Hydrodynamics of RN AdS$_4$ black hole and Holographic Optics

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Abstract

We consider the AdS$_4$ RN black hole and work out the momentum dependent hydrodynamic analysis for the vector modes. We also perform the spectral function calculation of the dual field theory. As an application, we consider the permittivity and permeability and find that for low frequency regime, the index of refraction is found to be negative, supporting the claim made in ref. [20] for AdS$_5$. We also find that at static limit the medium has the zero permeability, a character of the superconductivity .

Contents

1 Introduction

The gauge/gravity duality [1–3] has been proved to be the most fruitful idea in recent string theory and has been widely used in several different systems: from the quark-gluon plasma [6–13] and to condensed matter systems [14, 16, 17]. The super conductivity [18] and transport properties [15] were calculated for the 2+1 dimensional system near the critical point and dual background of the non-trivial scaling properties were also suggested [19]. In a very recent paper [20], a new connection between AdS gravity and optics we suggested: the negative refractive index of some meta-materials can be described by the holographic duality.

The most basic configuration is the asymptotically anti-de Sitter black hole with electric charge, which describe the finite temperature and chemical potential of the dual field theory. For the transport coefficient, one only needs to calculate the Green functions for the zero momentum case and corresponding analysis were done in [16, 17]. On the other hand, the permeability request momentum dependent hydrodynamic analysis even for the small frequency limit.

In this paper, we first perform the momentum dependent hydrodynamic analysis for four-dimensional Reissner-Nordström-Anti-de-Sitter black holes. The decoupled equations of motion and the Green functions have been worked out explicitly. We then study the spectral functions for the vector modes and as an application, refractive index for a strongly coupled medium were calculated.

In [16, 17], hydrodynamic analysis for the dyonic black hole for zero momentum case but with magnetic charge was performed. The zero momentum limit of our result is different from their result by taking the zero magnetic limit. This is because the limit of zero magnetic charge and that of zero momentum is not commuting. It is singular as was argued by the authors of ref. [15]. However our result gives consistent result for the conductivity in the literature [22, 23]. We also calculated the spectral functions, permittivity and permeability and for low frequency regime, the index of refraction is found to be negative supporting the claim made in ref. [20] for AdS$_5$. After finishing this work, we are informed that propagator structure with general quasi-normal modes has been studied in a very recent paper [28] and hydrodynamic analysis at zero temperature is described in [29].

2   Basic setup

The action for Einstein-Maxwell theory with a negative cosmological constant $\Lambda = -3/L^2$ is give by

$$S = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g}(R - 2\Lambda - L^2 F_{\mu\nu}F^{\mu\nu}) + \frac{1}{\kappa_4^2} \int d^3x \sqrt{-\gamma}\Theta + \frac{2}{\kappa_4^2 L} \int d^3x \sqrt{-\gamma}, \qquad (2.1)$$

where $F_{\mu\nu}$ is the field strength of Maxwell field $\mathcal{A}_\mu$, the second integral contains the Gibbons-Hawking term and the third one contains the counter term for the regularization. Here $\gamma_{\mu\nu}$ is the induced metric on the boundary and $\Theta = \gamma^{\mu\nu}\Theta_{\mu\nu}$ is the trace of the extrinsic curvature $\Theta_{\mu\nu} = -(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ with n the unit normal vector to the boundary directed outward.

The 4-dimensional Reissner-Nordström-Anti de Sitter black hole is the solution of Einstein equation and the metric is

$$ds^2 = \frac{l^2\alpha^2}{z^2}\left(-f(z)dt^2 + dx^2 + dy^2\right) + \frac{l^2}{z^2}\frac{dz^2}{f(z)}, \qquad (2.2)$$

$$A_t^{(0)} = -Q\alpha z. \qquad (2.3)$$

The function $f(z)$ has the form

$$f(z) = 1 + Q^2 z^4 - (1 + Q^2)z^3. \qquad (2.4)$$

The Hawking temperature of the black hole reads

$$T = \frac{(3 - Q^2)\alpha}{4\pi}. \qquad (2.5)$$

We consider the linearized perturbations h_{tx}, h_{xy} and A_x of the Einstein equation and the Maxwell equation, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$, $\mathcal{A}_\mu = A_\mu^{(0)} + A_\mu$. From the Maxwell equation we find

$$f(fA_x')' + \omega^2 A_x - Qfh_t^{x\prime} - k_y^2 f A_x = 0. \qquad (2.6)$$

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The linearized equations from the Einstein equation yield

$$-h_t^{x''} + \frac{2}{z}h_t^{x'} + 4Qz^2A_x' + \frac{k_y^2}{f}h_t^x + \frac{\omega k_y}{f}h_y^x = 0, \tag{2.7}$$

$$\omega h_t^{x'} - 4\omega Qz^2A_x + k_y f h_y^{x'} = 0, \tag{2.8}$$

$$h_y^{x''} + \frac{(z^{-2}f)'}{z^{-2}f}h_y^{x'} + \frac{\omega}{f^2}\left(k_y h_t^x + \omega h_y^x\right) = 0. \tag{2.9}$$

In these equations we have transformed the variables $A_x \to A_x\alpha$, $\omega \to \omega\alpha$ and $k \to k\alpha$, so that all the coordinate, momentum as well as field variables are dimensionless. Therefore that h_t^x and h_y^x were defined to be dimensionless from the beginning. At the end of the calculation, we can restore the original variables by reversing the scaling.

In order to have a decoupled equation, we define a new variable

$$\Phi_\pm = \frac{h_t^{x'}}{z^2} - 4QA_x + \frac{C_\pm}{z}A_x. \tag{2.10}$$

Then equations (2.6), (2.7) and (2.8) reduce to two decoupled equations for $\Phi_\pm$

$$\Phi_\pm'' + \frac{(z^2 f)'}{z^2 f}\Phi_\pm' + \frac{\omega^2 - k^2 f}{f^2}\Phi_\pm - \frac{C_\pm Qz}{f}\Phi_\pm = 0, \tag{2.11}$$

where $C_\pm = \frac{3 + 3Q^2 \pm \sqrt{9(Q^2+1)^2 + 16k^2Q^2}}{2Q}$.

3   Hydrodynamic analysis

We now seek a transformation

$$\Phi(z) = g(z)\Xi(z) \tag{3.1}$$

such that the coefficient of $\Xi(z)$ goes to zero when we take the hydrodynamic limit, $\omega, k \to 0$. Here Φ represent one of $\Phi_\pm$.

$$\left(\frac{\omega^2}{f(z)^2} - \frac{k^2 + C_\pm Qz}{f(z)} + \frac{2g'(z)}{zg(z)} + \frac{f'(z)g'(z)}{g(z)f(z)} + \frac{g''(z)}{g(z)}\right)\Xi(z) + \left(\frac{2}{z} + \frac{f'(z)}{f(z)} + 2\frac{g'(z)}{g(z)}\right)\Xi'(z) + \Xi''(z) = 0 \tag{3.2}$$

We need to find g such that Ξ's coefficient in eq.(3.2) is of order k^2 or higher in hydrodynamic limit. The constant $C_\pm$ can be expanded in a series of k^2

$$C_+ = \frac{3(1 + Q^2)}{Q} + \frac{4Qk^2}{3(1 + Q^2)} + \mathcal{O}(k^4) \tag{3.3}$$

$$C_- = -\frac{4Qk^2}{3(1 + Q^2)} + \mathcal{O}(k^4). \tag{3.4}$$

3.1 For Φ_+

We request that $g(z)$ satisfies following equation

$$-\frac{3\left(1+Q^2\right)z}{f(z)}g(z) + \left(\frac{2}{z} + \frac{f'(z)}{f(z)}\right)g'(z) + g''(z) = 0. \tag{3.5}$$

The solution for (3.5) regular at $z = 1$ is given by

$$g(z) = \frac{1}{z} - \frac{4Q^2}{3(1+Q^2)}, \tag{3.6}$$

where we used the linearity of equation of motion to normalize the solution.

By imposing the infalling boundary condition and taking out the near horizon limit,

$$\Xi(z) = (1-z)^\nu F(z), \tag{3.7}$$

with $\nu = -i\omega/4\pi T$. [1]

$$F(z)\left(\frac{(\nu-1)\nu}{(1-z)^2} + \frac{\omega^2 - k^2\left(1 + \frac{4Q^2 z}{3(1+Q^2)}\right)f(z)}{f(z)^2} - \frac{\nu}{1-z}\left(\frac{f'(z)}{f(z)} + \frac{2}{z} + \frac{2g'(z)}{g(z)}\right)\right) \tag{3.8}$$

$$+ F'(z)\left(\frac{-2\nu}{1-z} + \frac{f'(z)}{f(z)} + \frac{2}{z} + \frac{2g'(z)}{g(z)}\right) + F''(z) = 0 \tag{3.9}$$

In the long-wavelength, low-frequency limit, we can expand $F(z)$ in a double series with respect to ω and k

$$F(z) = C_0 + \omega F_1(z) + k^2 F_2(z) + \mathcal{O}(\omega^2, k^2) \tag{3.10}$$

$$0 = F_1''(z) + F_1'(z)\left(\frac{f'(z)}{f(z)} + \frac{2}{z} + \frac{2g'(z)}{g(z)}\right)$$
$$+ \frac{1}{3 - Q^2}\left(\frac{iC_0}{(-1+z)^2} + \frac{2iC_0}{z - z^2} + \frac{iC_0 f'(z)}{f(z)(1-z)} + \frac{2iC_0 g'(z)}{g(z)(1-z)}\right) \tag{3.11}$$

$$F_1 = iC_0 \int_1^z \left(\frac{1}{(3-Q^2)(x-1)} + \frac{g(1)^2}{f(x)x^2 g(x)^2}\right) dx := iC_0 H(z) \tag{3.12}$$

$$0 = F_2''(z) - + F_2'(z)\left(\frac{f'(z)}{f(z)} + \frac{2}{z} + \frac{2g'(z)}{g(z)}\right)\frac{C_0}{f(z)}\left(1 + \frac{4Q^2 z}{3(1+Q^2)}\right) \tag{3.13}$$

$$F_2 = \frac{C_0}{3(1+Q^2)}\int_1^z \frac{\int_1^y x^2\left(3 + Q^2(3+4x)\right)g(x)^2\,dx}{y^2 f(y)g(y)^2}\,dy := C_0 J(z) \tag{3.14}$$

---

[1]Here both ω and T are scaled by α to be dimensionless: $T = (3 - Q^2)/4\pi$ and $\nu = -i\omega/(3 - Q^2)$.

Later we will need following results for $H(z)$ and $J(z)$:

$$H(z) = H(0) + H_1 z + \cdots, \quad J(z) = J(0) + J_1 z + \cdots$$

$$\text{with} \quad H_1 = \frac{18 - 45Q^2 - Q^6}{9(3 - Q^2)(1 + Q^2)^2}, \tag{3.15}$$

$$J_1 = -\frac{27 + 63Q^2 + 29Q^4 + 9Q^6}{27(1 + Q^2)^3}. \tag{3.16}$$

3.2 For Φ_-

Since $C_- \sim O(k^2)$, we do not need transformation factor and we set $g(z) = 1$, unlike Φ_+ case. By doing as before, we get

$$\left(\frac{\nu(-2 + z + z\nu)}{z(z-1)^2} + \frac{\nu f'(z)}{(z-1)f(z)} + \frac{\left(w^2 - k^2 f(z) \left(1 - \frac{4Q^2 z}{3(1+Q^2)} \right) \right)}{f(z)^2} \right) F(z)$$

$$+ \left(\frac{2}{z} + \frac{f'(z)}{f(z)} + \frac{2\nu}{z-1} \right) F'(z) + F''(z) = 0 \tag{3.17}$$

Expanding $F(z) = \tilde{C}_0 + \omega F_1(z) + k^2 F_2(z) + \mathcal{O}(\omega^2, k^2)$, we have

$$\frac{i\tilde{C}_0}{(3 - Q^2)(1 - z)} \left(\frac{1}{1 - z} + \frac{2}{z} + \frac{f'(z)}{f(z)} \right) + \left(\frac{2}{z} + \frac{f'(z)}{f(z)} \right) F_1'(z) + F_1''(z) = 0, \tag{3.18}$$

$$-\frac{\tilde{C}_0}{f(z)} \left(1 - \frac{4Q^2 z}{3(1 + Q^2)} \right) + F_2'(z) \left(\frac{f'(z)}{f(z)} + \frac{2}{z} \right) + F_2''(z) = 0 \tag{3.19}$$

and their solutions are

$$F_1 = i\tilde{C}_0 \int_1^z \left(\frac{1}{(3 - Q^2)(x - 1)} + \frac{1}{f(x)x^2} \right) dx \tag{3.20}$$

$$F_2 = -\frac{\tilde{C}_0}{3(1 + Q^2)} \left(1 - \frac{1}{z} \right) \tag{3.21}$$

respectively.

Let us fix the constants C_0 and $\tilde{C}_0$ by imposing boundary conditions as

$$\lim_{z \to 0} h_t^x(z) = \hat{h}_t^x, \quad \lim_{z \to 0} h_x^y(z) = \hat{h}_t^x, \quad \lim_{z \to 0} A_x = \hat{A}_x \tag{3.22}$$

It would be simple by taking a derivative of $\Phi_\pm(z)$ and using equation (2.7). This gives the relation

$$z^2 \Phi_\pm' - C_\pm z A_x' = \frac{1}{f(z)} \left(k^2 h_t^x + \omega k h_x^y \right) - C_\pm A_x. \tag{3.23}$$

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In order to determine C_0 and $\tilde{C}_0$, we need to examine the above equation at the boundary

$$\lim_{z \to 0}(z^2 \Phi'_\pm - C_\pm z A'_x) = \left(k^2 \hat{h}^x_t + \omega k \hat{h}^x_t \right) - C_\pm \hat{A}_x := L_\pm \tag{3.24}$$

We have a few remarks on eq. (3.24).

- Only $\frac{1}{z}$ singularity of Φ near boundary can contribute to the left hand side.

- The boundary value of $z A_x$ is 0.

- The equation (3.24) is correct even for the finite k and ω.

Let

$$\Phi_\pm = \frac{\hat{\Phi}_\pm}{z} + \hat{\Pi}_\pm + T_\pm \log z \cdots \tag{3.25}$$

be the expansion near the boundary. Then

$$\hat{\Phi}_+ = C_0 \left(1 + i\omega H(0) + k^2 J(0) \right) = -L_+, \tag{3.26}$$

$$\hat{\Phi}_- = \tilde{C}_0 \left(-i\omega + \frac{k^2}{3(1+Q^2)} \right) = -L_-. \tag{3.27}$$

and

$$\hat{\Pi}_+ = -L_+ \left(\frac{i\omega}{3 - Q^2} - \frac{4}{3}\frac{Q^2}{1 + Q^2} + i\omega H_1 + k^2 J_1 \right) + \mathcal{O}(k\omega, \omega^2) \tag{3.28}$$

$$\hat{\Pi}_- = \tilde{C}_0 + \mathcal{O}(k\omega, \omega^2) \tag{3.29}$$

$$T_\pm = 0 \tag{3.30}$$

where H_1, J_1 are given in eq. (3.16) and eq. (3.16) respectively.

The constants C_0 and $\tilde{C}_0$ can be read off from the $\hat{\Phi}_\pm$ and $L_\pm$ parts, respectively

$$C_0 = \frac{-\left(k^2 \hat{h}^x_t + \omega k \hat{h}^x_t \right) + \left(\frac{3\left(1+Q^2\right)}{Q} + \frac{4Qk^2}{3(1+Q^2)} \right) \hat{A}_x}{1 + i\omega H(0) + k^2 J(0)} \tag{3.31}$$

$$\tilde{C}_0 = \frac{\left(k^2 \hat{h}^x_t + \omega k \hat{h}^x_t \right) + \frac{4Qk^2}{3(1+Q^2)} \hat{A}_x}{i\omega - \frac{k^2}{3(1+Q^2)}}, \tag{3.32}$$

where $H(z)$ and $J(z)$ were defined at eq. (3.12) and eq. (3.14) respectively.

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Let us turn to the $A'_x(z)$ and $h_t^{x'}$ near the boundary. From the definition of the master fields, we can solve $h_t^{x'}$ and $A_x(z)$ in terms of the $\Phi_\pm$:

$$h_t^{x'} = z^2\Phi_- + z^2(4Qz - C_-)\frac{\Phi_+ - \Phi_-}{C_+ - C_-}, \qquad A_x = z\frac{\Phi_+ - \Phi_-}{C_+ - C_-}. \tag{3.33}$$

For example $A'(\epsilon)$ can be calculated from the observation:

$$A'(\epsilon) = \frac{\hat{\Pi}_+ - \hat{\Pi}_-}{C_+ - C_-} + \mathcal{O}(k\omega, \omega^2, \epsilon) \tag{3.34}$$

where $\hat{\Pi}_\pm$ are defined in eq. (3.25). Now, we can evaluate the solutions for the fields near the boundary $z = \epsilon$:

$$h_t^{x'} = -\frac{\epsilon}{\alpha^2}\left(k^2\hat{h}_t^x + \omega k\hat{h}_t^x\right) + \frac{\epsilon^2}{\alpha} \cdot \frac{k^2\hat{h}_t^x + \omega k\hat{h}_t^x + 4i\omega Q\hat{A}_x}{i\omega - Dk^2} + \mathcal{O}(\epsilon^3), \tag{3.35}$$

$$h_x^{y'} = \frac{\epsilon}{\alpha^2}\left(k\omega\hat{h}_t^x + \omega^2\hat{h}_t^x\right) - \frac{\epsilon^2}{\alpha} \cdot \frac{k\omega\hat{h}_t^x + \omega^2\hat{h}_t^x + 4QDk\omega\hat{A}_x}{i\omega - Dk^2} + \mathcal{O}(\epsilon^3), \tag{3.36}$$

$$A'(\epsilon) = -QD\alpha \cdot \frac{\left(k^2\hat{h}_t^x + \omega k\hat{h}_t^x + 4i\omega Q\hat{A}_x\right)}{i\omega - Dk^2} + 4QD^2\alpha k^2 (h_t^x)^0 \tag{3.37}$$

$$+ \hat{A}_x\left(i\omega\frac{(3 - Q^2)^2}{9\alpha(1 + Q^2)^2} - \frac{9 + 21Q^2 - Q^4 + 3Q^6}{9(1 + Q^2)^3\alpha^2}k^2\right) + \mathcal{O}(\epsilon^3). \tag{3.38}$$

Notice that α factor were restored and D is the diffusion constant

$$D = \frac{1}{3\alpha(1 + Q^2)}. \tag{3.39}$$

For the Green functions, we need on-shell action that gives finite and quadratic function of the boundary values. Using the fact

$$\delta S = \int \partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi^i}\delta\phi^i\right)dz + \int E.O.M.\delta\phi^i dz, \tag{3.40}$$

and deleting all the contact terms, it is given by

$$S = \frac{l^2\alpha}{\kappa_4^2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\alpha^2}{4z^2}\left(h_t^x h_t^{x'} - f(z)h_x^y h_x^{y'}\right) - f(z)AA' + Q\alpha Ah_t^x\right]_{z\to 0}^1 + \cdots \tag{3.41}$$

The factor $\frac{1}{4}$ comes from $-3/4 + 1$ where $-3/4$ is from the original action while $+1$ is from the Gibbons Hawking term.

Following ref. [4], we discard the contribution of the horizon as a prescription for computing thermal Green's functions in Minkowski space. From the definition, $G_{i;j} = -\delta^2 S/\delta\phi_i\delta\phi_j$, one finds the correlators in the hydrodynamic approximation

$$G_{xt;xt} = \frac{l^2\alpha^2}{2\kappa_4^2} \cdot \frac{k^2}{i\omega - Dk^2}, \quad G_{xy;xy} = \frac{l^2\alpha^2}{2\kappa_4^2} \cdot \frac{\omega^2}{i\omega - Dk^2}, \tag{3.42}$$

$$G_{xt;xy} = \frac{l^2\alpha^2}{2\kappa_4^2} \cdot \frac{\omega k}{i\omega - Dk^2}, \tag{3.43}$$

$$G_{xt;x} = \frac{l^2\alpha^2}{\kappa_4^2} \left(\frac{2i\omega Q}{i\omega - Dk^2} - 4QD^2k^2 \right), \tag{3.44}$$

$$G_{xy;x} = \frac{l^2\alpha^2}{\kappa_4^2} \cdot \frac{2QDk\omega}{i\omega - Dk^2}, \tag{3.45}$$

$$G_{x;x} = \frac{2l^2\alpha}{\kappa_4^2} \left(\frac{4i\omega\alpha Q^2 D}{i\omega - Dk^2} - i\omega\frac{(3 - Q^2)^2}{9\alpha(1 + Q^2)^2} + k^2\frac{(9 + 21Q^2 - Q^4 + 3Q^6)}{9\alpha^2(1 + Q^2)^3} \right). \tag{3.46}$$

The transport coefficients can be read off from the green function. The DC conductivity σ is given by

$$\sigma = -\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}G_{xx}(\omega, k = 0) = \frac{l^2}{\kappa_4^2}\frac{(3 - Q^2)^2}{9(1 + Q^2)^2}. \tag{3.47}$$

Notice that this is dimensionless and vanishes as square of the temperature ($\sim T^2$) near the zero temperature while it is constant in high temperature.

4 Spectral function

To calculate spectral function of current-current operator, we need to know the boundary value of the fields and theirs conjugate momentum. For the numerical computation, it is better to introduce new master variable $\Psi_\pm$ in this section which is different from $\Phi_\pm$ by a factor $1/z$,

$$\Psi_\pm = z\Phi_\pm = \frac{1}{z}h_t^{x\prime} + (C_\pm - 4Qz) A_x. \tag{4.1}$$

We remark that the new master field $\Psi_\pm$ has nothing to do with Ψ in eq.(3.1). The equation of motion in terms of the new master variables are

$$\Psi_\pm'' + \frac{f'}{f}\Psi_\pm' + \frac{1}{f}\left(\frac{\omega^2 - k^2 f}{f} - C_\pm Qz - \frac{f'}{z}\right)\Psi_\pm = 0, \tag{4.2}$$

where $C_\pm$ is

$$C_\pm = \frac{3 + 3Q^2 \pm \sqrt{9(Q^2 + 1)^2 + 16k^2Q^2}}{2Q} = \frac{3(1 + Q^2)}{2Q}\left(1 \pm \sqrt{1 + \left(\frac{4Qk}{3(1 + Q^2)}\right)^2}\right) \tag{4.3}$$

From the equations for master field we can get the spectral function of master fields. We however need the spectral function of original variables not the master fields itself. So let us first find the series solution of h_t^x, A_x which defines the conjugate momentums π_h, π_a

$$
\begin{aligned}
h_t^x &= \hat{h}_t^x - \frac{k}{2}\hat{Z}_1 z^2 + \frac{1}{3}\hat{\pi}_h z^3 + \cdots \\
A_x &= \hat{A}_x + \hat{\pi}_a z + \cdots ,
\end{aligned}
\tag{4.4}
$$

where $\hat{Z}_1 = k\hat{h}_t^x + \omega\hat{h}_y^x$. The coefficient α_i, β_i is computed from eq. (2.7) : The master variables have series solution near the boundary,

$$
\Psi_\pm = \hat{\Psi}_\pm + \hat{\Pi}_\pm z + \cdots .
\tag{4.5}
$$

The eq. (4.1) can be written as a matrix form [21]

$$
\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \mathbf{R} \begin{pmatrix} h_t^{x\prime} \\ A_x \end{pmatrix}, \quad \text{with } \mathbf{R} = \begin{pmatrix} 1/z & C_+ - 4Qz \\ 1/z & C_- - 4Qz \end{pmatrix}.
\tag{4.6}
$$

Similarly, if we introduce the transformation matrix $\mathbf{R}_0$,

$$
\mathbf{R}_0 = \begin{pmatrix} 1 & C_+ \\ 1 & C_- \end{pmatrix},
\tag{4.7}
$$

the boundary values and of master fields and those of original fields are related by R_0.

$$
\begin{pmatrix} \hat{\Psi}_+ \\ \hat{\Psi}_- \end{pmatrix} = \mathbf{R}_0 \begin{pmatrix} -k\hat{Z}_1 \\ \hat{A}_x \end{pmatrix}, \quad \begin{pmatrix} \hat{\Pi}_+ \\ \hat{\Pi}_- \end{pmatrix} = \begin{pmatrix} \hat{\Psi}_+ \mathcal{G}_+ \\ \hat{\Psi}_- \mathcal{G}_- \end{pmatrix} = \mathbf{R}_0 \begin{pmatrix} \hat{\pi}_h \\ \hat{\pi}_a \end{pmatrix}
\tag{4.8}
$$

we use the linear response relation $\hat{\Pi}_\pm = \hat{\Psi}_\pm \mathcal{G}_\pm$ in eq. (4.8). Then the conjugate momentum $\hat{\pi}_h, \hat{\pi}_a$ are written as

$$
\begin{pmatrix} \hat{\pi}_h \\ \hat{\pi}_a \end{pmatrix} = \Lambda \begin{pmatrix} \mathcal{G}_+ \\ \mathcal{G}_- \end{pmatrix}, \quad \text{where} \quad \Lambda = \mathbf{R}_0^{-1}\text{Diag}(\Psi_+, \Psi_-)
$$

$$
\Lambda = \frac{1}{C_+ - C_-} \begin{pmatrix} C_-(k\hat{Z}_1 - C_+\hat{A}_x) & C_+(-k\hat{Z}_1 + C_-\hat{A}_x) \\ -k\hat{Z}_1 + C_+\hat{A}_x & k\hat{Z}_1 - C_-\hat{A}_x \end{pmatrix}
\tag{4.9}
$$

From the boundary action eq.(3.41), we compute two point function in terms of boundary values $(\hat{h}_t^x, \hat{A}_x)$ and conjugate momentum of master field $\hat{\Pi}_\pm$:

$$
G_{xtxt} = \frac{\delta^2 S_{bd}}{\delta\hat{h}_t^x \delta\hat{h}_t^x}, \quad G_{xyxy} = \frac{\delta^2 S_{bd}}{\delta\hat{h}_y^x \delta\hat{h}_y^x}, \quad G_{xx} = \frac{\delta^2 S_{bd}}{\delta\hat{A}_x \delta\hat{A}_x}
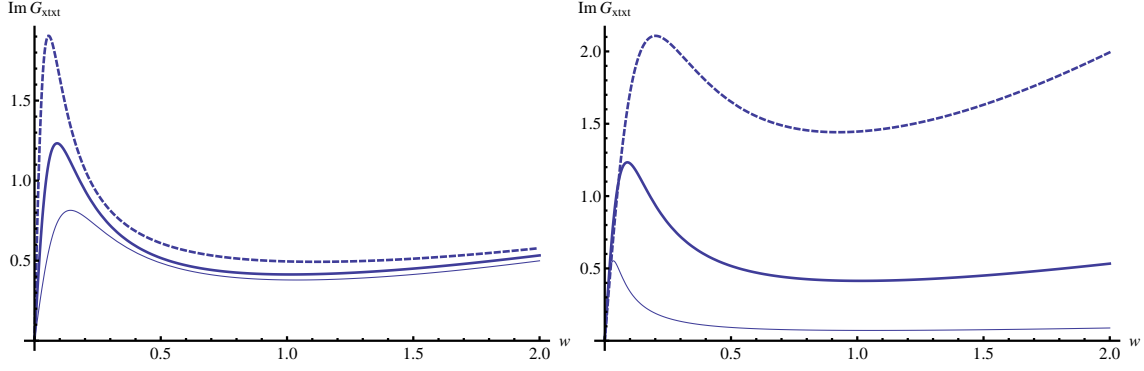\tag{4.10}
$$

Figure 1: Spectral function of xt,xt component, Im $\mathcal{G}_{xtxt}$. Left : with fixed k=0.5 varying Q=0.5(thin), 1(thick), 1.5(dashed). Right : with fixed Q=1, varying k=0.2(thin), 0.5(thick), 1(dashed)

The two point function for h_t^x and h_z^x is related by Ward identity, therefore the correlation functions for each components are

$$
\begin{aligned}
\frac{2\kappa_4^2}{l^2\alpha^3}G_{xt,xt} &= \mathcal{G}_{xt,xt} = k^2\frac{C_-\mathcal{G}_+ - C_+\mathcal{G}_-}{C_+ - C_-}, \quad \frac{2\kappa_4^2}{l^2\alpha^3}G_{xy,xy} = \mathcal{G}_{xy,xy} = \frac{\omega^2}{k^2}\mathcal{G}_{xt,xt} \\
\frac{\kappa_4^2}{l^2\alpha^3}G_{xt,x} &= \mathcal{G}_{x,xt} = k^2\frac{\mathcal{G}_+ - \mathcal{G}_-}{C_+ - C_-} \\
\frac{\kappa_4^2}{l^2\alpha}G_{xx} &= \mathcal{G}_{x,x} = \frac{C_+\mathcal{G}_+ - C_-\mathcal{G}_-}{C_+ - C_-}.
\end{aligned}
\tag{4.11}
$$

4.1   holographic recipe for Green function

In general, the second order differential equation has two independent solution and these two independent solution have two different integration constant. We represent the near the boundary solution by Φ_i and near horizon solution by ϕ_i. Then,

$$
\begin{aligned}
\Phi_1 &= u^{\Delta_-}(1+\cdots), \quad \phi_1 = (1-u)^{-\nu}(1+\cdots) \\
\Phi_2 &= u^{\Delta_+}(1+\cdots), \quad \phi_2 = (1-u)^{\nu}(1+\cdots)
\end{aligned}
\tag{4.12}
$$

where u = 0 is the UV boundary or AdS boundary and u=1 is IR boundary or black hole horizon when we consider finite temperature system. Near the boundary, u=0, Φ_1 and Φ_2 are the local solutions and $\Delta_\pm$ is the solution of indicial equation near the boundary, $\Delta_+ > \Delta_-$ where Δ_+ is the conformal
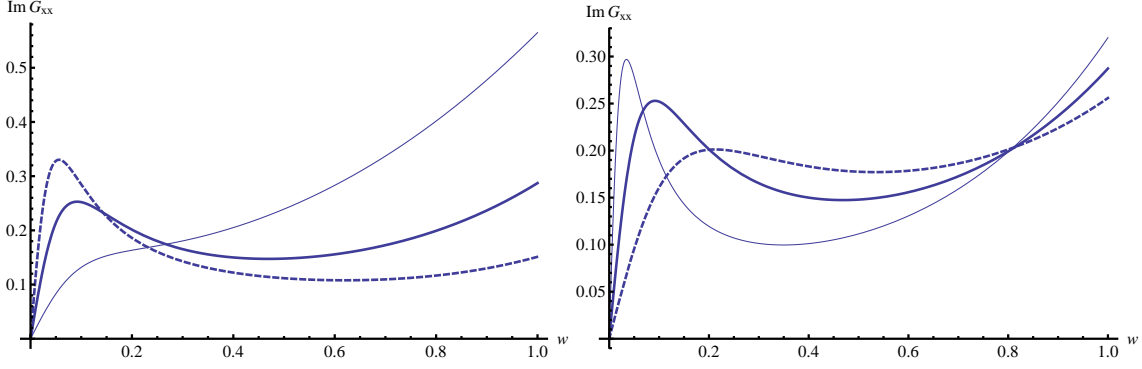
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Figure 2: Spectral function of xx component, $\text{Im}\mathcal{G}_{xx}$. Left : with fixed k=0.5 varying Q=0.5(thin), 1(thick), 1.5(dashed). Right : with fixed Q=1, varying k=0.2(thin), 0.5(thick), 1(dashed)

dimension of an operator and Δ_- is the dimension of the dual source field. Near the horizon, u=1, ϕ_1, ϕ_2 are the local solution and they are identified with infalling and outgoing solution respectively.

$$\mathcal{A}(\mathfrak{w}, \mathfrak{q})\Phi_1 + \mathcal{B}(\mathfrak{w}, \mathfrak{q})\Phi_2 = \mathcal{C}(\mathfrak{w}, \mathfrak{q})\phi_1 + \mathcal{D}(\mathfrak{w}, \mathfrak{q})\phi_2$$

$$E_\alpha = \phi_1(u) = \mathcal{A}(\mathfrak{w}, \mathfrak{q})\Phi_1(u) + \mathcal{B}(\mathfrak{w}, \mathfrak{q})\Phi_2(u). \tag{4.13}$$

where $\mathcal{A}(\mathfrak{w}, \mathfrak{q})$ and $\mathcal{B}(\mathfrak{w}, \mathfrak{q})$ are two integration constants and will be identified with the boundary value of the bulk field and its conjugate variable, respectively. To get the Green function it is first to evaluate the on-shell action,

$$\begin{aligned}
S_{\text{on-shell}} &= \lim_{\epsilon \to 0} \int d^d x \sqrt{g} g^{\alpha\beta} E_\alpha E_{\beta}{}'|_{u=\epsilon} \\
&= \lim_{\epsilon \to 0} \int d^d x \left[u^{-(\Delta_+ + \Delta_- - 1)} \mathcal{A}^2 \frac{\mathcal{A}\Delta_- u^{\Delta_- - 1} + \Delta_+ \mathcal{B}u^{\Delta_+ - 1}}{\mathcal{A}u^{\Delta_-} + \mathcal{B}u^{\Delta_+}} \right]_{u=\epsilon} \\
&= \lim_{\epsilon \to 0} \int d^d x \mathcal{A}^2 \left[\Delta_- u^{\Delta_- - \Delta_+} + \Delta_+ \frac{\mathcal{B}}{\mathcal{A}} \right]_{u=\epsilon}. \tag{4.14}
\end{aligned}$$

The first term in the last line is obviously divergent and it should be renormalized by introducing counter term. The second term is Green function which is obtained by differentiating on-shell action with boundary value.

More specifically, the local Frobenius solutions of our master variable near the boundary is

$$\begin{aligned}
\Psi_\pm(z, \omega, k) &= \hat{\Psi}_\pm(\omega, k)\Psi^A_\pm(z, \omega, k) + \hat{\Pi}_\pm(\omega, k)\Psi^B_\pm(z, \omega, k) \\
\Psi^A_\pm(z, \omega, k) &= 1 + \alpha_1 z + \alpha_2 z^2 + \cdots, \quad \Psi^B_\pm(z, \omega, k) = z(1 + \beta_1 z + \beta_2 z^2 + \cdots) \tag{4.15}
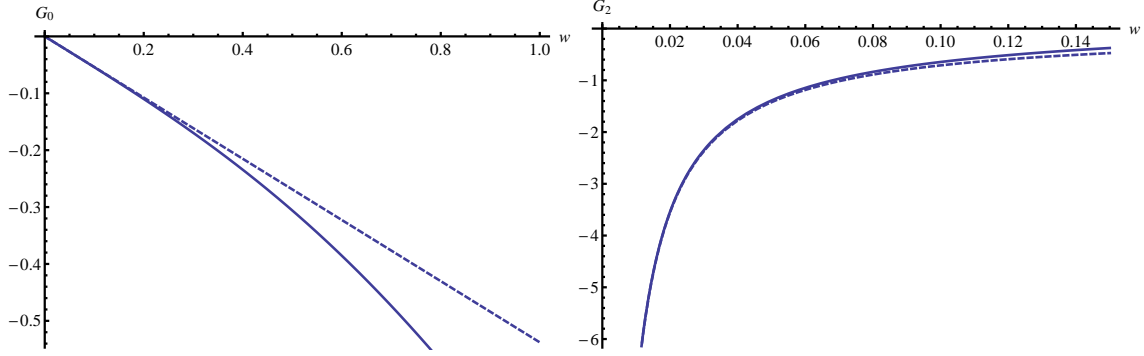\end{aligned}$$

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Figure 3: Comparison between numerically (solid) and analytically computed in hydrodynamic limit (dashed) spectral function with Q=0.5. Left is for $G^0(w)$ and right for $G^0(w)$. These two functions are well matched in small w region but deviated with each other.

What we want to calculate is Green function of each field and it is identified with conjugate momentum of boundary field [6]. In eq. (4.14), $\Delta_-=0$ for our master variable, so that kind of divergence is absent. Then

$$\frac{\hat{\Pi}_\pm(\omega, k)}{\hat{\Psi}_\pm(\omega, k)} = \frac{\partial_z \phi_1(w, k, z)}{\phi_1(w, k, z)}\bigg|_{z=0} \tag{4.16}$$

Eq. (4.11) and eq. (4.16) give us Green functions of original fields. Only remaining is to compute $\Pi_\pm$ numerically.

4.2 small k expansion

In this subsection, we will show how to compute $G_{xx}^{(2)}$ with our previous master field. From eq. (4.11), the expression for current-current Green function is

$$\mathcal{G}_{xx} = \frac{C_+\mathcal{G}_+ - C_-\mathcal{G}_-}{C_+ - C_-} \tag{4.17}$$

and G_{xx} is expanded near k=0 as $G_{xx} = G^{(0)} + k^2 G^{(2)} + \cdots$, so

$$\mathcal{G} = \mathcal{G}^{(0)} + k^2 \mathcal{G}^{(2)} + \cdots \tag{4.18}$$

where $\mathcal{G}^{(0)} = \mathcal{G}_+(w, k = 0)$ because $C_-(0) = 0$ and $\mathcal{G}^{(2)}$ is

$$G^{(2)} = \frac{C'_-(0)}{C_+(0)}\left(\mathcal{G}_+(0) - \mathcal{G}_-(0)\right) + \mathcal{G}'_+(0). \tag{4.19}$$

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To find second order solution for $\mathcal{G}_\pm$, we also need to expand $\mathcal{G}_\pm$ itself as

$$\mathcal{G}_\pm(w,k) = \mathcal{G}_\pm(w, k=0) + k^2 \frac{\partial \mathcal{G}_\pm(w,k)}{\partial k^2}\bigg|_{k=0} + \cdots = \mathcal{G}_\pm^{(0)}(w) + k^2\, \mathcal{G}_\pm^{(2)}(w) \tag{4.20}$$

First we find the infalling solution of the master variable which we call $\phi^{(0)}$ and expand it in momentum, k:

$$\Psi_\pm = \Psi_\pm^{(0)} + k^2 \Psi_\pm^{(2)} + \cdots = \phi_1^{(0)} + k^2 \phi_1^{(2)} + \cdots . \tag{4.21}$$

we then plug it into the eq. 4.1 and expand the equation of motion. The zeroth order equation is

$$\Psi_\pm^{(0)''} + \frac{f'}{f}\Psi_\pm^{(0)'} + \frac{1}{f}\left(\frac{w^2}{f} - Qz\, C_\pm(0) - \frac{f'}{z}\right)\Psi_\pm^{(0)} = 0 \tag{4.22}$$

and second order equation is

$$\Psi_\pm^{(2)''} + \frac{f'}{f}\Psi_\pm^{(2)'} + \frac{1}{f}\left(\frac{w^2}{f} - Qz\, C_\pm(0) - \frac{f'}{z}\right)\Psi_\pm^{(2)} - \frac{1 + Qz\, C_\pm'(0)}{f}\Psi_\pm^{(0)} = 0. \tag{4.23}$$

Numerically, $\mathcal{G}_\pm^{(0)}, \mathcal{G}_\pm^{(2)}$ are computed as

$$\mathcal{G}_\pm^{(0)} = \lim_{\epsilon \to 0} \frac{\phi_1^{(0)'}(\epsilon)}{\phi_1^{(0)}(\epsilon)}, \quad \mathcal{G}_\pm^{(2)} = \lim_{\epsilon \to 0} \frac{\phi_1^{(2)'}(\epsilon)}{\phi_1^{(2)}(\epsilon)}. \tag{4.24}$$

5   Negative index of refraction

At the beginning, let us first review the electric permittivity $\epsilon(\omega)$ and the magnetic permeability $\mu(\omega)$ and their dependence on the Green functions. At the leading order of the electromagnetic coupling, the optical properties of the medium can be described by the linear response to an external electromagnetic field. In an isotropic medium with spatial dispersion, the electric permittivity $\epsilon(\omega, k)$ and the magnetic permeability $\mu(\omega, k)$ depend on both the frequency and the wave vector k. The refractive index for the transverse modes has the form

$$n^2(\omega, k) = \epsilon(\omega, k)\mu(\omega, k) \tag{5.1}$$

Without the dissipation, $\epsilon(\omega, k)$ and $\mu(\omega, k)$ are real. However, in case of that the medium is dissipative, $\epsilon(\omega, k)$ and $\mu(\omega, k)$ will have imaginary part. So that the refractive index n is a complex quantity. The real part of n is the refraction index while the imaginary part encodes the information of dissipation. The refraction can be negative if $Re(\epsilon)$ and $Re(\mu)$ are not simultaneously negative. For the dissipative case, the negative refractive index is equivalent to require

$$n_{DL} = |\epsilon(\omega)|Re(\mu(\omega)) + |\mu(\omega)|Re(\epsilon(\omega)) < 0. \tag{5.2}$$
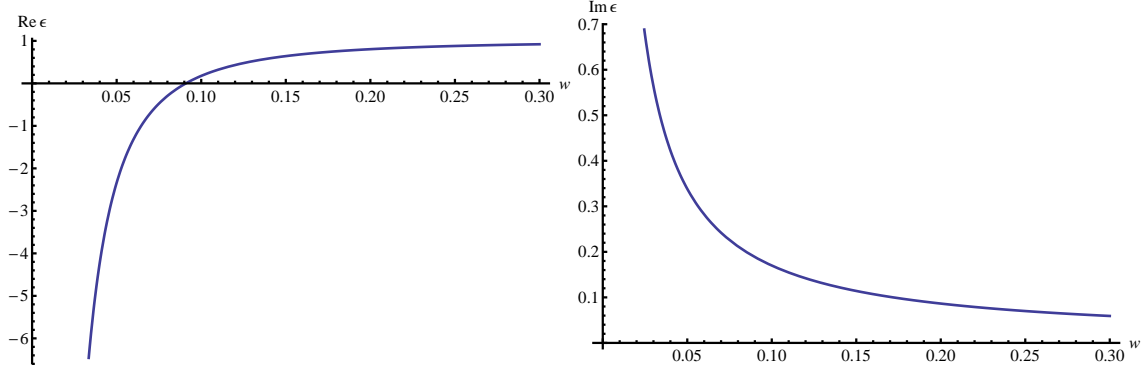
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Figure 4: Real(left) and imaginary(right) part of transverse electric permittivity, $\epsilon(w)$ with $e = 0.5$ and $Q=0.5$.

In the Laudau-Lifshitz approach to electrodynamics of continuous media [24–26], the transverse part of the dielectric tensor is determined by the transverse retarded Green function as follows [26, 27]

$$\epsilon_T = 1 - \frac{4\pi e^2}{\omega^2} G_T(\omega, k),$$

(5.3)

where e^2 is the 3-dimensional electromagnetic coupling and $G_T(\omega, k)$ is the transverse part of the Green function.

By expanding the Green function as $G_T(\omega, k) = G_T^{(0)}(\omega) + k^2 G_T^{(2)}(\omega)$,

$$\mathcal{G}^{(2)} = \frac{9 + 21Q^2 - Q^4 + 3Q^6}{9(1 + Q^2)^3} - \frac{4iQ^2}{9(1 + Q^2)^2 w}$$

(5.4)

the electric permittivity and effective magnetic permeability are given by

$$\epsilon(\omega) = 1 - \frac{4\pi e^2}{\omega^2} G_T^{(0)}(\omega, k) = 1 + i\frac{4\pi e^2 \sigma(\omega)}{\omega}$$

(5.5)

$$\mu(\omega) = \frac{1}{1 + 4\pi e^2 G_T^{(2)}(\omega)}.$$

(5.6)

we use the eq. (3.47) for the expression of $\epsilon(w)$. For sufficiently small value of e, the eq. (5.6) can be expanded as

$$\mu(\omega) = 1 - 4\pi e^2 G_T^{(2)}(\omega) + \mathcal{O}(e^4)$$

(5.7)

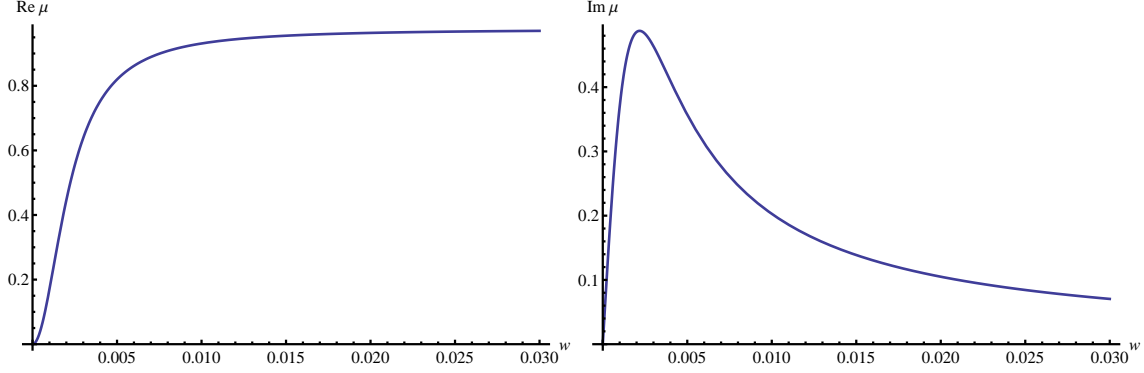From the above equations, we can see that at low enough frequencies, n_{DL} can be negative. From eq.

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Figure 5: Real(left) and imaginary(right) part of transverse magnetic permeability μ(w) with e =0.5 and Q=0.5.

(5.4), (5.5) and (5.6), in hydrodynamic limit, we find that

$$\epsilon(\omega) = 1 + \frac{4\pi e^2 l^2 \alpha}{(1+Q^2)\kappa_4^2}\left(\frac{i}{\omega}\frac{(3-Q^2)^2}{9\alpha(1+Q^2)} - \frac{1}{\omega^2}\frac{4Q^2}{3(1+Q^2)}\right), \tag{5.8}$$

$$\mu(\omega) = \left(1 + \frac{4\pi e^2 l^2 \alpha}{(1+Q^2)\kappa_4^2}\left(\frac{9+21Q^2-Q^4+3Q^6}{9(1+Q^2)^3} - \frac{4iQ^2}{9(1+Q^2)^2 w}\right)\right)^{-1} \tag{5.9}$$

$$= 1 + \frac{i}{\omega}\frac{16\pi e^2 Q^2}{9(1+Q^2)\alpha} + \mathcal{O}(e^4), \quad \text{for } e^2/\omega \ll 1. \tag{5.10}$$

The last equality can not be valid for small enough frequency, causing the difference of our figure for the permeability μ from that in the ref. [20]. Our result suggest that at static limit, the medium exhibiting the character of the super conductivity, i.e, the zero permeability.

It is clear that the value of $n_{DL} = |\epsilon(\omega)|Re(\mu(\omega)) + |\mu(\omega)|Re(\epsilon(\omega))$ can be negative at the low frequency regime, since the real part of $\epsilon(\omega)$ acquires negative value, while the imaginary part of $\mu(\omega)$ is positive.

6   Conclusion

In this paper, we worked out the momentum dependent hydrodynamic analysis for the vector modes of charged AdS$_4$ black hole. We also calculated the spectral function of the dual field theory and as an application, we calculated the permittivity and permeability and found that for low frequency regime, the index of refraction is found to be negative, which support the claim made in ref. [20] for AdS$_5$.

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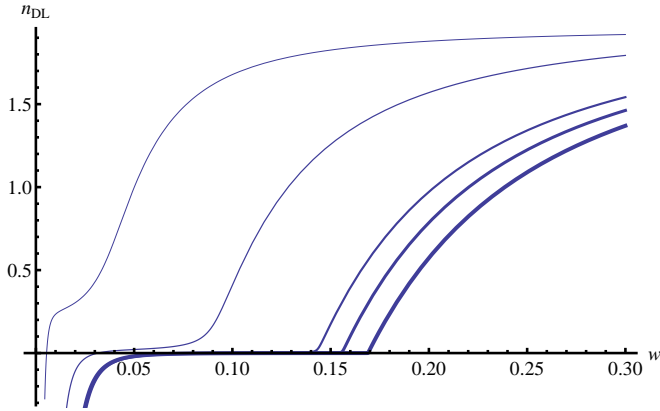Figure 6: Refractive index n_{DL} for Q=0.2, 0.5, 1, 1.2, 1.5 (from left to right).

It will be very interesting if one can extend our analysis to the sound modes, which will enable us to calculate the longitudinal permittivity. It will be also interesting to workout the the case involving the magnetic charge.

Acknowledgments

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