(3) Suppose $a \approx_H b$. Then there exists $p \in N$ such that $anti(a){*}b{*}neut(p) \in H$, where $anti(a) \in \{anti(a)\}$. By Definition 3, Definition 6 (1), Definition 4 (1) and Definition 5,

$$(anti(c){*}anti(a)){*}(b{*}c){*}neut(p) = anti(c){*}(anti(a){*}b{*}neut(p)){*}c$$

$$= anti(c){*}(anti(a){*}b{*}neut(p)){*}neut(c){*}anti(anti(c))$$

$$= (anti(c){*}(anti(a){*}b{*}neut(p)){*}anti(anti(c))){*}neut(c) \in H.$$

By Proposition 4 (2), $anti(c){*}anti(a) \in \{anti(a{*}c)\}$. Hence, there exists $anti(a{*}c) \in \{anti(a{*}c)\}$ and $p \in N$ such that $anti(a{*}c){*}(b{*}c){*}neut(p) \in H$. Applying Proposition 6, for any $anti(a{*}c) \in \{anti(a{*}c)\}$, there exists $p \in N$ such that $anti(a{*}c){*}(b{*}c){*}neut(p) \in H$. That is, $(a{*}c) \approx_H (b{*}c)$.

(4) Combining (1), (2) and (3), one can get (4). □

5. Homomorphism Theorems of Perfect Neutrosophic Extended Tripet Groups (PNETGs)

Proposition 7. *Let (N, *) be a weak commutative NETG. Then the following conditions are equivalent:*

(i) *for all a∈N, the opposite element of neut(a) is unique, that is, |{anti(neut(a))}| = 1;*
(ii) *for all a∈N, and any anti(neut(a))∈{anti(neut(a))}, anti(neut(a)) = neut(a).*

Proof. (i) ⇒ (ii): For all *a*∈*N*, by Proposition 1 (2) and Proposition 2 (2),

$$neut(a)*neut(a) = neut(a) = neut(neut(a)).$$

This means that *neut*(*a*)∈{*anti*(*neut*(*a*))}. Applying (i) we get *anti*(*neut*(*a*)) = *neut*(*a*).
(ii) ⇒ (i): Obviously. □

Definition 7. *Let (N, *) be a weak commutative NETG. Then N is called a perfect NETG if anti(neut(a)) = neut(a) for all a∈N.*

Proposition 8. *Let (N, *) be a perfect NETG. Then neut(anti(a)) = neut(a) for all a∈N.*

Proof. For all *a*∈*N*, and any *anti*(*a*)∈{*anti*(*a*)}, from *anti*(*neut*(*a*)) = *anti*(*a***anti*(*a*)), applying Proposition 4 (2) we have

$$anti(a)*anti(anti(a))\in\{anti(neut(a))\}.$$

Since *anti*(*a*)**anti*(*anti*(*a*)) = *neut*(*anti*(*a*)), thus *neut*(*anti*(*a*))∈{*anti*(*neut*(*a*))}. By Definition 7, we get *neut*(*anti*(*a*)) = *neut*(*a*). □

The following examples show that there exists commutative NETG which is not perfect, and there exists non-commutative NETG which is perfect.

Example 9. *Let (N, *) be the commutative NETG in Example 1. Then N is not perfect, since 1 = neut(1), {anti(neut(1))} = {1, 2}, 2 ≠ neut(1).*

Example 10. *Let N = {a, b, c, d, e, f, g}. The operation * on N is defined as Table 7. Then, (N, *) is a non-commutative perfect NETG.*

Table 7. Non-commutative perfect NETG.

*	a	b	c	d	e	f	g
a	a	b	c	d	e	f	a
b	b	a	f	e	d	c	b
c	c	e	a	f	b	d	c
d	d	f	e	a	c	b	d
e	e	c	d	b	f	a	e
f	f	d	b	c	a	e	f
g	a	b	c	d	e	f	g

Definition 8 ([21,22]). *Let (N_1, *) and (N_2, *) be two neutrosophic extended triplet groups (NETGs). A mapping f: N_1→N_2 is called a neutro-homomorphism if*

$$\forall x,y\in N_1, f(x*y) = f(x)*f(y).$$

The neutrosophic triplet kernel off is defined Ker(f) = {x∈N_1: there exists y∈N_2 such that f(x) = neut(y)}. A neutro-homomorphism f is called a neutro-monomorphism if it is only one to one (injective). A neutro-homomorphism f is called a neutro-epimorphism if it is only onto (surjective). If a neutro-homomorphism *f: $N_1 \to N_2$ is one to one and onto, then f is called neutro-isomorphism, and N_1 and N_2 are called neutro-isomorphic and denoted $N_1 \cong N_2$.*

It is easy to verify that the following proposition is true (the proof is omitted).

Proposition 9. *Let (N_1, *) and (N_2, *) be two NETGs and f: $N_1 \to N_2$ be a neutro-homomorphism. Then*

(1) *for any $x \in N_1$, f(neut(x)) = neut(f(x));*
(2) *for any $x \in N_1$ and any $anti(x) \in \{anti(x)\}$, $f(anti(x)) \in \{anti(f(x))\}$.*

Theorem 2. *Let (N_1, *) and (N_2, *) be two perfect NETGs and f: $N_1 \to N_2$ be a neutro-homomorphism. Then*

(1) *Ker(f) is a complete normal NT-subgroup of N_1;*
(2) *g is neutro-epimorphism, where g: $N_1 \to N_1/Ker(f)$; $g(a) = [a]_{Ker(f)}$ for all $a \in N_1$.*

Proof. (1) Assume $a, b \in Ker(f)$, then there exists $x, y \in N_2$, $f(a) = neut(x)$, $f(b) = neut(y)$. Thus

$$f(a*b) = f(a)*f(b) = neut(x)*neut(y) = neut(y*x). \text{ (By Proposition 4 (1))}$$

This means that $a*b \in Ker(f)$.

For any $anti(a) \in \{anti(a)\}$, by Proposition 9 (2), $f(anti(a)) \in \{anti(f(a))\} = \{anti(neut(x))\}$. Applying Definition 7, $\{anti(neut(x))\} = \{neut(x)\}$. Thus, $f(anti(a)) = neut(x)$. This means that $anti(a) \in Ker(f)$.

For any $p \in N_1$, by Proposition 9 (1), $f(neut(p)) = neut(f(p))$, then $neut(p) \in Ker(f)$. This means that $\{neut(p): p \in N_1\} \subseteq Ker(f)$.

Moreover, for any $p \in N_1$, by Definition 8, Proposition 9, Definitions 3 and 7, Proposition 4, we have

$$\begin{aligned} & f(anti(p)*a*p) \\ &= f(anti(p))*f(a)*f(p) \\ &= f(anti(p))*neut(x)*f(p) \\ &= f(anti(p))*f(p)*neut(x) \\ &= f(anti(p)*p)*neut(x) \\ &= f(neut(p))*neut(x) \\ &= neut(f(p))*neut(x) \\ &= neut(x*f(p)). \end{aligned}$$

It follows that $anti(p)*a*p \in Ker(f)$.

Combining above results, we get that $Ker(f)$ is a complete normal NT-subgroup of N_1.

(2) By (1) and Theorem 1 (4) we get (2). □

Remark 3. *Under the conditions of the above theorem, even if f is bijective, the related isomorphism theorem cannot be obtained. An example is given below.*

Example 11. *Let N = {1, 2, 3, 4, 5}. Define operation * on N as following Table 8. Then, (N, *) is a perfect NETG.*

Define f: N→N; for any $a \in N$, f(a) = a. Obviously, f is a neutro-isomorphism, Ker(f) = {2, 5}, N/Ker(f) = {{1}, {2, 5}, {3}, {4}}. It is easy to verify that g is neutro-epimorphism, where g: N→N/Ker(f); $g(a) = [a]_{Ker(f)}$ for all $a \in N$. But, N/Ker(f) is not isomorphic to N.

Table 8. Commutative perfect NETG.

*	1	2	3	4	5
1	2	1	4	3	1
2	1	2	3	4	2
3	4	3	1	2	3
4	3	4	2	1	4
5	1	2	3	4	5

Example 12. *Let N_1 = {1, 2, 3, 4, 5, 6, 7} and N_2 = {a, b, c, d}. The operations $*_1$, $*_2$ on N_1, N_2 are defined as Tables 9 and 10. Then, (N_1, $*_1$) and (N_2, $*_2$) are perfect NETGs.*

Table 9. The operation $*_1$ on commutative perfect NETG N_1.

$*_1$	1	2	3	4	5	6	7
1	2	1	4	3	1	1	1
2	1	2	3	4	2	2	2
3	4	3	1	2	3	3	3
4	3	4	2	1	4	4	4
5	1	2	3	4	5	6	2
6	1	2	3	4	6	5	2
7	1	2	3	4	2	2	7

Table 10. The operation $*_2$ on commutative perfect NETG N_2.

$*_2$	*a*	*b*	*c*	*d*
a	*b*	*a*	*d*	*c*
b	*a*	*b*	*c*	*d*
c	*d*	*c*	*a*	*b*
d	*c*	*d*	*b*	*a*

Define mapping ϕ: $N_1 \to N_2$; $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$, $4 \mapsto d$, $5 \mapsto b$, $6 \mapsto b$, $7 \mapsto b$. Then ϕ is a neutro-homomorphism, ker(ϕ) = {2, 5, 6, 7}, and N_1/ker(ϕ) = {{1}, {2, 5, 6, 7}, {3}, {4}}. Moreover, we can verify that N_1/Ker(ϕ) is isomorphic to N_2.

Example 13. *Let N_1 = {1, 2, 3, 4, 5, 6, 7, 8} and N_2 = {a, b, c, d, e, f}. The operations $*_1$, $*_2$ on N_1, N_2 are defined as Tables 11 and 12. Then, (N_1, $*_1$) and (N_2, $*_2$) are non-commutative perfect NETGs.*

Define mapping ϕ: $N_1 \to N_2$; $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto c$, $4 \mapsto d$, $5 \mapsto e$, $6 \mapsto f$, $7 \mapsto a$, $8 \mapsto a$. Then ϕ is a neutro-homomorphism, ker(ϕ) = {1, 7, 8}, and N_1/ker(ϕ) = {{1, 7, 8}, {2}, {3}, {4}, {5}, {6}}. Moreover, we can verify that N_1/Ker(ϕ) is isomorphic to N_2.

Table 11. The operation $*_1$ on non-commutative perfect NETG N_1.

$*_1$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	1	1
2	2	1	6	5	4	3	2	2
3	3	5	1	6	2	4	3	3
4	4	6	5	1	3	2	4	4
5	5	3	4	2	6	1	5	5
6	6	4	2	3	1	5	6	6
7	1	2	3	4	5	6	8	7
8	1	2	3	4	5	6	7	8

Table 12. The operation $*_2$ on non-commutative perfect NETG N_2.

$*_2$	***a***	***b***	***c***	***d***	***e***	***f***
a	a	b	c	d	e	f
b	b	a	f	e	d	c
c	c	e	a	f	b	d
d	d	f	e	a	c	b
e	e	c	d	b	f	a
f	f	d	b	c	a	e

6. Conclusions

This paper has further studied the neutrosophic extended triplet group (NETG) and obtained some important results. First, examples are given to show that some results in [21] are not true. Second, some new notions of normal NT-group and complete normal NT-group are introduced, and a quotient structure theorem of weak commutative NETG is proved. Third, the concept of perfect neutrosophic extended triplet group (PNETG)is proposed, a homomorphism theorem of PNETG is established. All these results are interesting for exploring the structure characterizations of NETG. As future research topics, we will discuss the integration of the related uncertainty theory, such as the combination of neutrosophic set, fuzzy set, rough set and logic algebras (see [23–26]).

Author Contributions: X.Z. and X.M. initiated the research and wrote the paper, F.S. and C.P. supervised the research work and provided helpful suggestions.

Funding: This research was funded by National Natural Science Foundation of China grant number 61573240.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Smarandache, F. Neutrosophic set—A generialization of the intuituionistics fuzzy sets. *Int. J. Pure Appl. Math.* **2005**, *3*, 287–297.
2. Borzooei, R.A.; Farahani, H.; Moniri, M. Neutrosophic deductive filters on BL-algebras. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2993–3004.
3. Jun, Y.B. Neutrosophic subalgebras of several types in BCK/BCI-algebras. *Ann. Fuzzy Math. Inform.* **2017**, *14*, 75–86.
4. Ye, J. Multiple attribute decision-making method using correlation coefficients of normal neutrosophic sets. *Symmetry* **2017**, *9*, 80. [CrossRef]
5. Ali, M.; Dat, L.Q.; Son, L.H.; Smarandache, F. Interval complex neutrosophic set: Formulation and applications in decision-making. *Int. J. Fuzzy Syst.* **2018**, *20*, 986–999. [CrossRef]
6. Ye, J. Multiple-attribute decision-making method using similarity measures of single-valued neutrosophic hesitant fuzzy sets based on least common multiple cardinality. *J. Intell. Fuzzy Syst.* **2018**, *34*, 4203–4211. [CrossRef]
7. Liu, P.D.; Liu, X. The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making. *Int. J. Mach. Learn. Cybern.* **2018**, *9*, 347–358. [CrossRef]
8. Wei, G.W.; Zhang, Z.P. Some single-valued neutrosophic Bonferroni power aggregation operators in multiple attribute decision making. *J. Ambient Intell. Hum. Comput.* **2018**, 1–20. [CrossRef]
9. Wang, J.Q.; Yang, Y.; Li, L. Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Comput. Appl.* **2018**, *30*, 1529–1547. [CrossRef]
10. Zhang, C.; Li, D.Y.; Broumi, S.; Sangaiah, A.K. Medical diagnosis based on single-valued neutrosophic probabilistic rough multisets over two universes. *Symmetry* **2018**, *10*, 213. [CrossRef]
11. Abdel-Basset, M.; Mohamed, M. The role of single valued neutrosophic sets and rough sets in smart city: Imperfect and incomplete information systems. *Measurement* **2018**, *124*, 47–55. [CrossRef]

12. Peng, X.D.; Dai, J.G. A bibliometric analysis of neutrosophic set: Two decades review from 1998 to 2017. *Artif. Intell. Rev.* **2018**. [CrossRef]
13. Zhang, X.H.; Bo, C.X.; Smarandache, F.; Dai, J.H. New inclusion relation of neutrosophic sets with applications and related lattice structure. *Int. J. Mach. Learn. Cybern.* **2018**. [CrossRef]
14. Zhang, X.H.; Bo, C.X.; Smarandache, F.; Park, C. New operations of totally dependent-neutrosophic sets and totally dependent-neutrosophic soft sets. *Symmetry* **2018**, *10*, 187. [CrossRef]
15. Smarandache, F. *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras, And Applications*; Pons Publishing House: Brussels, Belgium, 2017.
16. Smarandache, F.; Ali, M. Neutrosophic triplet group. *Neural Comput. Appl.* **2018**, *29*, 595–601. [CrossRef]
17. Zhang, X.H.; Smarandache, F.; Liang, X.L. Neutrosophic duplet semi-group and cancellable neutrosophic triplet groups. *Symmetry* **2017**, *9*, 275. [CrossRef]
18. Zhang, X.H.; Smarandache, F.; Ali, M.; Liang, X.L. Commutative neutrosophic triplet group and neutro-homomorphism basic theorem. *Ital. J. Pure Appl. Math.* **2018**, in press.
19. Zhang, X.H.; Hu, Q.Q.; Smarandache, F.; An, X.G. On neutrosophic triplet groups: Basic properties, NT-subgroups, and some notes. *Symmetry* **2018**, *10*, 289. [CrossRef]
20. Zhang, X.H.; Wu, X.Y.; Smarandache, F.; Hu, M.H. Left (right)-quasi neutrosophic triplet loops (groups) and generalized BE-algebras. *Symmetry* **2018**, *10*, 241. [CrossRef]
21. Çelik, M.; Shalla, M.M.; Olgun, N. Fundamental homomorphism theorems for neutrosophic extended triplet groups. *Symmetry* **2018**, *10*, 321. [CrossRef]
22. Bal, M.; Shalla, M.M.; Olgun, N. Neutrosophic triplet cosets and quotient groups. *Symmetry* **2018**, *10*, 126. [CrossRef]
23. Zhang, X.H. Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2017**, *33*, 1767–1774. [CrossRef]
24. Zhang, X.H.; Park, C.; Wu, S.P. Soft set theoretical approach to pseudo-BCI algebras. *J. Intell. Fuzzy Syst.* **2018**, *34*, 559–568. [CrossRef]
25. Ma, X.L.; Zhan, J.M.; Ali, M.I. A survey of decision making methods based on two classes of hybrid soft set models. *Artif. Intell. Rev.* **2018**, *49*, 511–529. [CrossRef]
26. Bucolo, M.; Fortuna, L.; La Rosa, M. Complex dynamics through fuzzy chains. *IEEE Trans. Fuzzy Syst.* **2004**, *12*, 289–295. [CrossRef]