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Weyl Type Theorems for Selfadjoint Operators on Krein Spaces

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Abstract. In this paper, we introduce a notion of the $\mathcal{J}$-kernel of a bounded linear operator on a Krein space and study the $\mathcal{J}$-Fredholm theory for Krein space operators. Using $\mathcal{J}$-Fredholm theory, we discuss when (a-)$\mathcal{J}$-Weyl's theorem or (a-)$\mathcal{J}$-Browder's theorem holds for bounded linear operators on a Krein space instead of a Hilbert space.

1. Introduction

The Fredholm theory is usually described by the spectral theory of bounded linear operators with finite dimensional kernel and cokernel (called *Fredholm operators*) and such operators naturally arise in the Fredholm integral equations. In particular, Fredholm operators are used to find an approximate inverse to a differential operator or a fundamental solution of a partial differential equation [9]. Hermann Weyl noticed that the elements in the spectrum of a normal operator N which can be removed by the compact perturbation of N are precisely the eigenvalues of finite multiplicity which are isolated points of the spectrum of N. It was proved by many people that there are several classes of operators including normal, hyponormal and Toeplitz operators for which Weyl's observation is true. In the local spectral theory, one of basic concepts is the single-valued extension property which arises in the spectral decomposition theory and was also known to be useful for the study of operators satisfying (a-)Weyl's theorem [7].

Let $\mathcal{K}$ be a Hilbert space with a positive definite inner product $\langle \cdot, \cdot \rangle$. Suppose that a selfadjoint involution J on $\mathcal{K}$, i.e., $J = J^{-1} = J^*$, is given to produce an indefinite inner product

$$\langle x, y \rangle_J = \langle Jx, y \rangle \quad (x, y \in \mathcal{K}).$$

In this case, we say that the pair $(\mathcal{K}, J)$ is a *Krein space* and that $\langle \cdot, \cdot \rangle_J$ is the *indefinite inner product given by* J. Due to such indefinite inner product, a Krein space $\mathcal{K}$ can be decomposed into a direct sum of a Hilbert space $\mathcal{K}_+$ and the anti-space $\mathcal{K}_-$ of a Hilbert space. The theory of a Krein space has attracted increasing attention in both mathematics and physics and has proved to be an effective tool for the research areas which are described by an indefinite inner product. For the study of massless or gauge fields, the positivity

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of an inner product in the space has to be abandoned, but we need an indefinite inner product because of useful properties such as Lorentz invariance. We refer the book [4] for a detailed information of Krein spaces.

The purpose of this paper is to introduce a notion of the $\mathcal{J}$-kernel of a bounded linear operator on a Krein space and to study the $\mathcal{J}$-Fredholm theory for operators on a Krein space instead of a Hilbert space. Using such a $\mathcal{J}$-Fredholm theory, we discuss for which Krein space operators $\mathcal{J}$-Weyl's theorem or $\mathcal{J}$-Browder's theorem hold. This paper is organized as follows. In section 2, we introduce basic notions of some operators on a Krein space and review various spectra of operators, (a-)Weyl's theorem and (a-)Browder's theorem for bounded linear operators on a Hilbert space. In section 3, we first introduce a notion of the $\mathcal{J}$-kernel for a Krein space operator. We also introduce notions of $\mathcal{J}$-Fredholm, $\mathcal{J}$-Weyl and $\mathcal{J}$-Browder operators and study their properties. Section 4 is devoted to the study of ($\mathcal{J}$-)Weyl's theorem and ($\mathcal{J}$-)Browder's theorem for $\mathcal{J}$-selfadjoint operators and $\mathcal{J}$-unitary operators.

2. Preliminaries

Let $\mathcal{K}$ and $\mathcal{H}$ be complex Hilbert spaces. We denote by $\mathcal{L}(\mathcal{K}, \mathcal{H})$ the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$, and abbreviate $\mathcal{L}(\mathcal{K}) = \mathcal{L}(\mathcal{K}, \mathcal{K})$. If $T \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we write $\ker(T)$ for the kernel of T and $\operatorname{ran}(T)$ for the range of T.

Let $(\mathcal{K}, J)$ be a Krein space equipped with an indefinite inner product $\langle \cdot, \cdot \rangle_J$. Throughout this paper, * denotes the Hilbert space adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$, whereas $^\#$ denotes the Krein space adjoint, called the *$\mathcal{J}$-adjoint*, with respect to the indefinite inner product $\langle \cdot, \cdot \rangle_J$. We easily see that the $\mathcal{J}$-adjoint $T^\#$ of T is given by

$$T^\# = JT^*J$$

for the selfadjoint involution J. We say that the operator J is a *fundamental symmetry*, and that both $P_+ = (I + J)/2$ and $P_- = (I - J)/2$ are *fundamental projections*. The direct sum $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ will be called the *fundamental decomposition*, where $\mathcal{K}_+ = P_+(\mathcal{K})$ and $\mathcal{K}_- = P_-(\mathcal{K})$. We easily see that fundamental projections P_+ and P_- are mutually orthogonal. Two important differences between Hilbert spaces and Krein spaces are the existence of nontrivial neutral and isotropic elements.

Now, we briefly review some notions of various spectra, which are used in this paper. We refer [2, 12] for more detailed information. Let $T \in \mathcal{L}(\mathcal{K})$ and $k \in \mathbb{N}$. The family $\{\ker(T^k)\}$ forms an ascending sequence of subspaces. We call the *ascent* of T for the smallest nonnegative integer k for which $\ker(T^k) = \ker(T^{k+1})$ holds. We also see that the family $\{\operatorname{ran}(T^k)\}$ forms a descending sequence. The smallest nonnegative integer k for which $\operatorname{ran}(T^k) = \operatorname{ran}(T^{k+1})$ is called the *descent* of T. We say that $T \in \mathcal{L}(\mathcal{K})$ is *upper semi-Fredholm* if it has closed range and finite dimensional kernel and *lower semi-Fredholm* if it has closed range and its range has finite codimension. If T is either upper or lower semi-Fredholm, then T is called *semi-Fredholm*, and the *index* of T is defined by $\operatorname{ind}(T) := \dim \ker(T) - \dim \ker(T^*)$. If both $\dim \ker(T)$ and $\dim \ker(T^*)$ are finite, then T is called *Fredholm*. An operator T is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm with finite ascent and finite descent.

For $T \in \mathcal{L}(\mathcal{K})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$, and $\sigma_{com}(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, and the compression spectrum of T, respectively. We review several spectra as follows;

(a) the *left essential spectrum* $\sigma_{le}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\}$,

(b) the *right essential spectrum* $\sigma_{re}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Fredholm}\}$,

(c) the *essential spectrum* $\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$,

(d) the *Weyl spectrum* $\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$,

(e) the *Browder spectrum* $\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$.

Evidently, we have the following inclusions

$$\sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T),$$

where we write acc $\sigma(T)$ for the set of all accumulation points of $\sigma(T)$.

Let iso $\sigma(T)$ be the set of all isolated points of $\sigma(T)$. We write $\pi_{00}(T) := \pi_{0f}(T) \cap \operatorname{iso}\sigma(T)$ where $\pi_{0f}(T) = \{\lambda \in \mathbb{C} : 0 < \dim\ker(T-\lambda) < \infty\}$. We also write $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$, which is the set of all *Riesz points of* T. We say that *Weyl's theorem holds* for T if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem holds* for T if $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$, equivalently, if $\sigma_w(T) = \sigma_b(T)$. Hermann Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to several classes of operators by many authors [3, 5, 7].

We recall the *Weyl essential approximate point spectrum* $\sigma_{ea}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T)$ given by

$$\sigma_{ea}(T) := \bigcap \{\sigma_a(T+K) : K \in \mathcal{C}(\mathcal{K})\},$$
$$\sigma_{ab}(T) := \bigcap \{\sigma_a(T+K) : TK = KT \text{ and } K \in \mathcal{C}(\mathcal{K})\}$$

where $\mathcal{C}(\mathcal{K})$ is the set of all compact operators on $\mathcal{K}$. We say that *a-Weyl's theorem holds* for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi^a_{00}(T)$ and that *a-Browder's theorem holds* for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = p^a_{00}(T)$, where $\pi^a_{00}(T) := \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \dim\ker(T-\lambda) < \infty\}$ and $p^a_{00}(T) := \sigma_a(T) \setminus \sigma_{ab}(T)$. Then we see obvious implications;

(a) a-Weyl's theorem $\Longrightarrow$ a-Browder's theorem $\Longrightarrow$ Browder's theorem,

(b) a-Weyl's theorem $\Longrightarrow$ Weyl's theorem $\Longrightarrow$ Browder's theorem.

3. $\mathcal{J}$-Fredholm Operators on Krein Spaces

In this section, we denote by $(\mathcal{K}, J)$ a Krein space equipped with an indefinite inner product $\langle\cdot,\cdot\rangle_J$. For an operator $T \in \mathcal{L}(\mathcal{K})$, let $\ker(T) = \{x \in \mathcal{K} : Tx = 0\}$. We define the *$\mathcal{J}$-kernel* of T, $\mathcal{J}\text{-}\ker(T)$, by

$$\mathcal{J}\text{-}\ker(T) := \{x \in \mathcal{K} : \langle Tx, Tx\rangle_J = 0\}.$$

We can easily see that the $\mathcal{J}$-kernel of T has the following properties;

Proposition 3.1. *Let $T \in \mathcal{L}(\mathcal{K})$. Then the following statements hold.*

(i) $\mathcal{J}\text{-}\ker(T)$ *is closed.*

(ii) $\ker(T) \subseteq \mathcal{J}\text{-}\ker(T)$.

In general, if T has a closed range, then the dimension of $\ker(T)$ is equal to the codimension of the range of the adjoint T^*. However, we see from Proposition 3.1 that even if the range of T is closed, that the dimension of $\mathcal{J}\text{-}\ker(T)$ may not be equal to the codimension of the range of the $\mathcal{J}$-adjoint $T^\#$. Indeed, it is observe that $\dim\mathcal{K}/\operatorname{ran}(T^\#) = \dim\mathcal{K}/\operatorname{ran}(T^*) = \dim\ker(T) \le \dim\mathcal{J}\text{-}\ker(T)$ since $\ker(T)$ may be properly contained in $\mathcal{J}\text{-}\ker(T)$. Moreover, the family $\{\ker(T^k)\}$ forms an ascending sequence of subspaces, but the sequence $\{\mathcal{J}\text{-}\ker(T^k)\}$ may not be ascending.

Example 3.2. *We consider the 3-dimensional Krein space $\mathcal{K}$ with an indefinite inner product given by the symmetric operator J acting on $\mathcal{K}$ defined by $J(x,y,z) = (-x,-y,z)$. Let T be a linear operator acting on $\mathcal{K}$ defined by $T(x_1,x_2,x_3) = (x_2,x_3,x_1)$. Then $(\sqrt{2},1,1)$ belongs to $\mathcal{J}\text{-}\ker(T)$, but does not belong to $\mathcal{J}\text{-}\ker(T^2)$. Indeed, since $T(\sqrt{2},1,1) = (1,1,\sqrt{2})$ and $T^2(\sqrt{2},1,1) = (1,\sqrt{2},1)$, we have that*

$$\langle T(\sqrt{2},1,1), T(\sqrt{2},1,1)\rangle_J = \langle(-1,-1,\sqrt{2}),(1,1,\sqrt{2})\rangle = 0,$$
$$\langle T^2(\sqrt{2},1,1), T^2(\sqrt{2},1,1)\rangle_J = \langle(-1,-\sqrt{2},1),(1,\sqrt{2},1)\rangle = -2.$$

Unlike the kernel $\ker(T)$, the $\mathcal{J}$-kernel $\mathcal{J}\text{-}\ker(T)$ is not an invariant subspace of T, that is, the inclusion $T(\mathcal{J}\text{-}\ker(T)) \subseteq \mathcal{J}\text{-}\ker(T)$ does not hold. $\square$

Though, in general, the sequence $\{\mathcal{J}\text{-}\ker(T^k)\}$ is not ascending, we would like to define a notion like an ascent. We define the *$\mathcal{J}$-ascent* by

$$\varphi(T) := \sup_k \dim(\mathcal{J}\text{-}\ker(T^k)).$$

If $\mathcal{J}\text{-}\ker(T)$ is an invariant subspace of T, it is obvious that $T^k(\mathcal{J}\text{-}\ker(T))$ is contained in $\mathcal{J}\text{-}\ker(T)$ for each nonnegative integer k, so that the collection $\{\mathcal{J}\text{-}\ker(T^k)\}$ is an ascending sequence. Therefore, if $\mathcal{J}\text{-}\ker(T)$ is an invariant subspace of T and $\varphi(T) := p < \infty$, then this means that $\mathcal{J}\text{-}\ker(T^p) = \mathcal{J}\text{-}\ker(T^{p+1})$ for such an integer $p > 0$.

We say that $T \in \mathcal{L}(\mathcal{K})$ is a *$\mathcal{J}$-Fredholm operator* if $\operatorname{ran}(T)$ is closed, $\dim \mathcal{J}\text{-}\ker(T) < \infty$ and $\operatorname{ran}(T)$ has finite codimension. In this case, we define the *$\mathcal{J}$-index of T* by

$$\mathcal{J}\text{-ind}(T) := \dim \mathcal{J}\text{-}\ker(T) - \dim(\mathcal{K}/\operatorname{ran}(T)).$$

It follows from Proposition 3.1 that $\operatorname{ind}(T) \le \mathcal{J}\text{-ind}(T)$. We also see that if T is $\mathcal{J}$-Fredholm, then it is automatically Fredholm. But, the converse does not hold, in general. We will give the example which is Fredholm, but not $\mathcal{J}$-Fredholm.

Example 3.3. *Let T be the unilateral shift on $l^2(\mathbb{N})$ and J be defined by*

$$Je_{2n-1} = e_{2n} \text{ and } Je_{2n} = e_{2n-1}.$$

Then J is a fundamental symmetry with $J^* = J^{-1} = J$ on $l^2(\mathbb{N})$. If $x := (x_n) \in \mathcal{J}$-ker$(T)$, then we have that

$$0 = \langle Tx, Tx\rangle_J = \langle JTx, Tx\rangle = \sum_{n=1}^{\infty} \left(x_{2n}\overline{x_{2n+1}} + x_{2n+1}\overline{x_{2n}}\right)$$

Hence, we have that $\dim \mathcal{J}$*-ker*$(T) = \infty$*, which implies that T is not $\mathcal{J}$-Fredholm. However, we see that* $\dim \ker(T) = 0$ *and* $\dim \ker(T^*) = 1$*, so that T is Fredholm.*

Lemma 3.4. *Let $T \in \mathcal{L}(\mathcal{K})$ and let $\mathcal{M}$ be a subspace of $\mathcal{K}$ such that* $\dim \mathcal{K}/\mathcal{M} = n < \infty$*. If T is $\mathcal{J}$-Fredholm, then so is the restriction $T|_{\mathcal{M}}$.*

Proof. It suffices to show this lemma only for $n = 1$ since we can inductively see the other cases. Suppose that $\dim \mathcal{K}/\mathcal{M} = 1$. Then we have the decomposition $\mathcal{K} = \mathcal{M} \oplus \operatorname{span}\{x_1\}$ for some $x_1 \neq 0$.

We first assume that $Tx_1 \notin \operatorname{ran}(T|_{\mathcal{M}})$. Since $\operatorname{ran}(T) = \operatorname{ran}(T|_{\mathcal{M}}) \oplus \operatorname{span}\{Tx_1\}$ and $\mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \subseteq \mathcal{J}\text{-}\ker(T)$, we have that

$$\dim \mathcal{K}/\operatorname{ran}(T|_{\mathcal{M}}) = \dim \mathcal{K}/\operatorname{ran}(T) + 1 \quad \text{and} \quad \dim \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \le \dim \mathcal{J}\text{-}\ker(T).$$

Suppose that $Tx_1 \in \operatorname{ran}(T|_{\mathcal{M}})$. Then we see that $\operatorname{ran}(T) = \operatorname{ran}(T|_{\mathcal{M}})$. and that there exists an element $u \in \mathcal{M}$ such that $Tx_1 = (T|_{\mathcal{M}})u$, which deduces that $\mathcal{J}\text{-}\ker(T) = \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \oplus \operatorname{span}\{x_1 - u\}$. Indeed, if $x \in \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \oplus \operatorname{span}\{x_1 - u\}$, then we can write

$$x = m + \lambda(x_1 - u) \quad \text{for some } m \in \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \text{ and } \lambda \in \mathbb{C}.$$

Thus, we have that $\langle Tx, Tx\rangle_J = \langle JTx, Tx\rangle = \langle Tm, Tm\rangle_J = 0$, which implies that $x \in \mathcal{J}\text{-}\ker(T)$. Conversely, if $x \in \mathcal{J}\text{-}\ker(T)$, then we can write $x = m + \lambda x_1$ for some $m \in \mathcal{M}$ and $\lambda \in \mathbb{C}$. Then we have that $0 = \langle Tx, Tx\rangle_J = \langle JT(m + \lambda u), T(m + \lambda u)\rangle = \langle T(m + \lambda u), T(m + \lambda u)\rangle_J$, so that $m + \lambda u \in \mathcal{J}\text{-}\ker(T|_{\mathcal{M}})$. This means that $x = (m + \lambda u) + \lambda(x_1 - u) \in \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) \oplus \operatorname{span}\{x_1 - u\}$. Therefore, we obtain that

$$\dim \mathcal{K}/\operatorname{ran}(T|_{\mathcal{M}}) = \dim \mathcal{K}/\operatorname{ran}(T) \quad \text{and} \quad \dim \mathcal{J}\text{-}\ker(T|_{\mathcal{M}}) = \dim \mathcal{J}\text{-}\ker(T) - 1,$$

which also implies that $T|_{\mathcal{M}}$ is $\mathcal{J}$-Fredholm. □

Remark 3.5. *Unlike the Fredholm index, in general, the index product formula does not hold for the $\mathcal{J}$-index. More precisely, even if $T, S \in \mathcal{L}(\mathcal{K})$ are $\mathcal{J}$-Fredholm, we may see that $\mathcal{J}$-ind$(ST) \neq \mathcal{J}$-ind$(S) + \mathcal{J}$-ind(T). We have an example which does not satisfy the index product formula for the $\mathcal{J}$-index as follows;*

The Krein space $\mathcal{K}$ given in Example 3.2 is finite dimensional, so that T is $\mathcal{J}$-Fredholm. We see that T^2 is also $\mathcal{J}$-Fredholm. If $x = (x_1, x_2, x_3)$ belongs to $\mathcal{J}$-$\ker(T)$, then we have

$$0 = \langle Tx, Tx\rangle_J = \langle JTx, Tx\rangle = -x_2^2 - x_3^2 + x_1^2,$$

so that $x_1^2 = x_2^2 + x_3^2$. This means that $\dim \mathcal{J}\text{-}\ker(T) = 2$. However, if x belongs to $\mathcal{J}$-$\ker(T^2)$, then we also have

$$0 = \langle T^2x, T^2x\rangle_J = \langle JT^2x, T^2x\rangle = -x_3^2 - x_1^2 + x_2^2.$$

Hence this implies that $\dim \mathcal{J}\text{-}\ker(T^2) = 2$, so that $\mathcal{J}$-ind$(T^2) = 2$. Thus the index product formula does not hold for the $\mathcal{J}$-index since $\mathcal{J}$-ind$(T) + \mathcal{J}$-ind$(T) = 4 \neq 2 = \mathcal{J}$-ind$(T^2)$. □

We say that $T \in \mathcal{L}(\mathcal{K})$ is *$\mathcal{J}$-Weyl* if it is $\mathcal{J}$-Fredholm and $\mathcal{J}$-*ind*(T)=0, and *$\mathcal{J}$-Browder* if it is $\mathcal{J}$-Fredholm and both the $\mathcal{J}$-ascent $\varphi(T)$ and the descent of T are finite. We define the *$\mathcal{J}$-essential spectrum*, *$\mathcal{J}$-Weyl spectrum*, and *$\mathcal{J}$-Browder spectrum* as follows;

$$\begin{aligned}
&\mathcal{J}\text{-}\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } \mathcal{J}\text{-Fredholm}\},\\
&\mathcal{J}\text{-}\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } \mathcal{J}\text{-Weyl}\},\\
&\mathcal{J}\text{-}\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } \mathcal{J}\text{-Browder}\}.
\end{aligned}$$

Proposition 3.6. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-Browder, then T is Browder.*

Proof. Suppose that T is $\mathcal{J}$-Browder and let $n \in \mathbb{N}$ be the descent of T. By Proposition 3.1, T has the finite ascent, so that the ascent of T is also n by [1, Theorem 3.3]. Since T is Fredholm, it is Browder. □

Corollary 3.7. *For any $T \in \mathcal{L}(\mathcal{K})$, the following implications hold.*

$$T \text{ is } \mathcal{J}\text{-Browder} \implies T \text{ is Browder} \implies T \text{ is Weyl}.$$

Moreover, it follows that $\sigma_w(T) \subseteq \sigma_b(T) \subseteq \mathcal{J}\text{-}\sigma_b(T)$.

Remark 3.8. *Unlike Proposition 3.6, even if $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-Weyl, it is not necessarily Weyl. Indeed, suppose that T is $\mathcal{J}$-Weyl and let $\dim \mathcal{K}/ran(T) = n < \infty$. Since $\mathcal{J}$-ind$(T) = 0$, we see that $\dim \mathcal{J}\text{-}\ker(T) = \dim \mathcal{K}/ran(T) = n < \infty$. Since $\ker(T) \subseteq \mathcal{J}\text{-}\ker(T)$, it follows that $\dim\ker(T) \leq n$. Thus, we have that $ind(T) = \dim\ker(T) - \dim \mathcal{K}/ran(T) \leq n - n = 0$. Moreover, even though T is Weyl, it is not $\mathcal{J}$-Weyl. For example, let U be the unilateral shift on $l^2(\mathbb{N})$ and T be defined by the next operator matrix*

$$T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \text{ on } l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$$

Then $\dim\ker(T) = \dim\ker(T^*) = 1$, so T is Weyl. Now we define a fundamental symmetry J on $l^2(\mathbb{N})$ as Example 3.3. Then since U has the infinite dimensional $\mathcal{J}$-kernel, it follows that $\dim \mathcal{J}\text{-}\ker(T) = \infty$ for a fundamental symmetry $\mathcal{J} := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. This means that T is not $\mathcal{J}$-Weyl. However, we observe the following implication;

$$T \text{ is } \mathcal{J}\text{-Weyl} \implies T \text{ is upper semi-Fredholm and } ind(T) \leq 0, \text{ that is, } 0 \notin \sigma_{ea}(T).$$

Moreover, we have that $\sigma_{ea}(T) \subseteq \mathcal{J}\text{-}\sigma_w(T)$. □

An operator $T \in \mathcal{L}(\mathcal{K})$ has *the single valued extension property at* $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0, the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0 \quad (\lambda \in U)$$

is the constant function $f \equiv 0$ on U. We say that T has *the single valued extension property* if it has the single valued extension property at every $\lambda \in \mathbb{C}$. Evidently, every operator T has the single valued extension property at every point of the boundary $\partial\sigma(T)$ of $\sigma(T)$, in particular, at every isolated point of $\sigma(T)$. We have that (see [1, Theorem 3.8] for details)

$$T - \lambda \text{ has finite ascent } \Longrightarrow T \text{ has the single valued extension property at } \lambda, \tag{1}$$

and that dually,

$$T - \lambda \text{ has finite descent } \Longrightarrow T^* \text{ has the single valued extension property at } \lambda. \tag{2}$$

If $T - \lambda$ is semi-Fredholm, then it is known that (1) and (2) are equivalent. Thus, if $T - \lambda$ is $\mathcal{J}$-Fredholm, the converses of (1) and (2) are also true.

We first consider the case where T is $\mathcal{J}$-Weyl and either selfadjoint or unitary.

Lemma 3.9. *Let $S, T \in \mathcal{L}(\mathcal{K})$. If S and T are $\mathcal{J}$-Fredholm operators, then both ST and TS are $\mathcal{J}$-Fredholm.*

Proof. We note that $T(\mathcal{J}\text{-}\ker(ST)) \subseteq \mathcal{J}\text{-}\ker(S)$ and $S(\mathcal{J}\text{-}\ker(TS)) \subseteq \mathcal{J}\text{-}\ker(T)$. Indeed, if $x \in \mathcal{J}\text{-}\ker(ST)$, then $\langle STx, STx\rangle_J = 0$, so that $Tx \in \mathcal{J}\text{-}\ker(S)$. This means the inclusion

$$T(\mathcal{J}\text{-}\ker(ST)) \subseteq \mathcal{J}\text{-}\ker(S).$$

Similarly, we can see the inclusion $S(\mathcal{J}\text{-}\ker(TS)) \subseteq \mathcal{J}\text{-}\ker(T)$. Since S and T are $\mathcal{J}$-Fredholm, it follows that $\dim \mathcal{J}\text{-}\ker(ST) < \infty$, $\dim \mathcal{J}\text{-}\ker(TS) < \infty$ and the ranges of ST and TS are finite codimensional, which completes the proof. □

Theorem 3.10. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-Weyl, and either selfadjoint or unitary, then the following statements are true;*

(i) $\mathcal{J}\text{-}\ker(T) = \ker(T)$.

(ii) *T^n is $\mathcal{J}$-Weyl for every positive integer n.*

(iii) *T^n is $\mathcal{J}$-Browder for every positive integer n.*

(iv) *T^n is Browder for every positive integer n.*

(v) *T^n is Weyl for every positive integer n.*

Proof. (i) If T is $\mathcal{J}$-Weyl and selfadjoint, then we have that $\dim \mathcal{J}\text{-}\ker(T) = \dim \ker(T)$. Since $\ker(T) \subseteq \mathcal{J}\text{-}\ker(T)$ by definition, we get the equality $\mathcal{J}\text{-}\ker(T) = \ker(T)$.

Suppose that T is $\mathcal{J}$-Weyl and unitary. Since T^* is also unitary, we get the equalities $\dim \mathcal{J}\text{-}\ker(T) = \dim \ker(T^*) = 0$, which implies that $\mathcal{J}\text{-}\ker(T) = \ker(T^*) = \{0\}$.

(ii) Suppose that T is selfadjoint and $\mathcal{J}$-Weyl. Since T is $\mathcal{J}$-Fredholm, it follows from Lemma 3.9 that T^n is also $\mathcal{J}$-Fredholm for every positive integer n. We will show that $\mathcal{J}\text{-ind}(T^n) = 0$. We note that if $\mathcal{J}\text{-}\ker(T) = \ker(T)$, then $\mathcal{J}\text{-}\ker(T^n) = \ker(T^n)$ for every positive integer n. Indeed, if $x \in \mathcal{J}\text{-}\ker(T^n)$, then $\langle T(T^{n-1})x, T(T^{n-1})x\rangle_J = \langle T^n x, T^n x\rangle_J = 0$. Thus, we have that $T^{n-1}x \in \mathcal{J}\text{-}\ker(T)$. By (i), we have that $\mathcal{J}\text{-}\ker(T) = \ker(T)$, so that $T^n x = 0$ and then $x \in \ker(T^n)$ for every positive integer n. Since T^n is selfadjoint, we have that

$$\mathcal{J}\text{-ind}(T^n) = \dim \mathcal{J}\text{-}\ker(T^n) - \dim \ker(T^n) = 0.$$

Therefore, T^n is $\mathcal{J}$-Weyl.

We assume that T is unitary and $\mathcal{J}$-Weyl. Since T^n is unitary for every integer n, we obtain that $\mathcal{J}\text{-ind}(T^n) = \dim \mathcal{J}\text{-}\ker(T^n)$ for each $n \geq 1$. If $x \in \mathcal{J}\text{-}\ker(T^n)$, then we see that $T^{n-1}x \in \mathcal{J}\text{-}\ker(T)$. But, T is injective, so that $T^{n-1}x = 0$. Since T^{n-1} is also injective, we have that $x = 0$. Hence, $\mathcal{J}\text{-ind}(T^n) = 0$, which implies that T^n is $\mathcal{J}$-Weyl.

(iii) We first show that T is $\mathcal{J}$-Browder. It is known that if T is either selfadjoint or unitary, then it has the single valued extension property. Since T is Fredholm, it has finite ascent and descent. By (i), this implies that $\dim \mathcal{J}\text{-}\ker(T^n) = \dim \ker(T^n) < \infty$ for every positive integer n. Thus, we have $\varphi(T) = \sup_n \dim(\mathcal{J}\text{-}\ker(T^n)) < \infty$, which implies that T is $\mathcal{J}$-Browder. We also obtain from the same argument that T^n is also $\mathcal{J}$-Browder for every positive integer n because T^n is also $\mathcal{J}$-Weyl and either selfadjoint or unitary.

(iv) and (v) are obvious from Corollary 3.7. □

Corollary 3.11. *Let T be selfadjoint. Consider the following conditions;*

(i) *T is $\mathcal{J}$-Weyl and $0 \in$ iso $\sigma(T)$.*

(ii) *T is $\mathcal{J}$-Browder but not invertible.*

(iii) *T is $\mathcal{J}$-Fredholm and $0 \in$ iso $\sigma(T)$.*

Then we have the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. (i)$\Rightarrow$(ii) It immediately follows from Theorem 3.10.
(ii)$\Rightarrow$(iii) Suppose that T is $\mathcal{J}$-Browder but not invertible. It is clear that T is $\mathcal{J}$-Fredholm. By Corollary 3.7, we have that T is Browder, so that T has finite ascent and descent. This implies that $0 \in \text{iso}\,\sigma(T)$. □

Like as notions of $\pi_{00}(T)$, $\pi_{00}^a(T)$ and $p_{00}(T)$, we define $\mathcal{J}\text{-}\pi_{00}(T)$, $\mathcal{J}\text{-}\pi_{00}^a(T)$ and $\mathcal{J}\text{-}p_{00}(T)$ as follows;

$$\begin{aligned}
&\mathcal{J}\text{-}\pi_{00}(T) = \{\lambda \in \text{iso}\,\sigma(T) :\ 0 < \dim \mathcal{J}\text{-}\ker(T-\lambda) < \infty\},\\
&\mathcal{J}\text{-}\pi_{00}^a(T) = \{\lambda \in \text{iso}\,\sigma_a(T) :\ 0 < \dim \mathcal{J}\text{-}\ker(T-\lambda) < \infty\},\\
&\mathcal{J}\text{-}p_{00}(T) = \sigma(T) \setminus \mathcal{J}\text{-}\sigma_b(T).
\end{aligned}$$

Then we get the relation between the spectrum and the $\mathcal{J}$-Weyl spectrum for selfadjoint operators.

Corollary 3.12. *If T is selfadjoint, then $\sigma(T) \setminus \mathcal{J}\text{-}\sigma_w(T) \subseteq \mathcal{J}\text{-}\pi_{00}(T)$.*

Proof. For $\lambda \in \sigma(T) \setminus \mathcal{J}\text{-}\sigma_w(T)$, $T - \lambda$ is $\mathcal{J}$-Weyl and selfadjoint since λ is real. Hence, $T - \lambda$ has finite ascent and descent, that is, $\lambda \in \text{iso}\,\sigma(T)$. By Theorem 3.10, we have that

$$0 < \dim \mathcal{J}\text{-}\ker(T-\lambda) < \infty.$$

Hence, $\lambda \in \mathcal{J}\text{-}\pi_{00}(T)$, which completes the proof. □

Remark 3.13. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-Fredholm, then we can find the closed subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{J}\text{-}\ker(T) \oplus \mathcal{K}_1 = T(\mathcal{K}) \oplus \mathcal{K}_2$. For the restriction $T_1 := T|_{\mathcal{K}_1}$, we have that $\ker(T_1) = \ker(T) \cap \mathcal{K}_1 \subset \mathcal{J}\text{-}\ker(T) \cap \mathcal{K}_1 = \{0\}$, Hence, T_1 is injective and we observe the implication*

$$T \text{ is } \mathcal{J}\text{-Fredholm} \Longrightarrow T \text{ is invertible modulo compact operator}$$

since a $\mathcal{J}$-Fredholm T is Fredholm. □

Lemma 3.14. *Let S, $T \in \mathcal{L}(\mathcal{K})$. Suppose that ST=TS and $T(\mathcal{J}\text{-}\ker(T)) \subseteq \mathcal{J}\text{-}\ker(S)$ and $S(\mathcal{J}\text{-}\ker(ST)) \subseteq \mathcal{J}\text{-}\ker(T)$. If ST is $\mathcal{J}$-Fredholm, then both S and T are $\mathcal{J}$-Fredholm.*

Proof. We observe two inclusions $\mathcal{J}\text{-}\ker(T) \subseteq \mathcal{J}\text{-}\ker(ST)$ and $\mathrm{ran}(ST) = \mathrm{ran}(TS) \subseteq \mathrm{ran}(T)$. Indeed, if $x \in \mathcal{J}\text{-}\ker(T)$, then $Tx \in T(\mathcal{J}\text{-}\ker(T)) \subseteq \mathcal{J}\text{-}\ker(S)$, so that $x \in \mathcal{J}\text{-}\ker(ST)$. Thus, we get the first inclusion and the second inclusion is trivial. If ST is $\mathcal{J}$-Fredholm, then $\dim \mathcal{J}\text{-}\ker(ST) < \infty$ and $\dim \mathcal{K}/\mathrm{ran}(ST) < \infty$. Hence, T is also $\mathcal{J}$-Fredholm. Similarly, we see that S is also $\mathcal{J}$-Fredholm. □

Theorem 3.15. *Let $T \in \mathcal{L}(\mathcal{K})$. Suppose that $\mathcal{J}\text{-}\ker(T)$ is an invariant subspace of T and that f is analytic in a bounded open neighborhood of $\sigma(T)$. Then the following equality holds :*

$$\mathcal{J}\text{-}\sigma_e(f(T)) = f(\mathcal{J}\text{-}\sigma_e(T)).$$

Proof. We assume that f is not identically zero on a bounded open neighborhood U of $\sigma(T)$ otherwise it is clear. We can write $f(z) = c(z-\alpha_1)\cdots(z-\alpha_n)g(z)$, where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ and $g(z)$ is analytic on U with no zeros. If $0 \notin f(\mathcal{J}\text{-}\sigma_e(T))$, then $c(\lambda-\alpha_1)\cdots(\lambda-\alpha_n)g(\lambda) \neq 0$ for any $\lambda \in \mathcal{J}\text{-}\sigma_e(T)$. Thus, $c(\lambda-\alpha_1)\cdots(\lambda-\alpha_n) \neq 0$, which implies that $\lambda \neq \alpha_i$ for any $\lambda \in \mathcal{J}\text{-}\sigma_e(T)$. Hence, each $T - \alpha_i$ is $\mathcal{J}$-Fredholm for $i = 1, 2, \cdots, n$. By Lemma 3.9, $f(T)$ is $\mathcal{J}$-Fredholm, so that $0 \notin \mathcal{J}\text{-}\sigma_e(f(T))$. Hence, $\mathcal{J}\text{-}\sigma_e(f(T)) \subseteq f(\mathcal{J}\text{-}\sigma_e(T))$.

To prove the converse, we first take any $\lambda \notin \mathcal{J}\text{-}\sigma_e(f(T))$. Then $f(T) - \lambda$ is $\mathcal{J}$-Fredholm and we have the decomposition as follows;

$$f(T) - \lambda = d(T-\lambda_1)\cdots(T-\lambda_m)h(T), \tag{3}$$

where $d, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $h(T)$ is invertible. Since the right side of (3) commute, it follows from Lemma 3.14 that each $T - \lambda_i$ is $\mathcal{J}$-Fredholm. Therefore, we see that $\lambda \notin f(\mathcal{J}\text{-}\sigma_e(T))$, which implies that $f(\mathcal{J}\text{-}\sigma_e(T)) \subseteq \mathcal{J}\text{-}\sigma_e(f(T))$. □

4. Weyl Type Theorems for $\mathcal{J}$-Selfadjoint Operators

We consider the several spectra of $\mathcal{J}$-selfadjoint operators and the spectra of their Hilbert space adjoint operators. In the remaining of the section, $(\mathcal{K}, J)$ denote a Krein space with an indefinite inner product $\langle\cdot,\cdot\rangle_J$.

Lemma 4.1. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-selfadjoint, then the following statements hold.*

(i) $\sigma_p(T) = \sigma_p(T^*)$, $\sigma_a(T) = \sigma_a(T^*)$, $\sigma_s(T) = \sigma_s(T^*)$, *and* $\sigma_{com}(T) = \sigma_{com}(T^*)$.

(ii) $\sigma(T) = \sigma(T^*)$, *that is, a spectrum of T is symmetric with respect to the real line.*

(iii) $\sigma_{le}(T) = \sigma_{le}(T^*)$, $\sigma_{re}(T) = \sigma_{re}(T^*)$ *and* $\sigma_e(T) = \sigma_e(T^*)$.

(iv) $J\Big(\ker(T^* - \lambda)\Big) = \ker(T - \lambda)$ *and* $J\Big(\ker(T - \lambda)\Big) = \ker(T^* - \lambda)$.

(v) $J\Big(\mathcal{J}\text{-}\ker(T^* - \lambda)\Big) = \mathcal{J}\text{-}\ker(T - \lambda)$ *and* $J\Big(\mathcal{J}\text{-}\ker(T - \lambda)\Big) = \mathcal{J}\text{-}\ker(T^* - \lambda)$.

Proof. (i) We first show that $\sigma_a(T) = \sigma_a(T^*)$. If $\lambda \in \sigma_a(T)$, then there exists a sequence $\{x_n\}$ in $\mathcal{K}$ with $\|x_n\| = 1$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty$. Thus we obtain that

$$\lim_{n\to\infty} \|(T^* - \lambda)Jx_n\| = \lim_{n\to\infty} \|J(T-\lambda)x_n\| = 0. \tag{4}$$

Since $\|Jx_n\| = \|x_n\| = 1$ for every n, we have that $\lambda \in \sigma_a(T^*)$, which implies the inclusion $\sigma_a(T) \subseteq \sigma_a(T^*)$. Similarly, the converse inclusion holds, so that $\sigma_a(T) = \sigma_a(T^*)$.

We claim that $\sigma_p(T) = \sigma_p(T^*)$. Indeed, since T is $\mathcal{J}$-selfadjoint, we have that

$$T - \lambda = T^\# - \lambda = J(T^* - \lambda)J.$$

Thus, for some vector $x \in \mathcal{K}$, $(T-\lambda)x = 0$ if and only if $(T^* - \lambda)Jx = 0$. This means that λ is an eigenvalue of T if and only if λ is an eigenvalue of T^*. Furthermore, for the surjective spectra and the compression spectra, we also have that

$$\sigma_s(T) = \overline{\sigma_a(T^*)} = \overline{\sigma_a(T)} = \sigma_s(T^*) \text{ and } \sigma_{com}(T) = \overline{\sigma_p(T^*)} = \overline{\sigma_p(T)} = \sigma_{com}(T^*).$$

(ii) Since $\sigma(T) = \sigma_a(T) \cup \sigma_s(T)$ for any $T \in \mathcal{L}(\mathcal{K})$, it is clear from (i) that $\sigma(T) = \sigma(T^*)$. Hence, $\lambda \in \sigma(T)$ if and only if $\overline{\lambda} \in \sigma(T)$, which implies that the spectrum of T is symmetric with respect to the real line.

(iii) We note that $\sigma_{re}(T^*) = \overline{\sigma_{le}(T)}$ and $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$ for any $T \in \mathcal{L}(\mathcal{K})$. We will only show that $\sigma_{le}(T) = \sigma_{le}(T^*)$ since the others are similar. It follows from [6] that $\lambda \in \sigma_{le}(T)$ if and only if there exists a sequence (x_n) of unit vectors in $\mathcal{K}$ such that (x_n) weakly converges to 0 and $\lim_{n\to\infty} \|(T-\lambda)x_n\| = 0$. By (4), it suffices to show that if a sequence (x_n) weakly converges to 0, then (Jx_n) weakly converges to 0. Since (x_n) weakly converges to 0, $\lim_{n\to\infty}\langle x_n, y\rangle = 0$ for any $y \in \mathcal{K}$. Hence we have that

$$\lim_{n\to\infty}\langle Jx_n, y\rangle = \lim_{n\to\infty}\langle x_n, Jy\rangle = 0 \text{ for any } y \in \mathcal{K},$$

so that a sequence (Jx_n) weakly converges to 0. Therefore, we have $\sigma_{le}(T) = \sigma_{le}(T^*)$.

(iv) For $x \in \ker(T-\lambda)$, we have $(T^*-\lambda)Jx = J(T-\lambda)x = 0$, so that $Jx \in \ker(T^*-\lambda)$. Hence, $J\Big(\ker(T-\lambda)\Big) \subset \ker(T^*-\lambda)$ and we also have $\ker(T-\lambda) \subset J\Big(\ker(T^*-\lambda)\Big)$. We similarly see that the reverse inclusions are satisfied.

(v) Suppose that $x \in \mathcal{J}\text{-}\ker(T-\lambda)$. Since J is a fundamental symmetry, there exists a vector $y \in \mathcal{K}$ such that $x = Jy$. Then we have that

$$\begin{aligned} 0 &= \langle (T-\lambda)x, (T-\lambda)x\rangle_J = \langle (T^*-\lambda)Jx, (T-\lambda)x\rangle \\ &= \langle (T^*-\lambda)y, J(T^*-\lambda)y\rangle = \langle (T^*-\lambda)y, (T^*-\lambda)y\rangle_J, \end{aligned}$$

which means that $Jx = y \in \mathcal{J}\text{-}\ker(T^*-\lambda)$. Hence, we obtain that

$$J\Big(\mathcal{J}\text{-}\ker(T-\lambda)\Big) \subseteq \mathcal{J}\text{-}\ker(T^*-\lambda) \text{ and } \mathcal{J}\text{-}\ker(T-\lambda) \subseteq J\Big(\mathcal{J}\text{-}\ker(T^*-\lambda)\Big).$$

Since the reverse inclusions can similarly be shown, we skip the proof. □

Lemma 4.2. *Suppose that $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-selfadjoint and $\lambda \in \mathbb{C}$.*

(i) $\dim\ker(T-\lambda) = \dim\ker(T^*-\lambda)$.

(ii) $\dim\mathcal{J}\text{-}\ker(T-\lambda) = \dim\mathcal{J}\text{-}\ker(T^*-\lambda)$.

(iii) *$T-\lambda$ has closed range if and only if $T^*-\lambda$ has closed range.*

Proof. (i) Since T is $\mathcal{J}$-selfadjoint, it follows from (iv) in Lemma 4.1 that

$$J\Big(\ker(T^*-\lambda)\Big) = \ker(T-\lambda).$$

Since J is invertible, we see that $\dim\ker(T-\lambda) = \dim\ker(T^*-\lambda)$.

(ii) By (v) in Lemma 4.1, we also have that $\dim\mathcal{J}\text{-}\ker(T-\lambda) = \dim\mathcal{J}\text{-}\ker(T^*-\lambda)$.

(iii) Suppose that $T-\lambda$ has closed range. If $y \in \overline{\operatorname{ran}(T^*-\lambda)}$, then there exists a sequence (y_n) in $\operatorname{ran}(T^*-\lambda)$ such that $\lim_{n\to\infty}\|y_n - y\| = 0$. We choose a sequence (x_n) in $\mathcal{K}$ such that $y_n = (T^*-\lambda)x_n$ for each $n \in \mathbb{N}$. Since $Jy_n = J(T^*-\lambda)x_n = (T-\lambda)Jx_n \in \operatorname{ran}(T-\lambda)$ and $\lim_{n\to\infty}\|Jy_n - Jy\| = \lim_{n\to\infty}\|y_n - y\| = 0$, we have that $Jy \in \overline{\operatorname{ran}(T-\lambda)} = \operatorname{ran}(T-\lambda)$. Putting $Jy = (T-\lambda)z$ for some $z \in \mathcal{K}$, we have that

$$y = J(T-\lambda)z = (T^*-\lambda)Jz \in \operatorname{ran}(T^*-\lambda).$$

Thus, we see that $\overline{\operatorname{ran}(T^*-\lambda)} = \operatorname{ran}(T^*-\lambda)$, so that $\operatorname{ran}(T^*-\lambda)$ is closed. Similarly, we can see that the converse is true. □

Lemma 4.3. *Let $T \in \mathcal{L}(\mathcal{K})$ be $\mathcal{J}$-selfadjoint and $\lambda \in \mathbb{C}$.*

(i) *$T-\lambda$ has finite ascent if and only if $T^*-\lambda$ has finite ascent.*

(ii) *$T - \lambda$ has finite descent if and only if $T^* - \lambda$ has finite descent.*

(iii) *$\varphi(T - \lambda) < \infty$ if and only if $\varphi(T^* - \lambda) < \infty$.*

Proof. (i) Suppose that $T - \lambda$ has finite ascent, that is, $\ker(T - \lambda)^p = \ker(T - \lambda)^{p+1}$ for some positive integer p. We will prove that $\ker(T^* - \lambda)^{p+1} \subset \ker(T^* - \lambda)^p$. If $x \in \ker(T^* - \lambda)^{p+1}$, then $(T - \lambda)^{p+1}Jx = J(T^* - \lambda)^{p+1}x = 0$. Hence, $Jx \in \ker(T - \lambda)^{p+1} = \ker(T - \lambda)^p$, so that we have that

$$(T^* - \lambda)^p x = J(T - \lambda)^p Jx = 0.$$

Therefore $x \in \ker(T^* - \lambda)^p$, which implies that $\ker(T^* - \lambda)^{p+1} \subset \ker(T^* - \lambda)^p$. Since in general, the reverse inclusion is true, we have that $T^* - \lambda$ has finite ascent. The converse implication similarly holds.

(ii) Suppose that $T - \lambda$ has finite descent, that is, $\mathrm{ran}(T - \lambda)^q = \mathrm{ran}(T - \lambda)^{q+1}$ for some positive integer q. For any $y \in \mathrm{ran}(T^* - \lambda)^q$, there exists $x \in \mathcal{K}$ such that $(T^* - \lambda)^q x = y$. Then we have that

$$Jy = J(T^* - \lambda)^q x = (T - \lambda)^q Jx \in \mathrm{ran}(T - \lambda)^q = \mathrm{ran}(T - \lambda)^{q+1}.$$

Thus, there exists a vector $z \in \mathcal{K}$ such that $Jy = (T - \lambda)^{q+1}z$, so that

$$y = J(T - \lambda)^{q+1}z = (T^* - \lambda)^{q+1}Jz \in \mathrm{ran}(T^* - \lambda)^{q+1}.$$

Hence we have the inclusion $\mathrm{ran}(T^* - \lambda)^q \subseteq \mathrm{ran}(T^* - \lambda)^{q+1}$. However, the reverse inclusion is trivial, so that $T^* - \lambda$ has finite descent. A similar argument shows that the converse implication also holds.

(iii) Since $(T - \lambda)^k$ is $\mathcal{J}$-selfadjoint for every positive integer k, it follows from Lemma 4.2 that the statements are equivalent. □

Proposition 4.4. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-selfadjoint, then the following statements hold.*

(i) *T^n is $\mathcal{J}$-selfadjoint for every nonnegative integer n.*

(ii) *If T is invertible, then T^{-1} is also $\mathcal{J}$-selfadjoint.*

(iii) *T is left invertible if and only if T is right invertible.*

(iv) *If $\dim\ker(T) < \infty$, then ind$(T) = 0$, that is, T is Weyl.*

Proof. (i) Since $JT^nJ = (JTJ)^n = (T^*)^n = (T^n)^*$ for any $n \geq 0$, T^n is $\mathcal{J}$-selfadjoint.

(ii) It follows from $J = J^{-1}$ that $JT^{-1}J = (JTJ)^{-1} = (T^*)^{-1} = (T^{-1})^*$.

(iii) If T is left invertible, then $\dim\ker(T) = 0$ and $\mathrm{ran}(T)$ is closed. Thus, it follows from Lemma 4.2 that $\dim\ker(T^*) = 0$ and the range $\mathrm{ran}(T^*)$ is closed. This means that T^* is also left invertible. By a principle of duality, T is right invertible. Similarly, we can show that the converse is true.

(iv) In the proof of Lemma 4.2, we see that $\dim\ker(T) = \dim\ker(T^*)$. Since $\dim\ker(T) < \infty$, it is satisfied that $\mathrm{ind}(T) = 0$, so that T is Weyl. □

Theorem 4.5. *Suppose that $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-selfadjoint and $\mathcal{J}$-Weyl. Then the following statements are true;*

(i) $\mathcal{J}$-$\ker(T) = \ker(T)$.

(ii) *T^n is $\mathcal{J}$-Weyl for every positive integer n.*

(iii) *T^n is Weyl for every positive integer n.*

(iv) *T^n is $\mathcal{J}$-Browder for every positive integer n.*

Proof. (i) If T is $\mathcal{J}$-selfadjoint and $\mathcal{J}$-Weyl, then

$$\dim\ker(T) = \dim\ker(T^*) = \dim\mathcal{J}\text{-}\ker(T) < \infty.$$

By Proposition 3.1, we have that $\mathcal{J}\text{-}\ker(T) = \ker(T)$.

(ii) Since T is $\mathcal{J}$-Weyl, it is $\mathcal{J}$-Fredholm. By Lemma 3.9, we have that T^n is $\mathcal{J}$-Fredholm for every positive integer n. We will show that $\mathcal{J}\text{-ind}(T^n) = 0$ for $n \geq 1$. Since $\mathcal{J}\text{-}\ker(T) = \ker(T)$, we obtain that $\mathcal{J}\text{-}\ker(T^n) = \ker(T^n)$ for every positive integer n. Indeed, if $x \in \mathcal{J}\text{-}\ker(T^n)$, then $\langle T(T^{n-1})x, T(T^{n-1})x\rangle_J = \langle T^n x, T^n x\rangle_J = 0$, which means that $T^{n-1}x \in \mathcal{J}\text{-}\ker(T)$. Since $\mathcal{J}\text{-}\ker(T) = \ker(T)$, we get $T^n x = 0$ and $x \in \ker(T^n)$. Since T^n is $\mathcal{J}$-selfadjoint, we have that

$$\mathcal{J}\text{-ind}(T^n) = \dim\mathcal{J}\text{-}\ker(T^n) - \dim\ker(T^n) = 0,$$

which implies that T^n is $\mathcal{J}$-Weyl.

(iii) It follows from (ii) and Proposition 4.4 that T^n is $\mathcal{J}$-selfadjoint and $\mathcal{J}$-Weyl for every positive integer n. Since $\text{ind}(T^n) = \mathcal{J}\text{-ind}(T^n) = 0$, T^n is also Weyl.

(iv) We first show that T is $\mathcal{J}$-Browder. Since T^n is $\mathcal{J}$-selfadjoint, we obtain from (i), (ii) and (iii) that

$$\varphi(T) = \sup_k \dim\mathcal{J}\text{-}\ker(T^k) = \sup_k \dim\ker(T^k) < \infty.$$

Since $\dim\ker(T) = \dim\ker(T^*)$ and T has finite ascent, T has also finite decent by [1, Theorem 3.4]. Thus, T is $\mathcal{J}$-Browder. By the same argument, we see that T^n is also $\mathcal{J}$-Browder for every positive integer n because T^n is also $\mathcal{J}$-selfadjoint and $\mathcal{J}$-Weyl. □

We note that, in general, a $\mathcal{J}$-selfadjoint operator does not satisfy Weyl's theorem. Indeed, let $(\mathcal{H}\oplus\mathcal{H}, J)$ be a Krein space where $\mathcal{H}$ is a Hilbert space, I is the identity on $\mathcal{H}$ and $J := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Let T be the operator $\begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix}$ acting on $\mathcal{H}\oplus\mathcal{H}$, where U is the unilateral shift on $\mathcal{H}$. It is not hard to see that while T is a $\mathcal{J}$-selfadjoint operator on $(\mathcal{H}\oplus\mathcal{H}, J)$, neither Weyl's theorem nor Browder's theorem holds for T.

Now we consider the set $\mathcal{J}\text{-}F_+(\mathcal{K}) := \{T \in \mathcal{L}(\mathcal{K}) :\ \text{ran}(T) \text{ is closed and } \dim\mathcal{J}\text{-}\ker(T) < \infty\}$. From the definition, we have that

$$\mathcal{J}\text{-}\sigma_{ea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{J}\text{-}F_+(\mathcal{K}) \text{ or } \mathcal{J}\text{-ind}(T-\lambda) > 0\}$$

is the *$\mathcal{J}$-Weyl essential approximate point spectrum*,

$$\mathcal{J}\text{-}\sigma_{ab}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{J}\text{-}F_+(\mathcal{K}) \text{ or } \varphi(T) = \infty\}$$

is the *$\mathcal{J}$-Browder essential approximate point spectrum*. We introduce another Weyl's theorems and Browder's theorems in terms of the $\mathcal{J}$-kernel on a Krein space $(\mathcal{K}, J)$, which may be regarded as extensions of Weyl's theorem and Browder's theorem for Hilbert space operators.

Definition 4.6. *Let T be any operator in $\mathcal{L}(\mathcal{K})$.*

1. *$\mathcal{J}$-Weyl's theorem holds for T if $\sigma(T) \setminus \mathcal{J}\text{-}\sigma_w(T) = \mathcal{J}\text{-}\pi_{00}(T)$.*
2. *$\mathcal{J}$-Browder's theorem holds for T if $\sigma(T) \setminus \mathcal{J}\text{-}\sigma_w(T) = \mathcal{J}\text{-}p_{00}(T)$, equivalently, if $\mathcal{J}\text{-}\sigma_w(T) = \mathcal{J}\text{-}\sigma_b(T)$.*
3. *a-$\mathcal{J}$-Weyl's theorem holds for T if $\sigma_a(T) \setminus \mathcal{J}\text{-}\sigma_{ea}(T) = \mathcal{J}\text{-}\pi^a_{00}(T)$.*
4. *a-$\mathcal{J}$-Browder's theorem holds for T if $\mathcal{J}\text{-}\sigma_{ea}(T) = \mathcal{J}\text{-}\sigma_{ab}(T)$.*

We observe that in general, there is no relation between Weyl's theorem and $\mathcal{J}$-Weyl's theorem. We give two examples such that one satisfies Weyl's theorem, but not $\mathcal{J}$-Weyl's theorem and another satisfies $\mathcal{J}$-Weyl's theorem, but not Weyl's theorem.

Example 4.7. *Let J be an operator defined by $Je_1 = e_2$, $Je_2 = e_1$ and $Je_n = e_n$ $(n \geq 3)$. We see that J is a fundamental symmetry on $l^2(\mathbb{N})$, which gives an indefinite inner product.*

1. *Let U be the unilateral shift on $l^2(\mathbb{N})$ and $x \in \mathcal{J}$-ker(U). Then we have that*
$$0 = \langle JUx, Ux\rangle = \sum_{n=1}^{\infty} |x_{n+1}|^2,$$
so that $\dim \mathcal{J}$-*ker*$(U) = 1$. *Since* $\dim$ *ker*$(U^*) = 1$, *we have* $\mathcal{J}$-*ind*$(U) = 0$, *so that U is $\mathcal{J}$-Weyl. We see that* $0 \in \sigma(U) \setminus \mathcal{J}\text{-}\sigma_w(U)$, *but* $\mathcal{J}\text{-}\pi_{00}(U) = \emptyset$. *Therefore, $\mathcal{J}$-Weyl's theorem does not hold for U. However, Weyl's theorem holds for U since* $\sigma(U) = \sigma_w(U) = \overline{\mathbb{D}}$ *and* $\pi_{00}(U) = \emptyset$, *where $\overline{\mathbb{D}}$ denotes the unit disc.*
2. *Let V be an operator on $l^2(\mathbb{N})$ defined by* $V(x_1, x_2, x_3, \cdots) = (x_1, 0, x_2, x_3, \ldots)$ *for* $(x_n) \in l^2(\mathbb{N})$. *Then we have that* $\sigma(V) = \sigma_w(V) = \pi_{00}(V) = \{0, 1\}$, *so that Weyl's theorem does not hold.*
Indeed, $\dim$ *ker*$(V) = 0$, $\dim$ *ker*$(V^*) = \dim$ *ker*$(V - I) = 1$, *and* $\dim$ *ker*$(V^* - I) = 2$. *If* $x \in \mathcal{J}$-*ker*(V), *then* $0 = \langle JVx, Vx\rangle = \sum_{n=1}^{\infty} |x_{n+1}|^2$, *so that* $\dim \mathcal{J}$-*ker*$(V) = 1$. *Moreover, if* $x \in \mathcal{J}$-*ker*$(V - I)$, *then we have that*
$$0 = \langle J(V - I)x, (V - I)x\rangle = \sum_{n=1}^{\infty} |x_{n+1} - x_{n+2}|^2.$$
Thus, we see that $\dim \mathcal{J}$-*ker*$(V - I) = 2$, *so that* $\sigma(V) \setminus \mathcal{J}\text{-}\sigma_w(V) = \mathcal{J}\text{-}\pi_{00}(V)$. *This means that $\mathcal{J}$-Weyl's theorem holds for V.* □

In the following theorem, we discuss $\mathcal{J}$-selfadjoint operators which satisfy the Weyl's theorem or the Browder's theorem.

Theorem 4.8. *Let $T \in \mathcal{L}(\mathcal{K})$ be $\mathcal{J}$-selfadjoint.*

(i) *Weyl's theorem holds for T if and only if so does for T^*.*

(ii) *Browder's theorem holds for T if and only if so does for T^*.*

(iii) *$\mathcal{J}$-Weyl's theorem holds for T if and only if so does for T^*.*

(iv) *$\mathcal{J}$-Browder's theorem holds for T if and only if so does for T^*.*

Proof. (i) We first prove that $\sigma_w(T) = \sigma_w(T^*)$. If $T - \lambda$ is Weyl for some $\lambda \in \mathbb{C}$, then it is Fredholm and $\dim \ker(T - \lambda) = \dim \ker(T^* - \overline{\lambda}) < \infty$. We see from Lemma 4.2 that
$$\begin{aligned}\operatorname{ind}(T^* - \lambda) &= \dim \ker(T^* - \lambda) - \dim \ker(T - \overline{\lambda})\\ &= \dim \ker(T - \lambda) - \dim \ker(T^* - \overline{\lambda}) = 0.\end{aligned}$$
Moreover, we have $\sigma_e(T) = \sigma_e(T^*)$, so that $T^* - \lambda$ is Fredholm. Thus, $T^* - \lambda$ is Weyl, which implies that $\sigma_w(T) \supseteq \sigma_w(T^*)$. By symmetry, the reverse inclusion is also true.

Since T is $\mathcal{J}$-selfadjoint, Lemmas 4.1 and 4.2 give the following equivalences;
$$\begin{aligned}\lambda \in \pi_{00}(T) &\Longleftrightarrow \lambda \in \operatorname{iso} \sigma(T) \text{ and } 0 < \dim \ker(T - \lambda) < \infty\\ &\Longleftrightarrow \lambda \in \operatorname{iso} \sigma(T^*) \text{ and } 0 < \dim \ker(T^* - \lambda) < \infty\\ &\Longleftrightarrow \lambda \in \pi_{00}(T^*).\end{aligned}$$
Thus, we have that $\pi_{00}(T) = \pi_{00}(T^*)$. Hence we see that Weyl's theorem holds for T if and only if so does T^*.

(ii) By Lemma 4.3, we have that $\sigma_b(T) = \sigma_b(T^*)$. It is clear to see that
$$\sigma_b(T) = \sigma_w(T) \Longleftrightarrow \sigma_b(T^*) = \sigma_w(T^*).$$
Indeed, if $\sigma_b(T) = \sigma_w(T)$, then $\sigma_b(T^*) = \sigma_b(T) = \sigma_w(T) = \sigma_w(T^*)$ where the third equality follows from (i). Conversely, if $\sigma_b(T^*) = \sigma_w(T^*)$, then we also have $\sigma_b(T) = \sigma_w(T)$. Therefore, Browder's theorem holds for T if and only if so does T^*.

(iii) We claim that $\mathcal{J}$-$\sigma_w(T) = \mathcal{J}$-$\sigma_w(T^*)$. If $T-\lambda$ is $\mathcal{J}$-Weyl, then it is $\mathcal{J}$-Fredholm and $\dim \mathcal{J}\text{-}\ker(T-\lambda) = \dim \mathcal{J}\text{-}\ker(T^* - \overline{\lambda}) < \infty$. By Lemma 4.2, we have that

$$\begin{aligned}\mathcal{J}\text{-ind}(T^* - \lambda) &= \dim \mathcal{J}\text{-}\ker(T^* - \lambda) - \dim \ker(T - \overline{\lambda}) \\ &= \dim \mathcal{J}\text{-}\ker(T - \lambda) - \dim \ker(T^* - \overline{\lambda}) = 0.\end{aligned}$$

Hence $T^* - \lambda$ is $\mathcal{J}$-Weyl, which means that $\mathcal{J}$-$\sigma_w(T) = \mathcal{J}$-$\sigma_w(T^*)$. We obtain from Lemmas 4.1 and 4.2 that $\mathcal{J}$-$\pi_{00}(T) = \mathcal{J}$-$\pi_{00}(T^*)$. Indeed, we have that

$$\begin{aligned}\lambda \in \mathcal{J}\text{-}\pi_{00}(T) &\Longleftrightarrow \lambda \in \operatorname{iso} \sigma(T) \text{ and } 0 < \dim \mathcal{J}\text{-}\ker(T - \lambda) < \infty \\ &\Longleftrightarrow \lambda \in \operatorname{iso} \sigma(T^*) \text{ and } 0 < \dim \mathcal{J}\text{-}\ker(T^* - \lambda) < \infty \\ &\Longleftrightarrow \lambda \in \mathcal{J}\text{-}\pi_{00}(T^*).\end{aligned}$$

Therefore, $\mathcal{J}$-Weyl's theorem holds for T if and only if so does T^*.

(iv) From Lemma 4.3, we have that $\mathcal{J}$-$\sigma_b(T) = \mathcal{J}$-$\sigma_b(T^*)$. Like (ii), we see that

$$\mathcal{J}\text{-}\sigma_b(T) = \mathcal{J}\text{-}\sigma_w(T) \Longleftrightarrow \mathcal{J}\text{-}\sigma_b(T^*) = \mathcal{J}\text{-}\sigma_w(T^*).$$

Therefore, $\mathcal{J}$-Browder's theorem holds for T if and only if so does T^*. □

Example 4.9. *We assume that T and J are matrices of the form*

$$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{in } \ \mathcal{L}(\mathbb{C} \oplus \mathbb{C}).$$

Then J is a fundamental symmetry with $J = J^* = J^{-1}$ on $\mathbb{C} \oplus \mathbb{C}$ and $T^*J = JT^*$, that is, T is a $\mathcal{J}$-selfadjoint operator. We observe that $\sigma(T) = \{i, -i\}$ and $\mathcal{J}$-$\sigma_w(T) = \emptyset$. Indeed, $\mathcal{J}$-ker$(T - iI) = $ ker$(T^* + iI) = \{(x, 0) : x \in \mathbb{C}\}$ and $\mathcal{J}$-ker$(T + iI) = $ ker$(T^* - iI) = \{(0, x) : x \in \mathbb{C}\}$. Moreover, we get that $\mathcal{J}$-$\pi_{00}(T) = \{i, -i\}$. Therefore $\mathcal{J}$-Weyl's theorem holds for T, and it follows from Theorem 4.8 that T^* satisfies also $\mathcal{J}$-Weyl's theorem.

Remark 4.10. *In the proof of Theorem 4.8, we observe that if T is $\mathcal{J}$-selfadjoint, then the equalities $\sigma_w(T) = \overline{\sigma_w(T)}$, $\sigma_b(T) = \overline{\sigma_b(T)}$, and $p_{00}(T) = \overline{p_{00}(T)}$ are true. However, we observe that $\pi_{00}(T) \neq \overline{\pi_{00}(T)}$. On the other hand, if a $\mathcal{J}$-selfadjoint operator T is reguloid, that is, there is a generalized inverse of $T - \lambda$ for $\lambda \in$ iso $\sigma(T)$, that is*

$$T - \lambda = (T - \lambda)S_\lambda(T - \lambda) \ \text{ for some } \ S_\lambda \in \mathcal{L}(\mathcal{K}),$$

then the equality $\pi_{00}(T) = \overline{\pi_{00}(T)}$ also holds. Indeed, if $\lambda \in \pi_{00}(T)$, then $\lambda \in$ iso $\sigma(T)$ and $0 < \dim \ker(T - \lambda) < \infty$. Since T is reguloid, $T - \lambda$ has closed range, so that T is upper semi-Fredholm. However, both T and T^* have the single valued extension property at λ, equivalently, $T - \lambda$ has finite ascent and descent. Hence, T is Browder and we have that

$$\pi_{00}(T) \subseteq \sigma(T) \setminus \sigma_b(T) = \overline{\sigma(T) \setminus \sigma_b(T)} = \overline{p_{00}(T)} \subseteq \overline{\pi_{00}(T)}.$$

Similarly, we see that the reverse inclusion also holds. □

The following theorem may be regarded as a generalized version of Theorem 4.8 for Weyl or Browder essential approximate spectrum.

Theorem 4.11. *Let $T \in \mathcal{L}(\mathcal{K})$ be $\mathcal{J}$-selfadjoint. Then the following statements hold.*

(i) *a-Weyl's theorem holds for T if and only if so does for T^*.*

(ii) *a-Browder's theorem holds for T if and only if so does for T^*.*

(iii) *a-$\mathcal{J}$-Weyl's theorem holds for T if and only if so does for T^*.*

(iv) *a-$\mathcal{J}$-Browder's theorem holds for T if and only if so does for T^*.*

Proof. (i) We first claim that $\sigma_{ea}(T) = \sigma_{ea}(T^*)$. Suppose that $T-\lambda$ is upper semi-Fredholm and $\mathrm{ind}(T-\lambda) \le 0$. Then it follows from (iii) in Lemma 4.1 that $T^* - \lambda$ is also upper semi-Fredholm. We will only prove that $\mathrm{ind}(T^* - \lambda) \le 0$. Since $\mathrm{ind}(T - \lambda) \le 0$, we have that $\mathrm{ind}(T^* - \overline{\lambda}) \ge 0$ by duality. This implies from (i) in Lemma 4.2 that

$$\dim\ker(T-\overline{\lambda}) = \dim\ker(T^*-\overline{\lambda}) \ge \dim \mathcal{K}\big/\mathrm{ran}(T^*-\overline{\lambda}) = \dim\ker(T-\lambda).$$

Hence we obtain that

$$\begin{aligned}\mathrm{ind}(T^*-\lambda) &= \dim\ker(T^*-\lambda) - \dim\mathcal{K}\big/\mathrm{ran}(T^*-\lambda)\\ &= \dim\ker(T-\lambda) - \dim\ker(T-\overline{\lambda}) \ \le\ 0.\end{aligned}$$

Therefore, we have that $\sigma_{ea}(T^*) \subseteq \sigma_{ea}(T)$. Similarly, we get the reverse inclusion.

On the other hand, by Lemma 4.1 (i) and Lemma 4.2 (i) we have that $\pi_{00}^{a}(T) = \pi_{00}^{a}(T^*)$. Thus, a-Weyl's theorem holds for T if and only if so does T^*.

(ii) We see from Lemma 4.1 and Lemma 4.3 that $\sigma_{ab}(T) = \sigma_{ab}(T^*)$. In the proof of (i), we also proved that $\sigma_{ea}(T) = \sigma_{ea}(T^*)$, which implies that a-Browder's theorem holds for T if and only if so does T^*.

(iii) We observe the equality $\mathcal{J}$-$\sigma_{ea}(T) = \mathcal{J}$-$\sigma_{ea}(T^*)$. Indeed, we suppose that $T-\lambda$ is upper semi $\mathcal{J}$-Fredholm and $\mathcal{J}$-$\mathrm{ind}(T) \le 0$. From the equality $\mathrm{ind}(T) \le \mathcal{J}$-$\mathrm{ind}(T) \le 0$, we have that $\dim\ker(T-\lambda) \le \dim\ker(T^*-\overline{\lambda})$. By (iii) in Lemma 4.2, we have that

$$\begin{aligned}\mathcal{J}\text{-}\mathrm{ind}(T^*-\lambda) &= \dim\mathcal{J}\text{-}\ker(T-\lambda) - \dim\ker(T-\overline{\lambda})\\ &\le \dim\ker(T-\lambda) - \dim\ker(T^*-\overline{\lambda}) \ \le\ 0.\end{aligned}$$

Hence we have that $\lambda \notin \mathcal{J}$-$\sigma_{ea}(T^*)$, which implies that $\mathcal{J}$-$\sigma_{ea}(T^*) \subseteq \mathcal{J}$-$\sigma_{ea}(T)$. We can similarly get the reverse inclusion.

Moreover, it follows from Lemma 4.1 and Lemma 4.2 that

$$\begin{aligned}\lambda \in \mathcal{J}\text{-}\pi_{00}^{a}(T) &\Longleftrightarrow \lambda \in \mathrm{iso}\,\sigma_a(T) \text{ and } 0 < \dim\mathcal{J}\text{-}\ker(T-\lambda) < \infty\\ &\Longleftrightarrow \lambda \in \mathrm{iso}\,\sigma_a(T^*) \text{ and } 0 < \dim\mathcal{J}\text{-}\ker(T^*-\lambda) < \infty\\ &\Longleftrightarrow \lambda \in \mathcal{J}\text{-}\pi_{00}^{a}(T^*).\end{aligned}$$

Thus, we have that a-$\mathcal{J}$-Weyl's theorem holds for T if and only if so does T^*.

(iv) Lemma 4.1 and Lemma 4.3 give the relation $\mathcal{J}$-$\sigma_{ab}(T) = \mathcal{J}$-$\sigma_{ab}(T^*)$, so that by the same argument in the proof of Theorem 4.8, we have

$$\mathcal{J}\text{-}\sigma_{ab}(T) = \mathcal{J}\text{-}\sigma_{ea}(T) \Longleftrightarrow \mathcal{J}\text{-}\sigma_{ab}(T^*) = \mathcal{J}\text{-}\sigma_{ea}(T^*).$$

Therefore, a-$\mathcal{J}$-Browder's theorem holds for T if and only if so does T^*. $\square$

We observe that the spectrum of J is the set $\sigma(J) = \{-1, 1\}$ since $J = J^* = J^{-1}$. If T is $\mathcal{J}$-unitary, then $T^*JT = TJT^* = J$ and we see that $\sigma(T^*JT) = \sigma(TJT^*) = \sigma(J) = \{-1, 1\}$. While T is invertible, it does not hold that the equality $T^* = T^{-1}$, in general. In the following lemma, we investigate various spectra of a $\mathcal{J}$-unitary operator.

Lemma 4.12. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-unitary, then we have the following properties;*

(i) *$\sigma_p(T^{-1}) = \sigma_p(T^*)$, $\sigma_a(T^{-1}) = \sigma_a(T^*)$, $\sigma_s(T^{-1}) = \sigma_s(T^*)$, and $\sigma_{com}(T^{-1}) = \sigma_{com}(T^*)$.*

(ii) *$\sigma(T^{-1}) = \sigma(T^*)$, which implies that for every $\lambda \in \sigma(T)$, $\sigma(T)$ also contains its inverse point $\frac{1}{\lambda}$ with respect to the unit circle.*

(iii) $\sigma_{le}(T^{-1}) = \sigma_{le}(T^*)$, $\sigma_{re}(T^{-1}) = \sigma_{re}(T^*)$, *and* $\sigma_e(T^{-1}) = \sigma_e(T^*)$.

(iv) $J\Big(\ker(T^* - \lambda)\Big) = \ker(T^{-1} - \lambda)$ *and* $J\Big(\ker(T^{-1} - \lambda)\Big) = \ker(T^* - \lambda)$.

(v) $J\Big(\mathcal{J}\text{-}\ker(T^* - \lambda)\Big) = \mathcal{J}\text{-}\ker(T^{-1} - \lambda)$ *and* $J\Big(\mathcal{J}\text{-}\ker(T^{-1} - \lambda)\Big) = \mathcal{J}\text{-}\ker(T^* - \lambda)$.

Proof. The proof is routine, so that we omit it. □

For a $\mathcal{J}$-unitary operator T, we easily see the relations about kernels;

$$\begin{aligned}\dim\ker(T^{-1} - \lambda) &= \dim\ker(T^* - \lambda) < \infty,\\ \dim\mathcal{J}\text{-}\ker(T^{-1} - \lambda) &= \dim\mathcal{J}\text{-}\ker(T^* - \lambda) < \infty.\end{aligned}$$

Moreover, we have that $T^{-1} - \lambda$ has closed range if and only if $T^* - \lambda$ has closed range. Like the case of a $\mathcal{J}$-selfadjoint operator, we also observe the following lemma for a $\mathcal{J}$-unitary operator.

Lemma 4.13. *Let $T \in \mathcal{L}(\mathcal{K})$ be $\mathcal{J}$-unitary and $\lambda \in \mathbb{C}$. Then the following statements are true;*

(i) *$T^{-1} - \lambda$ has finite ascent if and only if $T^* - \lambda$ has finite ascent.*

(ii) *$T^{-1} - \lambda$ has finite descent if and only if $T^* - \lambda$ has finite descent.*

(iii) *$\varphi(T^{-1} - \lambda) < \infty$ if and only if $\varphi(T^* - \lambda) < \infty$.*

Proof. The proofs are very similar to those of Lemma 4.3. □

Theorem 4.14. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-unitary and $\mathcal{J}$-Weyl, then the followings are true;*

(i) $\mathcal{J}\text{-}\ker(T) = \ker(T)$.

(ii) *T^n is $\mathcal{J}$-Weyl for every positive integer n.*

(iii) *T^n is Weyl for every positive integer n.*

(iv) *T^n is $\mathcal{J}$-Browder for every positive integer n.*

Proof. The proof is very similar to that of Theorem 4.5. □

Theorem 4.15. *If $T \in \mathcal{L}(\mathcal{K})$ is $\mathcal{J}$-unitary, then the followings are true;*

(i) *Weyl's theorem holds for T^{-1} if and only if so does for T^*.*

(ii) *Browder's theorem holds for T^{-1} if and only if so does for T^*.*

(iii) *$\mathcal{J}$-Weyl's theorem holds for T^{-1} if and only if so does for T^*.*

(iv) *$\mathcal{J}$-Browder's theorem holds for T^{-1} if and only if so does for T^*.*

(v) *a-Weyl's theorem holds for T^{-1} if and only if so does for T^*.*

(vi) *a-Browder's theorem holds for T^{-1} if and only if so does for T^*.*

(vii) *a-$\mathcal{J}$-Weyl's theorem holds for T^{-1} if and only if so does for T^*.*

(viii) *a-$\mathcal{J}$-Browder's theorem holds for T^{-1} if and only if so does for T^*.*

Proof. This is comparable to Theorem 4.8 and Theorem 4.11, and its proof is again very similar. □

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