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On the stability of a Cauchy type functional equation

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Abstract: In this work, the Hyers-Ulam type stability and the hyperstability of the functional equation

$$f\left(\frac{x+y}{2}+xy\right)+f\left(\frac{x-y}{2}-xy\right)=f(x)$$

are proved.

Keywords: Hyers-Ulam stability, additive mapping, hyperstability, topological vector space

MSC: 39B82, 34K20, 26D10

1 Introduction

The functional equation (ξ) is called stable if any function g satisfying the equation (ξ) *approximately*, is near to a true solution of (ξ). Ulam, in 1940 [1], introduced the stability of homomorphisms between two groups. More precisely, he proposed the following problem: given a group $(G_1, .)$, a metric group $(G_2, \star, d)$ and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x.y), f(x) \star f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? If this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [2] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Aoki [3] and Rassias [4] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. During the last decades, several stability problems of functional equations have been investigated by several mathematicians. A large list of references concerning the stability of functional equations can be found in [5–15].

In this paper, we deal with the functional equations

$$f\Big(\frac{x+y}{2}+xy\Big)+f\Big(\frac{x-y}{2}-xy\Big)=f(x), \tag{1.1}$$

$$f\Big(\frac{x+y}{2}+xy\Big)+g\Big(\frac{x-y}{2}-xy\Big)=h(x), \tag{1.2}$$

$$f\Big(\frac{x+y}{2}+\alpha xy\Big)+f\Big(\frac{x-y}{2}+\beta xy\Big)=f(x)+(\alpha+\beta)f(xy). \tag{1.3}$$

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2 Solutions of the functional equations (1.1), (1.2) and (1.3)

Theorem 2.1. *Let X be a vector space. A mapping $f : \mathbb{R} \to X$ satisfies equation* (1.1) *if and only if f is additive.*

Proof. Let f satisfy equation (1.1). Letting $x = y = 0$ in equation (1.1), we get $f(0) = 0$. Letting $x = 0$ in equation (1.1), we obtain that f is odd. Setting $x = \frac{1}{2}$ and replacing y by $y + \frac{1}{4}$ in equation (1.1) and using the oddness of f, we obtain

$$f\Big(y + \frac{1}{2}\Big) = f(y) + f\Big(\frac{1}{2}\Big), \quad y \in \mathbb{R}. \tag{2.1}$$

Let $a, b \in \mathbb{R}$ with $2a + 2b \neq -1$. Let $x, y \in \mathbb{R}$ such that $x = a + b$ and $y = \frac{a-b}{2a+2b+1}$. Since f satisfies equation (1.1), we get

$$f(a + b) = f(a) + f(b) \tag{2.2}$$

for all $a, b \in \mathbb{R}$ with $2a+2b \neq -1$. Since f is odd, it follows from equations (2.1) and (2.2) that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Therefore f is additive.

Conversely, if f is additive, it is easy to check that f satisfies equation (1.1). □

Theorem 2.2. *Let X be a vector space. Suppose that mappings $f, g, h : \mathbb{R} \to X$ satisfy equation* (1.2). *Then*

(*i*) $f(x) + g\big(-x - \frac{1}{2}\big) = h\big(-\frac{1}{2}\big), \quad x \in \mathbb{R}$,

(*ii*) $f(x) + g(y) = h(x + y), \quad x, y \in \mathbb{R}$,

(*iii*) *$f - f(0)$, $g - g(0)$ and $h - h(0)$ are additive.*

Proof. Letting $x = 0$ and replacing y by $2y$ in equation equation (1.2), we get

$$f(y) + g(-y) = h(0), \quad y \in \mathbb{R}. \tag{2.3}$$

Letting $y = 0$ in equation (1.2), we get

$$f\Big(\frac{x}{2}\Big) + g\Big(\frac{x}{2}\Big) = h(x), \quad x \in \mathbb{R}. \tag{2.4}$$

It follows from equations (2.3) and (2.4) that

$$\begin{aligned} h(x) + h(-x) &= \Big[f\Big(\frac{x}{2}\Big) + g\Big(\frac{x}{2}\Big)\Big] + \Big[f\Big(-\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)\Big] \\ &= \Big[f\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big)\Big] + \Big[f\Big(-\frac{x}{2}\Big) + g\Big(\frac{x}{2}\Big)\Big] \\ &= h(0) + h(0) = 2h(0), \quad x \in \mathbb{R}. \end{aligned}$$

In particular,

$$h\Big(\frac{1}{2}\Big) + h\Big(-\frac{1}{2}\Big) = 2h(0). \tag{2.5}$$

Setting $x = \frac{1}{2}$ and replacing y by $-y - \frac{1}{4}$ in equation (1.2), we obtain

$$f(-y) + g\Big(y + \frac{1}{2}\Big) = h\Big(\frac{1}{2}\Big), \quad y \in \mathbb{R}. \tag{2.6}$$

It follows from equations (2.3), (2.5) and (2.6) that

$$f\Big(-y - \frac{1}{2}\Big) + g(y) = h\Big(-\frac{1}{2}\Big), \quad y \in \mathbb{R}. \tag{2.7}$$

Letting $x = -y - \frac{1}{2}$ in equation (2.7), we get

$$f(x) + g\Big(-x - \frac{1}{2}\Big) = h\Big(-\frac{1}{2}\Big), \quad x \in \mathbb{R}. \tag{2.8}$$

Replacing x and y by $x + y$ and $\frac{x-y}{1+2x+2y}$ in equation (1.2), respectively, we get

$$f(x) + g(y) = h(x + y), \quad x, y \in \mathbb{R},\ 2x + 2y \neq -1. \tag{2.9}$$

By equations (2.8) and (2.9), we obtain that $f(x) + g(y) = h(x + y)$ for all $x, y \in \mathbb{R}$. Then $h(x) = f(x) + g(0) = f(0) + g(x)$ for all $x \in \mathbb{R}$. This implies $f - f(0)$, $g - g(0)$ and $h - h(0)$ are additive. □

We need the following theorem from [16] to find the general solution of equation (1.3).

Theorem 2.3. *[16] Let X be a vector space and α be a real number. If a function $f:\mathbb{R}\to X$ satisfies $f(x+y-\alpha xy)+f(\alpha xy)=f(x)+f(y)$ and $f(0)=0$, then f is additive.*

Theorem 2.4. *Let X be a vector space and α,β be real numbers. If a mapping $f:\mathbb{R}\to X$ satisfies equation (1.3) with $f(0)=0$ or $\alpha+\beta\neq 1$, then f is additive.*

Proof. Letting $x=y=0$ in equation (1.3), we get $(\alpha+\beta)f(0)=0$. Therefore we may assume that $f(0)=0$. Letting $x=0$ in equation (1.3), we obtain that f is odd. Letting $y=0$ and replacing x by $2x$ in equation (1.3), we get $f(2x)=2f(x)$ for all $x\in\mathbb{R}$. Replacing y by $-y$ in equation (1.3), we obtain

$$f\Big(\frac{x-y}{2}-\alpha xy\Big)+f\Big(\frac{x+y}{2}-\beta xy\Big)=f(x)+(\alpha+\beta)f(-xy) \tag{2.10}$$

for all $x,y\in\mathbb{R}$. Adding the equations (1.3) and (2.10) and using the oddness of f, we have

$$\Big[f\Big(\frac{x+y}{2}+\alpha xy\Big)+f\Big(\frac{x-y}{2}-\alpha xy\Big)\Big]+\Big[f\Big(\frac{x+y}{2}-\beta xy\Big)+f\Big(\frac{x-y}{2}+\beta xy\Big)\Big]=2f(x) \tag{2.11}$$

for all $x,y\in\mathbb{R}$. Interchanging x with y in equation (1.3), we obtain

$$f\Big(\frac{x+y}{2}+\alpha xy\Big)+f\Big(\frac{y-x}{2}+\beta xy\Big)=f(y)+(\alpha+\beta)f(-xy) \tag{2.12}$$

for all $x,y\in\mathbb{R}$. Replacing y by $-y$ in equation (2.12), and then adding the resulting equation to equation (2.12), we have

$$\Big[f\Big(\frac{x+y}{2}+\alpha xy\Big)+f\Big(\frac{x-y}{2}-\alpha xy\Big)\Big]=\Big[f\Big(\frac{x+y}{2}+\beta xy\Big)+f\Big(\frac{x-y}{2}-\beta xy\Big)\Big] \tag{2.13}$$

for all $x,y\in\mathbb{R}$. Using equation (2.13), we rewrite equation (2.11) as

$$\Big[f\Big(\frac{x+y}{2}+\beta xy\Big)+f\Big(\frac{x+y}{2}-\beta xy\Big)\Big]+\Big[f\Big(\frac{x-y}{2}-\beta xy\Big)+f\Big(\frac{x-y}{2}+\beta xy\Big)\Big]=2f(x) \tag{2.14}$$

for all $x,y\in\mathbb{R}$. Interchanging x with y in equation (2.14), and then adding the resulting equation to (2.14), we obtain

$$f\Big(\frac{x+y}{2}+\beta xy\Big)+f\Big(\frac{x+y}{2}-\beta xy\Big)=f(x)+f(y) \tag{2.15}$$

for all $x,y\in\mathbb{R}$. Replacing x and y by $2x$ and $2y$ in equation (2.15), respectively, and using $f(2t)=2f(t)$, we get

$$f(x+y+4\beta xy)+f(x+y-4\beta xy)=2f(x)+2f(y),\quad x,y\in\mathbb{R}. \tag{2.16}$$

Let $\beta=0$. It follows from equation (2.16) that f is additive. Let $\beta\neq 0$ and let $\gamma=4\beta$. Letting $y=1/\gamma$ in equation (2.16), we have

$$f(2x+1/\gamma)=2f(x)+f(1/\gamma),\quad x\in\mathbb{R}. \tag{2.17}$$

Replacing y by $2y+1/\gamma$ in equation (2.16), we get

$$f\big(2(x+y+\gamma xy)+1/\gamma\big)+f\big(2(y-\gamma xy)+1/\gamma\big)=2f(x)+2f(2y+1/\gamma) \tag{2.18}$$

for all $x,y\in\mathbb{R}$. Using (2.17), we rewrite equation (2.18) as

$$f(x+y+\gamma xy)+f(y-\gamma xy)=f(x)+2f(y),\quad x\in\mathbb{R}. \tag{2.19}$$

Letting $y=1/\gamma$ in equation (2.19) and using equation (2.17), we obtain

$$f(1/\gamma-x)=f(1/\gamma)-f(x),\quad x\in\mathbb{R}. \tag{2.20}$$

Replacing y by $1/\gamma-y$ in equation (2.19), we get

$$f\big(1/\gamma-(x+y-\gamma xy)\big)-f(1/\gamma-y)=f(\gamma xy)-f(x),\quad x,y\in\mathbb{R}. \tag{2.21}$$

Using equation (2.19), we rewrite equation (2.21) as

$$f(x+y-\gamma xy)+f(\gamma xy)=f(x)+f(y),\quad x,y\in\mathbb{R}.$$

Then by Theorem 2.3, we obtain that f is additive. □

Using the ideas from Theorem 2.4, we have

Theorem 2.5. *Let X be a vector space and α, β be real numbers. If a mapping $f : \mathbb{R} \to X$ with $f(0) = 0$ satisfies*

$$f\Big(\frac{x+y}{2} + \alpha xy\Big) + f\Big(\frac{x-y}{2} + \beta xy\Big) = f(x) + f\big((\alpha+\beta)xy\big) \tag{2.22}$$

for all $x, y \in \mathbb{R}$, then f is additive.

Proof. Letting $x = 0$ in equation (2.22), we obtain that f is odd, since $f(0) = 0$. Letting $y = 0$ and replacing x by $2x$ in equation (2.22), we get $f(2x) = 2f(x)$ for all $x \in \mathbb{R}$. Replacing y by $-y$ in equation (2.22), we obtain

$$f\Big(\frac{x-y}{2} - \alpha xy\Big) + f\Big(\frac{x+y}{2} - \beta xy\Big) = f(x) + f\big(-(\alpha+\beta)xy\big) \tag{2.23}$$

for all $x, y \in \mathbb{R}$. Adding the equations (2.22) and (2.23) and using the oddness of f, we have

$$\Big[f\Big(\frac{x+y}{2} + \alpha xy\Big) + f\Big(\frac{x-y}{2} - \alpha xy\Big)\Big]\Big[f\Big(\frac{x+y}{2} - \beta xy\Big) + f\Big(\frac{x-y}{2} + \beta xy\Big)\Big] = 2f(x)$$

for all $x, y \in \mathbb{R}$. Interchanging x with y in equation (2.22), we obtain

$$f\Big(\frac{x+y}{2} + \alpha xy\Big) + f\Big(\frac{y-x}{2} + \beta xy\Big) = f(y) + f\big((\alpha+\beta)xy\big) \tag{2.24}$$

for all $x, y \in \mathbb{R}$. Replacing y by $-y$ in equation (2.24), and then adding the resulting equation to equation (2.24), we have

$$\Big[f\Big(\frac{x+y}{2} + \alpha xy\Big) + f\Big(\frac{x-y}{2} - \alpha xy\Big)\Big] = \Big[f\Big(\frac{x+y}{2} + \beta xy\Big) + f\Big(\frac{x-y}{2} - \beta xy\Big)\Big]$$

for all $x, y \in \mathbb{R}$.

By the same method as in the proof of Theorem 2.4, one can complete the proof. □

3 Stability of the functional equation (1.1)

In this section, we investigate the Hyers-Ulam stability problem for the functional equation (1.1). We assume that X is a Banach space.

Theorem 3.1. *Let $\varepsilon \geqslant 0$ be fixed and let $f : \mathbb{R} \to X$ be a mapping satisfying*

$$\Big\|f\Big(\frac{x+y}{2} + xy\Big) + f\Big(\frac{x-y}{2} - xy\Big) - f(x)\Big\| \leqslant \varepsilon \tag{3.1}$$

for all $x, y \in \mathbb{R}$. Then there exists a unique additive mapping $A : \mathbb{R} \to X$ satisfying

$$\|f(x) - A(x)\| \leqslant 5\varepsilon \tag{3.2}$$

for all $x, y \in \mathbb{R}$.

Proof. Letting $x = y = 0$ in inequality (3.1), we get $\|f(0)\| \leqslant \varepsilon$. Putting $x = 0$ and replacing y by $2y$ in inequality (3.1), we have

$$\|f(y) + f(-y) - f(0)\| \leqslant \varepsilon, \quad y \in \mathbb{R}. \tag{3.3}$$

Setting $x = \frac{1}{2}$ and replacing y by $-y - \frac{1}{4}$ in inequality (3.1), we obtain

$$\Big\|f(-y) + f\Big(y + \frac{1}{2}\Big) - f\Big(\frac{1}{2}\Big)\Big\| \leqslant \varepsilon, \quad y \in \mathbb{R}. \tag{3.4}$$

It follows from inequalities (3.3) and (3.4) that

$$\Big\|f(y) + f\Big(-y - \frac{1}{2}\Big) - f\Big(-\frac{1}{2}\Big)\Big\| \leqslant 5\varepsilon, \quad y \in \mathbb{R}. \tag{3.5}$$

Replacing x and y by $x+y$ and $\frac{x-y}{1+2x+2y}$ in inequality (3.1), respectively, we get

$$\|f(x)+f(y)-f(x+y)\| \leqslant \varepsilon, \quad x,y \in \mathbb{R},\ 2x+2y \neq -1. \tag{3.6}$$

It follows from inequalities (3.5) and (3.6) that

$$\|f(x)+f(y)-f(x+y)\| \leqslant 5\varepsilon, \quad x,y \in \mathbb{R}.$$

By the Hyers' theorem, the $\lim_{n\to\infty} 2^{-n}f(2^n x)$ exists for each $x \in \mathbb{R}$ and the mapping $A : \mathbb{R} \to X$ given by $A(x) := \lim_{n\to\infty} 2^{-n}f(2^n x)$ is the unique additive mapping satisfying inequality (3.2). □

Theorem 3.2. *Let $\varepsilon \geqslant 0$, $0 < p < 1$, $q > 0$ be fixed and let $f : \mathbb{R} \to X$ be a mapping satisfying*

$$\left\|f\Big(\frac{x+y}{2}+xy\Big)+f\Big(\frac{x-y}{2}-xy\Big)-f(x)\right\| \leqslant \varepsilon(|x|^p+|y|^q) \tag{3.7}$$

for all $x,y \in \mathbb{R}$. Then there exists a unique additive mapping $A : \mathbb{R} \to X$ satisfying

$$\|f(x)-A(x)\| \leqslant \begin{cases} \dfrac{2^p}{2-2^p}\varepsilon|x|^p, & \text{if} \quad x \in \mathbb{R}\setminus S \\ \theta(x), & \text{if} \quad x \in S \end{cases} \tag{3.8}$$

for all $x \in \mathbb{R}$, where $S = \left\{-\frac{1}{2^{m+2}} : \ m \in \mathbb{N}\cup\{0\}\right\}$ and

$$\theta(x) = \left[\frac{2^p}{2-2^p} - \frac{2^{(m+1)p}}{2^{(m+1)}}\right]\varepsilon|x|^p + \frac{1+\frac{2}{2^q}+\frac{1}{2^p}}{2^{m+1}}\varepsilon,$$

when $2^{m+2}x = -1$ for some integer $m \geqslant 0$.

Proof. Letting $x = y = 0$ in inequality (3.7), we get $f(0) = 0$. Setting $x = \frac{1}{2}$ and $y = 0$, we obtain

$$\left\|f\Big(\frac{1}{2}\Big)-2f\Big(\frac{1}{4}\Big)\right\| \leqslant \frac{\varepsilon}{2^p}. \tag{3.9}$$

Putting $x = 0$ and replacing y by $2y$ in inequality (3.7), we have

$$\|f(y)+f(-y)\| \leqslant 2^q\varepsilon|y|^q, \quad y \in \mathbb{R}. \tag{3.10}$$

Using inequalities (3.9) and (3.10), we obtain

$$\left\|f\Big(-\frac{1}{2}\Big)-2f\Big(-\frac{1}{4}\Big)\right\| \leqslant \left[1+\frac{2}{2^q}+\frac{1}{2^p}\right]\varepsilon. \tag{3.11}$$

Replacing x and y by $x+y$ and $\frac{x-y}{1+2x+2y}$ in inequality (3.7), respectively, we get

$$\|f(x)+f(y)-f(x+y)\| \leqslant \varepsilon\left[|x+y|^p + \left|\frac{x-y}{1+2x+2y}\right|^q\right] \tag{3.12}$$

for all $x,y \in \mathbb{R}$ with $2x+2y \neq -1$. Letting $y = x$ in inequality (3.12)

$$\|f(2x)-2f(x)\| \leqslant 2^p\varepsilon|x|^p, \quad x \in \mathbb{R},\ 4x \neq -1.$$

For $x \in \mathbb{R}$, there exists $m \geqslant 1$ such that $2^{n+2}x \neq -1$ for all $n \geqslant m$. Therefore

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}\right\| \leqslant \Big(\frac{2^p}{2}\Big)^{n+1}\varepsilon|x|^p, \quad n \geqslant m. \tag{3.13}$$

Hence the sequence $\{2^{-n}f(2^n x)\}$ is Cauchy. Let $x \in \mathbb{R}$ such that $2^{m+2}x \neq -1$ for all integers $m \geqslant 0$. Then inequality (3.13) implies that

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}}-f(x)\right\| \leqslant \varepsilon\sum_{k=0}^{n}\Big(\frac{2^p}{2}\Big)^{k+1}|x|^p. \tag{3.14}$$

If $x \in \mathbb{R}$ such that $2^{m+2}x = -1$ for some integer $m \geqslant 0$, then inequalities (3.11) and (3.13) imply that

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - f(x)\right\| \leqslant \varepsilon \sum_{\substack{k=0 \\ k\neq m}}^{n} \left(\frac{2^p}{2}\right)^{k+1} |x|^p + \frac{1 + \frac{2}{2^q} + \frac{1}{2^p}}{2^{m+1}} \varepsilon. \tag{3.15}$$

Letting $n \to \infty$ in inequalities (3.14) and (3.15), we get inequality (3.8). □

One can obtain a similar result for the case $p > 1$. The proof can be achieved similarly as in that of Theorem 3.2. In the following proposition, by using Gajda's function (see [17]), we show that Theorem 3.2 is false for $p = 1$.

Proposition 3.3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } |x| \geqslant 1. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x).$$

Then f satisfies

$$\left|f\left(\frac{x+y}{2} + xy\right) + f\left(\frac{x-y}{2} - xy\right) - f(x)\right| \leqslant 12(|x| + |y|) \tag{3.16}$$

for all $x, y \in \mathbb{R}$, and the range of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive function $A : \mathbb{R} \to \mathbb{R}$.

Proof. It is clear that f is bounded by 2 on $\mathbb{R}$. If $|x| + |y| = 0$ or $|x| + |y| \geqslant \frac{1}{2}$, then

$$\left|f\left(\frac{x+y}{2} + xy\right) + f\left(\frac{x-y}{2} - xy\right) - f(x)\right| \leqslant 6 \leqslant 12(|x| + |y|).$$

Now suppose that $0 < |x| + |y| < \frac{1}{2}$. Then there exists an integer $k \geqslant 1$ such that

$$\frac{1}{2^{k+1}} \leqslant |x| + |y| < \frac{1}{2^k}. \tag{3.17}$$

Therefore

$$2^m\left|\frac{x+y}{2} + xy\right|, 2^m\left|\frac{x-y}{2} - xy\right|, 2^m|x| < 1$$

for all $m = 0, 1, \ldots, k-1$. From the definition of f and inequality (3.17), we have

$$\begin{aligned}\left|f\left(\frac{x+y}{2} + xy\right) + f\left(\frac{x-y}{2} - xy\right) - f(x)\right| &\leqslant \sum_{n=k}^{\infty} 2^{-n}\left[\left|\phi\left(2^n\left(\frac{x+y}{2} + xy\right)\right)\right| + \left|\phi\left(2^n\left(\frac{x-y}{2} - xy\right)\right)\right| + |\phi(2^n(x))|\right] \\ &\leqslant \frac{6}{2^k} \leqslant 12(|x| + |y|).\end{aligned}$$

Therefore f satisfies inequality (3.16). To complete the proof, we assume that there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - A(x)| \leqslant \beta|x|, \quad x \in \mathbb{R}.$$

Since A is additive, there exists a constant $c \in \mathbb{R}$ such that $A(x) = cx$ for all rational numbers x. Then we have

$$|f(x)| \leqslant (\beta + |c|)|x| \tag{3.18}$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > \beta + |c|$ and let x be a rational number in $(0, 2^{1-m})$. Then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. So

$$f(x) \geqslant \sum_{n=0}^{m-1} 2^{-n}\phi(2^n x) = mx > (\beta + |c|)x,$$

which contradicts inequality (3.18). □

Theorem 3.4. *Let $\varepsilon \geqslant 0$, $p, q > 0$ and let $f : \mathbb{R} \to X$ be a mapping satisfying*

$$\left\| f\left(\frac{x+y}{2} + xy\right) + f\left(\frac{x-y}{2} - xy\right) - f(x) \right\| \leqslant \varepsilon |x|^p |y|^q \tag{3.19}$$

for all $x, y \in \mathbb{R}$. Then f is additive.

Proof. Letting $x = y = 0$ in inequality (3.19), we get $f(0) = 0$. Letting $x = 0$ in inequality (3.19), we obtain that f is odd. Setting $y = 0$ in inequality (3.19), we get that $f(2x) = 2f(x)$, and so $f(2^n x) = 2^n f(x)$ for all $x \in \mathbb{R}$ and all positive integers n. Replacing x and y by $x + y$ and $\frac{x-y}{1+2x+2y}$ in inequality (3.19), we get

$$\|f(x) + f(y) - f(x+y)\| \leqslant \varepsilon |x+y|^p \left| \frac{x-y}{1+2x+2y} \right|^q \tag{3.20}$$

for all $x, y \in \mathbb{R}$ with $2x + 2y \neq -1$. We may assume that $p < 1$ (for the case $p \geqslant 1$ we have a similar proof). For $x, y \in \mathbb{R}$, there exists $m \geqslant 1$ such that $2^{n+1}(x+y) \neq -1$ for all $n \geqslant m$. Replacing x an y by $2^n x$ and $2^n y$ in inequality (3.20), respectively, and using $f(2^n t) = 2^n f(t)$ for $t = x, y, x+y$, we get

$$\|f(x) + f(y) - f(x+y)\| \leqslant \varepsilon \left(\frac{2^p}{2}\right)^n \frac{|x+y|^p |x-y|^q}{|2^{-n} + 2x + 2y|^q} \tag{3.21}$$

for all $n \geqslant m$. Taking the limit as $n \to \infty$ in inequality (3.21), we obtain $f(x+y) = f(x) + f(y)$. □

4 Stability of the functional equation (1.1) in topological vector spaces

In this section, E is a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers.

Theorem 4.1. *Let V be a nonempty bounded convex subset of E containing the origin. Suppose that $f : \mathbb{R} \to E$ satisfies*

$$f\left(\frac{x+y}{2} + xy\right) + f\left(\frac{x-y}{2} - xy\right) - f(x) \in V \tag{4.1}$$

for all $x, y \in \mathbb{R}$. Then there exists a unique additive mapping $A : \mathbb{R} \to E$ such that

$$f(x) - A(x) \in \overline{4V - V} \tag{4.2}$$

for all $x \in \mathbb{R}$, where $\overline{4V - V}$ denotes the sequential closure of $4V - V$.

Proof. Letting $x = y = 0$ in (4.1), we get $f(0) \in V$. Putting $x = 0$ and replacing y by $2y$ in (4.1), we have

$$f(y) + f(-y) - f(0) \in V, \quad y \in \mathbb{R}. \tag{4.3}$$

Setting $x = \frac{1}{2}$ and replacing y by $-y - \frac{1}{4}$ in (4.1), we obtain

$$f(-y) + f\left(y + \frac{1}{2}\right) - f\left(\frac{1}{2}\right) \in V, \quad y \in \mathbb{R}. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$f(y) + f\left(-y - \frac{1}{2}\right) - f\left(-\frac{1}{2}\right) \in 4V - V, \quad y \in \mathbb{R}. \tag{4.5}$$

Replacing x and y by $x + y$ and $\frac{x-y}{1+2x+2y}$ in (4.1), we get

$$f(x) + f(y) - f(x+y) \in V, \quad x, y \in \mathbb{R},\ 2x + 2y \neq -1. \tag{4.6}$$

Since $V \subseteq 4V - V$, it follows from (4.5) and (4.6) that

$$f(x) + f(y) - f(x+y) \in 4V - V, \quad x, y \in \mathbb{R}. \tag{4.7}$$

It is easy to prove that

$$\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}} \in \frac{1}{2^{n+1}} W \subseteq W, \tag{4.8}$$

$$f(x) - \frac{f(2^n x)}{2^n} \in \sum_{k=1}^{n} \frac{1}{2^k} W \subseteq W \tag{4.9}$$

for all $x \in \mathbb{R}$ and all integers $n \geqslant 1$, where $W = 4V - V$. Since V is a nonempty bounded convex subset of E containing the origin, W is a nonempty bounded convex subset of E containing the origin. It follows from (4.8) that

$$\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} = \sum_{k=n}^{m-1} \left[\frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}} \right] \in \sum_{k=n}^{m-1} \frac{1}{2^{k+1}} W \subseteq \frac{1}{2^n} W \tag{4.10}$$

for all $x \in \mathbb{R}$ and all integers $m > n \geqslant 0$. Let U be an arbitrary neighborhood of the origin in E. Since W is bounded, there exists a rational number $t > 0$ such that $tW \subseteq U$. Choose $n_0 \in \mathbb{N}$ such that $2^{n_0} t > 1$. Let $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$ with $m \geqslant n \geqslant n_0$. Then (4.10) implies that

$$\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \in U.$$

Thus the sequence $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence in E. By the sequential completeness of E, the limit $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in \mathbb{R}$. So (4.2) follows from (4.9) by letting $n \to \infty$.

To show that $A : \mathbb{R} \to E$ is additive, replacing x and y by $2^n x$ and $2^n y$, respectively, in (4.7) and then dividing by 2^n, we obtain

$$\frac{f(2^n x)}{2^n} + \frac{f(2^n y)}{2^n} - \frac{f(2^n (x+y))}{2^n} \in \frac{1}{2^n} W$$

for all $x, y \in \mathbb{R}$ and all integers $n \geqslant 0$. Since W is bounded, on taking the limit as $n \to \infty$, we get that A is additive.

To prove the uniqueness of A, assume on the contrary that there is another additive mapping $T : \mathbb{R} \to E$ satisfying (4.2) and there is an $a \in \mathbb{R}$ such that $x = T(a) - A(a) \neq 0$. So there is a neighborhood U of the origin in E such that $x \notin U$, since E is Hausdorff. Since A and T satisfy (4.2), we get $T(b) - A(b) \in \overline{W} - \overline{W}$ for all $b \in \mathbb{R}$. Since W is bounded, $\overline{W} - \overline{W}$ is bounded. Hence there exists a positive integer m such that $\overline{W} - \overline{W} \subseteq mU$. Therefore $mx = T(ma) - A(ma) \in mU$ which is a contradiction to $x \notin U$. This completes the proof. □

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