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Stability and hyperstability of a quadratic functional equation and a characterization of inner product spaces

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Abstract: We have proved the Hyers-Ulam stability and the hyperstability of the quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z)=4[f(x)+f(y)+f(z)]$$

in the class of functions from an abelian group G into a Banach space.

Keywords: Hyers-Ulam stability, hyperstability, quadratic functional equation, fixed point theorem

MS: 39B82, 39B52, 47H14, 47H10

1 Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms:

Let $(G, \cdot)$ be a group and let $(G', \cdot, d)$ be a metric group with the metric d. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a mapping $h : G \to G'$ satisfies the inequality

$$d(h(x \cdot y), h(x) \cdot h(y)) \le \delta$$

for all $x, y \in G$, then there is a homomorphism $H : G \to G'$ with

$$d(h(x), H(x)) \le \epsilon$$

for all $x \in G$?

Ulam's problem was partially solved by Hyers [2] in 1941.

Theorem 1. [2] *Let E be a normed vector space, F a Banach space and suppose that the mapping $f : E \to F$ satisfies the inequality*

$$\left\|f(x+y)-f(x)-f(y)\right\| \le \epsilon$$

for all $x, y \in E$, where ϵ is a constant. Then the limit

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

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exists for each $x \in E$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in E$. Also, if for each x the function $t \to f(tx)$ from $\mathbb{R}$ to F is continuous for each fixed x, then T is linear. If f is continuous at a single point of E, then T is continuous in E.

Bourgin [3], Aoki [4], Rassias [5] and Gajda [6] treated this problem for approximate additive mappings controlled by unbounded functions.

Theorem 2. *Let $f : E \to F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E \setminus \{0\}$, where θ and p are constants with $\theta > 0$ and $p \neq 1$. Then there exists a unique additive mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \le \frac{\theta}{|1 - 2^{p-1}|}\|x\|^p \tag{1.2}$$

for all $x \in E \setminus \{0\}$.

Theorem 2 is due to Aoki [4] for $0 < p < 1$ (see also [5]); Gajda [6] for $p > 1$; Hyers [2] for $p = 0$ and Rassias [7] for $p < 0$ (see [3, 8]).

In 1994, Găvruta [9] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions, i.e., he replaced $\theta(\|x\|^p + \|y\|^p)$ with a general control function $\varphi(x, y)$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10–13]).

Recently, interesting results concerning quadratic functional equation

$$f(x+y+z) + f(x+y-z) + f(x-y+z) + f(-x+y+z) = 4[f(x) + f(y) + f(z)] \tag{1.3}$$

have been obtained in [14, 15].

Lemma 1. [14] *Let X and Y be vector spaces over fields of characteristic different from 2, respectively. A mapping $f : X \to Y$ satisfies* (1.3) *if and only if the mapping $f : X \to Y$ is a solution of the quadratic equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.*

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ is approximately a true solution of $\mathfrak{D}$. The term hyperstability was used for the first time probably in [16]. However, it seems that the first hyperstability result was published in [3] and concerned the ring homomorphisms. The hyperstability results for Cauchy equation were investigated by Brzdęk [17–19]. Gselmann [20] studied the hyperstability of the parametric fundamental equation of information. In [21], Bahyrycz and Piszczek provided the hyperstability of the Jensen functional equation. For more information on hyperstability of functional equations, see [22].

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}^\star$ be the set of positive integers. We denote that $\mathbb{N}_{m_0}$ (with $m_0 \in \mathbb{N}^\star$) the set of all integers greater than or equal to m_0. Let $\mathbb{R}_+ := [0, \infty)$ be the set of nonnegative real numbers and Y^X denote the family of all functions mapping from a nonempty set X into a nonempty set Y.

In this paper, we present the stability and hyperstability results for the quadratic functional equation (1.3) in the class of functions from a commutative group $(G, +)$ into a Banach space E.

The method of the proof of the main results is motivated by an idea used in [17–19, 23, 24]. It is based on a fixed point theorem for functional spaces obtained by Brzdęk et al. (see [25, Theorem 1]).

First, we take the following three hypotheses (all notations come from [25]):
(H1) U is a nonempty set, V is a Banach space, $f_1, \ldots, f_k : U \to U$ and $L_1, \ldots, L_k : U \to \mathbb{R}_+$ are given.
(H2) $\mathcal{T} : V^U \to V^U$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\right\| \le \sum_{i=1}^{k} L_i(x) \left\|\xi(f_i(x)) - \mu(f_i(x))\right\|$$

for all $\xi, \mu \in V^U$, $x \in U$.
(H3) $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x))$$

for all $\delta \in \mathbb{R}_+^U$, $x \in U$.

The mentioned fixed point theorem is stated as follows.

Theorem 3. *Let* **(H1)–(H3)** *be valid and functions $\varepsilon : U \to \mathbb{R}_+$ and let $\varphi : U \to V$ fulfil the following two conditions:*

$$\left\|\mathcal{T}\varphi(x) - \varphi(x)\right\| \le \varepsilon(x), \quad x \in U,$$

$$\varepsilon^\star(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in U.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\left\|\varphi(x) - \psi(x)\right\| \le \varepsilon^\star(x), \quad x \in U.$$

Moreover

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \quad x \in U.$$

2 Main results

The following theorems are the main results in this paper and concern the stability of the functional equation (1.3).

Theorem 4. *Let $(G, +)$ be an abelian group and E be a Banach space. Let $f : G \to E$, $\varphi : G^3 \to [0, \infty)$ and $u : \mathbb{Z}^\star = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions satisfying the following three conditions*

$$\mathcal{M} := \{m \in \mathbb{Z}^\star : 2u(2m+1) + 8u(-m) + u(-4m-1) < 1\} \neq \emptyset, \tag{2.1}$$

$$\varphi(tx, ty, tz) \le u(t)\varphi(x, y, z) \tag{2.2}$$

and

$$\|f(x+y+z) + f(x+y-z) + f(x-y+z) + f(-x+y+z) - 4[f(x) + f(y) + f(z)]\| \le \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in G$, $t \in \{2m+1, -m, -4m-1\}$ and $m \in \mathcal{M}$. Then there exists a unique mapping $Q : G \to E$ satisfying (1.3) *and*

$$\left\|f(x) - Q(x)\right\| \le \phi(x), \tag{2.4}$$

where

$$\phi(x) := \inf\left\{\frac{\varphi((2m+1)x, -mx, -mx)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} : m \in \mathcal{M}\right\}$$

for all $x \in G$.

Proof. Replacing (x, y, z) by $((2m+1)x, -mx, -mx)$ in (2.3), we get

$$\left\|2f((2m+1)x) + 8f(-mx) - f((-4m-1)x) - f(x)\right\| \le \varphi((2m+1)x, -mx, -mx) := \varepsilon_m(x) \tag{2.5}$$

for all $x \in G$ and $m \in \mathbb{Z}^\star$. Further put

$$\mathcal{T}\xi(x) := 2\xi((2m+1)x) + 8\xi(-mx) - \xi((-4m-1)x), \quad x \in G, \; \xi \in E^G, \; m \in \mathbb{Z}^\star.$$

Then the inequality (2.5) takes the form

$$\left\|\mathcal{T}f(x) - f(x)\right\| \le \varepsilon_m(x), \quad x \in G.$$

Now, we define an operator $\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G$ for $m \in \mathbb{Z}^\star$ by

$$\Lambda\delta(x) := 2\delta((2m+1)x) + 8\delta(-mx) + \delta((-4m-1)x), \; x \in G, \quad \delta \in \mathbb{R}_+^G. \tag{2.6}$$

This operator has the form described in (**H3**) with $k = 4$ and $f_1(x) = (2m+1)x$, $f_2(x) = -mx$, $f_3(x) = (-4m-1)x$, $L_1(x) = 2$, $L_2(x) = 8$ and $L_3(x) = 1$ for $x \in G$.
Moreover, for every $\xi, \mu \in E^G$ and $x \in G$, we obtain

$$\begin{aligned}\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| &= \|2(\xi-\mu)(f_1(x)) + 8(\xi-\mu)(f_2(x)) - (\xi-\mu)(f_3(x))\| \\ &\le 2\left\|(\xi-\mu)(f_1(x))\right\| + 8\left\|(\xi-\mu)(f_2(x))\right\| + \|(\xi-\mu)(f_3(x))\| \\ &= \sum_{i=1}^{4} L_i(x)\left\|(\xi-\mu)(f_i(x))\right\|,\end{aligned}$$

where $(\xi-\mu)(y) = \xi(y) - \mu(y)$ for all $y \in G$. So **(H2)** is valid. It is easy to check that

$$\begin{aligned}\Lambda\varepsilon_k(x) &= 2\varepsilon_k((2m+1)x) + 8\varepsilon_k(-mx) + \varepsilon_k((-4m-1)x) \\ &\le 2u(2m+1)\varepsilon_k(x) + 8u(-m)\varepsilon_k(x) + u(-4m-1)\varepsilon_k(x) \\ &= [2u(2m+1) + 8u(-m) + u(-4m-1)]\varepsilon_k(x)\end{aligned} \tag{2.7}$$

for all $x \in G$, $k \in \mathbb{Z}^\star$ and $m \in \mathcal{M}$. Therefore, since the operator Λ is linear, we have

$$\begin{aligned}\varepsilon^\star(x) :&= \sum_{n=0}^{\infty} \Lambda^n \varepsilon_m(x) \\ &\le \sum_{n=0}^{\infty} (2u(2m+1) + 8u(-m) + u(-4m-1))^n \varepsilon_m(x) \\ &= \frac{\varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} < \infty\end{aligned} \tag{2.8}$$

for all $x \in G$ and $m \in \mathcal{M}$. Thus, according to Theorem 3, for each $m \in \mathcal{M}$ there exists a unique mapping $Q_m : G \to E$ such that

$$Q_m(x) = 2Q_m((2m+1)x) + 8Q_m(-mx) - Q_m((-4m-1)x), \quad x \in G$$

$$\left\|f(x) - Q_m(x)\right\| \le \frac{\varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} \tag{2.9}$$

for all $x \in G$ and $m \in \mathcal{M}$. Moreover

$$Q_m(x) = \lim_{n\to\infty} \mathcal{T}^n f(x), \quad x \in G, \quad m \in \mathcal{M}. \tag{2.10}$$

Next, we show that

$$\|\mathcal{T}^n f(x+y+z) + \mathcal{T}^n f(x+y-z) + \mathcal{T}^n f(x-y+z) + \mathcal{T}^n f(y-x+z) - 4\mathcal{T}^n f(x)$$

$$- 4\mathcal{T}^n f(y) - 4\mathcal{T}^n f(z)\| \le (2u(2m+1) + 8u(-m) + u(-4m-1))^n \varphi(x,y,z) \tag{2.11}$$

Fix $m \in \mathcal{M}$. Indeed, if $n = 0$, then (2.11) is simply (2.3). So, fix $n \in \mathbb{N}$ and suppose that (2.11) holds for n. Then

$$\begin{aligned}
&\|\mathcal{T}^{n+1} f(x+y+z) + \mathcal{T}^{n+1} f(x+y-z) + \mathcal{T}^{n+1} f(x-y+z) + \mathcal{T}^{n+1} f(y-x+z) \\
&- 4\mathcal{T}^{n+1} f(x) - 4\mathcal{T}^{n+1} f(y) - 4\mathcal{T}^{n+1} f(z)\| \\
&= \|2\mathcal{T}^n f((2m+1)(x+y+z)) + 8\mathcal{T}^n f(-m(x+y+z)) \\
&- \mathcal{T}^n f((-4m-1)(x+y+z)) + 2\mathcal{T}^n f((2m+1)(x+y-z)) \\
&+ 8\mathcal{T}^n f(-m(x+y-z)) - \mathcal{T}^n f((-4m-1)(x+y-z)) \\
&+ 2\mathcal{T}^n f((2m+1)x-y+z) + 8\mathcal{T}^n f(-m(x-y+z)) \\
&- \mathcal{T}^n f((-4m-1)x-y+z) + 2\mathcal{T}^n f((2m+1)(-x+y+z)) \\
&+ 8\mathcal{T}^n f(-m(-x+y+z)) - \mathcal{T}^n f((-4m-1)(-x+y+z)) \\
&- 4[2\mathcal{T}^n f((2m+1)x) + 8\mathcal{T}^n f(-mx) - \mathcal{T}^n f((-4m-1)x)] \\
&- 4[2\mathcal{T}^n f((2m+1)y) + 8\mathcal{T}^n f(-my) - \mathcal{T}^n f((-4m-1)y)] \\
&- 4[2\mathcal{T}^n f((2m+1)z) + 8\mathcal{T}^n f(-mz) - \mathcal{T}^n f((-4m-1)z)]\| \\
&\le 2(2u(2m+1) + 8u(-m) + u(-4m-1))^n \varphi((2m+1)x, (2m+1)y, (2m+1)z) \\
&+ 8(2u(2m+1) + 8u(-m) + u(-4m-1))^n \varphi(-mx, -my, -mz) \\
&+ (2u(2m+1) + 8u(-m) + u(-4m-1))^n \varphi((-4m-1)x, (-4m-1)y, (-4m-1)z) \\
&\le (2u(2m+1) + 8u(-m) + u(-4m-1))^{n+1} \varphi(x,y,z)
\end{aligned}$$

for all $x, y, z \in G$.
Thus, by induction, we obtain that (2.11) holds for all $x, y, z \in G$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.11), we obtain that

$$Q_m(x+y+z) + Q_m(x+y-z) + Q_m(x-y+z) + Q_m(-x+y+z) = 4[Q_m(x) + Q_m(y) + Q_m(z)] \tag{2.12}$$

for all $x, y, z \in G$ and $m \in \mathcal{M}$ such that

$$\|f(x) - Q_m(x)\| \le \frac{\varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)}.$$

Now, we prove that $Q_m = Q_k$ for all $m, k \in \mathcal{M}$. Let us fix $m, k \in \mathcal{M}$ and note that Q_k satisfie (2.9) with m replaced by k. Hence, by replacing (x, y, z) by $((2m+1)x, -mx, -mx)$ in (2.12), we get $\mathcal{T} Q_j = Q_j$ for $j = m, k$ and

$$\|Q_m(x) - Q_k(x)\| \le \frac{\varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} + \frac{\varepsilon_k(x)}{1 - 2u(2k+1) - 8u(-k) - u(-4k-1)}$$

for all $x \in G$. It follows from the linearity of Λ and (2.7) that

$$\begin{aligned}
\|Q_m(x) - Q_k(x)\| &= \|\mathcal{T}^n Q_m(x) - \mathcal{T}^n Q_k(x)\| \\
&\le \frac{\Lambda^n \varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} + \frac{\Lambda^n \varepsilon_k(x)}{1 - 2u(2k+1) - 8u(-k) - u(-4k-1)} \\
&\le \big(2u(2m+1) + 8u(-m) + u(-4m-1)\big)^n A_m(x) + \big(2u(2k+1) + 8u(-k) + u(-4k-1)\big)^n A_k(x),
\end{aligned}$$

where

$$A_m(x) := \frac{\varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we get $Q_m = Q_k =: Q$. Thus we have

$$\|f(x) - Q(x)\| \le A_m(x), \quad x \in G, \quad m \in \mathcal{M}$$

and so we obtain (2.4).

In view of 2.12, it is easy to notice that Q is a solution of (1.3).

To prove the uniqueness of the mapping Q, let us assume that there exists a mapping $Q' : G \to E$ which satisfies (1.3) and the inequality

$$\left\|f(x) - Q'(x)\right\| \leq \phi(x), \quad x \in G.$$

Then

$$\left\|Q(x) - Q'(x)\right\| \leq 2\phi(x), \quad x \in G.$$

Further $\mathcal{T}Q'(x) = Q'(x)$ for all $x \in G$. Consequently, with a fixed $m \in \mathcal{M}$

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \|\mathcal{T}^n Q(x) - \mathcal{T}^n Q'(x)\| \\ &\leq 2\Lambda^n \phi(x) \\ &\leq \frac{2\Lambda^n \varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} \\ &\leq \frac{2[2u(2m+1) + 8u(-m) + u(-4m-1)]^n \varepsilon_m(x)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)}\end{aligned}$$

for all $x \in G$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we get $Q = Q'$. The proof of the theorem is complete. □

In a similar way, we can prove Theorem 5 if the inequality (2.3) is defined on $G\backslash\{0\} := G_0$.

Theorem 5. *Let $(G, +)$ be an abelian group and E be a Banach space. Let $f : G \to E$, $\varphi : G_0^3 \to [0, \infty)$ and $u : \mathbb{Z}^\star = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions satisfying the following three conditions*

$$\mathcal{M} := \{m \in \mathbb{Z}^\star : 2u(2m+1) + 8u(-m) + u(-4m-1) < 1\} \neq \emptyset, \tag{2.13}$$

$$\varphi(tx, ty, tz) \leq u(t)\varphi(x, y, z) \tag{2.14}$$

and

$$\|f(x+y+z) + f(x+y-z) + f(x-y+z) + f(-x+y+z) - 4[f(x) + f(y) + f(z)]\| \leq \varphi(x, y, z) \tag{2.15}$$

for all $x, y, z \in G_0$, $t \in \{2m+1, -m, -4m-1\}$ and $m \in \mathcal{M}$ with $f(0) = 0$. Then there exists a unique mapping $Q : G \to E$ satisfying (1.3) and

$$\left\|f(x) - Q(x)\right\| \leq \phi(x), \tag{2.16}$$

where

$$\phi(x) := \inf\left\{\frac{\varphi\big((2m+1)x, -mx, -mx\big)}{1 - 2u(2m+1) - 8u(-m) - u(-4m-1)} : m \in \mathcal{M}\right\}$$

for all $x \in G_0$.

Corollary 1. *Let $(G, +)$ be a commutative group and E be a Banach space. Let $f : G \to E$, $\varphi : G^3 \to [0, \infty)$ and $u : \mathbb{Z}^\star = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ be functions and the conditions (2.1), (2.2) and (2.3) be valid. Assume that*

$$\inf\{\varphi((2m+1)x, -mx, -mx) : m \in \mathcal{M}\} = 0, \tag{2.17}$$

$$\lim_{m\to\infty} (2u(2m+1) + 8u(-m) + u(-4m-1)) = 0 \tag{2.18}$$

for all $x \in G$. Then f satisfies (1.3) on G.

Proof. Suppose that

$$\inf\{\varphi((2m+1)x, -mx, -mx) : m \in \mathcal{M}\} = 0$$

for all $x \in G$. Hence from Theorem 4 we have $\phi(x) = 0$ for all $x \in G$. Then f satisfies (1.3) on G. □

Remark 1. *In Theorem 4, if*

$$\liminf_{m\to\infty}(2u(2m+1)+8u(-m)+u(-4m-1))=0$$

(*this is the case when, e.g.,* $\lim_{|m|\to\infty} u(m)=0$), *then* (2.1) *holds and*

$$\phi(x)=\inf_{m\in\mathcal{M}}\varphi((2m+1)x,-mx,-mx)$$

for all $x\in G$.

Now we give some applications of Theorem 5 to some cases:

$$\varphi_1(x,y,z)=\theta\|x\|^p\cdot\|y\|^q\cdot\|z\|^r,\quad p+q+r<0$$

and

$$\varphi_2(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p),\quad p<0,$$

where $\varphi(x,y,z)=\varphi_j(x,y,z)$ for $j\in\{1,2\}$, $\theta\in\mathbb{R}_+$, $p,q,r\in\mathbb{R}$ and $x,y,z\neq 0$.

Corollary 2. *Let* E_1 *and* E_2 *be a normed space and a Banach space, respectively. Assume* $S:=(S,+)$ *is a subgroup of the group* $(E_1,+)$, $p,q,r\in\mathbb{R}$, $p+q+r<0$ *and* $\theta\geq 0$. *If* $f(0)=0$ *and* $f:S\to E_2$ *satisfies*

$$\|f(x+y+z)+f(x+y-z)+f(x-y+z)+f(y-x+z)-4[f(x)+f(y)+f(z)]\|\leq\theta\|x\|^p\|y\|^q\|z\|^r$$

for all $x,y,z\in S\setminus\{0\}$, *then* f *is a solution of* (1.3) *on* S.

Proof. Let $\varphi_1(x,y,z)=\theta\|x\|^p\cdot\|y\|^q\cdot\|z\|^r$ and $u(t)=|t|^{p+q+r}$ in Theorem 5 where $p,q,r\in\mathbb{R}$, $p+q+r<0$ and $t\in\mathbb{Z}^\star$. Then we observe that condition (2.14) is valid. Obviously, (2.17) and (2.18) hold and there exists $m_0\in\mathbb{N}^\star$ such that

$$2|2m+1|^{p+q+r}+8|m|^{p+q+r}+|4m+1|^{p+q+r}<1,\quad m\geq m_0.$$

So we obtain (2.13), as well. Consequently, by Corollary 1, every mapping $f:S\to E_2$ fulfilling (2.15) satisfies (1.3) on S. □

Corollary 3. *Let* E_1 *and* E_2 *be a normed space and a Banach space, respectively. Assume* $S:=(S,+)$ *is a subgroup of the group* $(E_1,+)$, $p\in\mathbb{R}$, $p<0$ *and* $\theta\geq 0$. *If* $f(0)=0$ *and* $f:S\to E_2$ *satisfies*

$$\|f(x+y+z)+f(x+y-z)+f(x-y+z)+f(y-x+z)-4[f(x)+f(y)+f(z)]\|\leq\theta(\|x\|^p+\|y\|^p+\|z\|^p)$$

for all $x,y,z\in S\setminus\{0\}$, *then* f *is a solution of* (1.3) *on* S.

Proof. Let $\varphi_2(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ and $u(t)=|t|^p$ in Theorem 5 where $p\in\mathbb{R}$, $p<0$ and $t\in\mathbb{Z}^\star$. Then we observe that condition (2.14) is valid. Obviously, (2.17) and (2.18) hold and there exists $m_0\in\mathbb{N}^\star$ such that

$$2|2m+1|^p+8|m|^p+|4m+1|^p<1,\quad m\geq m_0.$$

So we obtain (2.13), as well. Consequently, by Corollary 1, every mapping $f:S\to E_2$ fulfilling (2.15) satisfies (1.3) on S. □

We know that any norm that satisfies the parallelogram law is bound to have been originated from a scalar product. The following corollary gives a characterization of the inner product space, which is one of the applications of Corollaries 2 and 3.

Corollary 4. *Let* X *be a normed space and* $X_0=X\setminus\{0\}$. *Write*

$$\Delta(x,y,z)=\Big|\|x+y+z\|^2+\|x+y-z\|^2+\|x-y+z\|^2+\|y-x+z\|^2-4[\|x\|^2+\|y\|^2+\|z\|^2]\Big|$$

for all $x,y\in X$. *Assume that one of the following two hypotheses is valid* *(i)* $\sup_{x,y,z \in X_0} \frac{\Delta(x,y,z)}{\|x\|^p \|y\|^q \|z\|^r} < \infty$ *for* $p + q + r < 0$,

(ii) $\sup_{x,y,z \in X_0} \frac{\Delta(x,y,z)}{\|x\|^p + \|y\|^p + \|z\|^p} < \infty$ *for* $p < 0$.

Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$. Then from Corollaries 2 and 3, we easily derive f which is a solution of the functional equation (1.3). That implies $\Delta(x, y) = 0$. Thus the norm $\|\cdot\|$ on X satisfies the parallelogram low:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.$$

Therefore, X is an inner product space. □

Corollary 5. *Let G be a commutative group and E be a Banach space. Let* $\varphi : G^3 \to [0, \infty)$ *and* $u : \mathbb{Z}^\star = \mathbb{Z}\backslash\{0\} \to [0, \infty)$ *be functions and the conditions* (2.1), (2.2), (2.17) *and* (2.18) *be valid. If* $F : G^3 \to E$ *is a mapping such that* $F(x_0, y_0, z_0) \neq 0$ *for some* $x_0, y_0, z_0 \in G$ *and*

$$\|F(x, y, z)\| \le \varphi(x, y, z)$$

for all $x, y, z \in G$*, then the functional equation*

$$g(x + y + z) + g(x - y + z) + g(y - x + z) + g(x + y - z) = F(x, y, z) + 4[g(x) + g(y) + g(z)], \quad x, y, z \in G \quad (2.19)$$

has no solution in the class of functions $g : G \to E$.

Proof. Suppose that $g : G \to E$ is a solution to (2.19). Then (2.3) holds, and consequently, according to Corollary 1, g satisfies (1.3) on G, which means that $F(x_0, y_0, z_0) = 0$. This is a contradiction. □

3 Conclusions

We have proved the Hyers-Ulam stability and the hyperstability of the quadratic functional equation

$$f(x + y + z) + f(x + y - z) + f(x - y + z) + f(-x + y + z) = 4[f(x) + f(y) + f(z)]$$

in the class of functions from an abelian group G into a Banach space.

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