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Connected Primitive Disk Complexes and Genus Two Goeritz Groups of Lens Spaces

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Given a stabilized Heegaard splitting of a three-manifold, the primitive disk complex for the splitting is the subcomplex of the disk complex for a handlebody in the splitting spanned by the vertices of the primitive disks. In this work, we study the structure of the primitive disk complex for the genus-2 Heegaard splitting of each lens space. In particular, we show that the complex for the genus-2 splitting for the lens space $L(p,q)$ with $1 \leq q \leq p/2$ is connected if and only if $p \equiv \pm 1 \pmod{q}$, and describe the combinatorial structure of each of those complexes. As an application, we obtain a finite presentation of the genus-2 Goeritz group of each of those lens spaces, the group of isotopy classes of orientation preserving homeomorphisms of the lens space that preserve the genus-2 Heegaard splitting of it.

1 Introduction

Every closed orientable three-manifold can be decomposed into two handlebodies of the same genus, which is called a *Heegaard splitting* of the manifold. The genus of the handlebodies is called the *genus* of the splitting. The three-sphere admits a Heegaard splitting of each genus $g \geq 0$, and lens spaces and $\mathbb{S}^2 \times \mathbb{S}^1$ admit Heegaard splittings of each genus $g \geq 1$.

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There is a well-known simplicial complex, called the *disk complex*, for a handlebody and in general for an arbitrary irreducible three-manifold with compressible boundary. The vertices of a disk complex are the isotopy classes of essential disks in the manifold. When a given Heegaard splitting is stabilized, we can define the *primitive disk complex* for the splitting, which is the full subcomplex of the disk complex for a handlebody in the splitting spanned by the vertices represented by the primitive disks in the handlebody. Strictly speaking, for each stabilized Heegaard splitting, there are exactly two primitive disk complexes depending on the choice of a handlebody of the splitting. However, for all the Heegaard splittings we will consider in this article, the two primitive disk complexes are isomorphic. So we simply call it *the* primitive disk complex for the splitting.

The first goal of this work is to reveal the combinatorial structure of the primitive disk complex for the genus-2 Heegaard splitting of each lens space $L(p,q)$. For the three-sphere and $\mathbb{S}^2 \times \mathbb{S}^1$, the structure of the primitive disk complex for the genus-2 splitting is well understood from the works [4] and [6]. They are both contractible, and further the complex for the three-sphere is two-dimensional and deformation retracts to a tree in its barycentric subdivision, whereas the complex for $\mathbb{S}^2 \times \mathbb{S}^1$ itself is a tree. In [5], the structure of the primitive disk complex for the genus-2 splitting of the lens space $L(p,1)$ was fully studied. In addition, a generalized version of a primitive disk complex is also studied in [13] for a genus-2 handlebody embedded in the three-sphere. In this work, including the case of $L(p,1)$, we describe the structure of the primitive disk complex for the genus-2 splitting in detail for every lens space. An interesting fact is that not all lens spaces admit connected primitive disk complexes for their genus-2 splitting. In Section 4, we find all lens spaces having connected primitive disk complexes for their genus-2 splittings (Theorem 4.2) and then describe the structure of the complex for each lens spaces (Theorem 4.5).

The next goal is to show that the genus-2 Goeritz group of the lens space having connected primitive disk complex is finitely presented by giving an explicit presentation of each of them. Given a Heegaard splitting of a three-manifold, the *Goeritz group* of the splitting is the group of isotopy classes of orientation preserving homeomorphisms of the manifold that preserve the splitting. When a genus-g Heegaard splitting for a manifold is unique up to isotopy, we call the Goeritz group of the splitting the *genus-g Goeritz group* of the manifold without mentioning a specific splitting of the manifold. The presentations of those groups have been obtained for some manifolds. For example, from the works [9], [16], [1], and [4], a finite presentation of the genus-2 Goeritz group of the three-sphere was obtained and from [6], that of $\mathbb{S}^2 \times \mathbb{S}^1$ was obtained. We refer the reader to [11], [12], [17], [7], and [8] for finite presentations or finite generating sets of the Goeritz groups of several Heegaard splittings. For the genus-2 Goeritz groups of lens spaces, the finite presentations are obtained only for the lens spaces $L(p, 1)$ in [5]. In this work, we show that the genus-2 Goeritz group of each lens space having connected primitive disk complex is finitely presented and obtain a presentation of each of them (Theorem 5.7). Such a lens space $L(p, q)$ with $1 \leq q \leq p/2$ is exactly the one satisfying $p \equiv \pm 1 \pmod{q}$, which includes the case of $L(p, 1)$. The basic idea is to investigate the action of the Georitz group on the connected primitive disk complex of each of the lens spaces, and then calculate the isotropy subgroups of its simplices up to the action of the Goeritz group.

We use the standard notation $L = L(p, q)$ for a lens space in standard textbooks. For example, we refer [15] to the reader. That is, there is a genus-1 Heegaard splitting of L such that an oriented meridian circle of a solid torus in the splitting is identified with a (p, q)-curve on the boundary torus of the other solid torus (fixing oriented longitude and meridian circles of the torus), where $\pi_1(L(p, q))$ is isomorphic to the cyclic group of order $|p|$. The integer p can be assumed to be positive, and it is well known that two lens spaces $L(p, q)$ and $L(p', q')$ are homeomorphic if and only if $p = p'$ and $q'q^{\pm 1} \equiv \pm 1 \pmod{p}$. Thus we will assume $1 \leq q \leq p/2$ for the lens space $L(p, q)$, or $0 < q < p$ sometimes. Further, there is a unique integer q' satisfying $1 \leq q' \leq p/2$ and $qq' \equiv \pm 1 \pmod{p}$, and so, for any other genus-1 Heegaard splitting of $L(p, q)$, we may assume that an oriented meridian circle of a solid torus of the splitting is identified with a $(p, \bar{q})$-curve on the boundary torus of the other solid torus for some $\bar{q} \in \{q, q', p - q', p - q\}$.

Throughout the paper, $(V, W; \Sigma)$ will denote a genus-2 Heegaard splitting of a lens space $L = L(p, q)$. That is, V and W are genus-2 handlebodies such that $V \cup W = L$ and $V \cap W = \partial V = \partial W = \Sigma$ is a genus-2 closed orientable surface, which is called a Heegaard surface in L. Any disks in a handlebody are always assumed to be properly embedded, and their intersection is transverse and minimal up to isotopy. In particular, if a disk D intersects a disk E, then $D \cap E$ is a collection of pairwise disjoint arcs that are properly embedded in both D and E. For convenience, we will not distinguish disks (or union of disks) and homeomorphisms from their isotopy classes in their notation. Finally, $\mathrm{Nbd}(X)$ will denote a regular neighborhood of X and $\mathrm{cl}(X)$ the closure of X for a subspace X of a polyhedral space, where the ambient space will always be clear from the context.

2 Primitive Disk Complexes

Let M be an irreducible three-manifold with compressible boundary. The *disk complex* of M is a simplicial complex defined as follows. The vertices are the isotopy classes of essential disks in M, and a collection of $k+1$ vertices spans a k-simplex if and only if it admits a collection of representative disks which are pairwise disjoint. In particular, if M is a handlebody of genus $g \geq 2$, then the disk complex is $(3g-4)$-dimensional and is not locally finite.

Let D and E be essential disks in M, and suppose that D intersects E transversely and minimally. Let $C \subset D$ be a disk cut off from D by an outermost arc α of $D \cap E$ in D such that $C \cap E = \alpha$. We call such a C an *outermost subdisk* of D cut off by $D \cap E$. The arc α cuts E into two disks, say G and H. Then, we have two disjoint disks E_1 and E_2 which are isotopic to disks $G \cup C$ and $H \cup C$, respectively. We call E_1 and E_2 the *disks from surgery* on E along the outermost subdisk C of D. Since E and D are assumed to intersect minimally, E_1 (and E_2) is isotopic to neither E nor D. Also at least one of E_1 and E_2 is non-separating if E is non-separating. Observe that each of E_1 and E_2 has fewer arcs of intersection with D than E had since at least the arc α no longer counts. For an essential disk D in M intersecting transversely and minimally the union of two disjoint essential disks E and F, we define similarly the disks from surgery on $E \cup F$ along an outermost subdisk of D cut off by $D \cap (E \cup F)$. The following is a key property of a disk complex.

Theorem 2.1. If $\mathcal{K}$ is a full subcomplex of the disk complex satisfying the following condition, then $\mathcal{K}$ is contractible.

- Let E and D be disks in M representing vertices of $\mathcal{K}$. If they intersect each other transversely and minimally, then at least one of the disks from surgery on E along an outermost subdisk of D cut off by $D \cap E$ represents a vertex of $\mathcal{K}$. □

In [4], the above theorem is proved in the case where M is a handlebody, but the proof is still valid for an arbitrary irreducible manifold with compressible boundary. From the theorem, we see that the disk complex itself is contractible, and the *non-separating disk complex* is also contractible, which is the full subcomplex spanned by the vertices of non-separating disks. We denote by $\mathcal{D}(M)$ the non-separating disk complex of M.

Consider the case that M is a genus-2 handlebody V. Then the complex $\mathcal{D}(V)$ is two-dimensional, and every edge of $\mathcal{D}(V)$ is contained in infinitely but countably many two-simplices. For any two non-separating disks in V which intersect each other transversely and minimally, it is easy to see that “both” of the two disks obtained from surgery on one along an outermost subdisk of another cut off by their intersection are

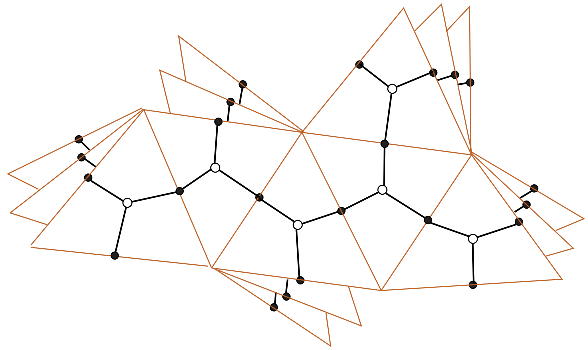


Fig. 1. A portion of the non-separating disk complex $\mathcal{D}(V)$ of a genus-2 handlebody V with its dual complex, a tree.

non-separating. This implies, from Theorem 2.1, that $\mathcal{D}(V)$ and the link of any vertex of $\mathcal{D}(V)$ are all contractible. Thus the complex $\mathcal{D}(V)$ deformation retracts to a tree in the barycentric subdivision of it. Actually, this tree is a dual complex of $\mathcal{D}(V)$. A portion of the non-separating disk complex of V together with its dual tree is described in Figure 1.

Now we return to the genus-2 Heegaard splitting $(V, W; \Sigma)$ of a lens space $L = L(p, q)$. An essential disk E in V is called *primitive* if there exists an essential disk E' in W such that ∂E intersects $\partial E'$ transversely in a single point. Such a disk E' is called a *dual disk* of E, which is also primitive in W having a dual disk E. Note that both $W \cup \mathrm{Nbd}(E)$ and $V \cup \mathrm{Nbd}(E')$ are solid tori. Primitive disks are necessarily non-separating.

The *primitive disk complex* $\mathcal{P}(V)$ for the splitting $(V, W; \Sigma)$ is defined to be the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices of primitive disks in V. From the structure of $\mathcal{D}(V)$, we observe that every connected component of any full subcomplex of $\mathcal{D}(V)$ is contractible. Thus, $\mathcal{P}(V)$ is contractible if it is connected or each of its connected components is contractible otherwise. In Section 4, we describe the complete combinatorial structure of the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of each lens space. In particular, we find all lens spaces whose primitive disk complexes for the genus-2 splittings are connected, and so contractible. We first develop several properties of the primitive disks in the following section, which will play a key role throughout the article.

3 Primitive Disks

3.1 Primitive elements of the free group of rank two

The fundamental group of the genus-2 handlebody is the free group $\mathbb{Z} * \mathbb{Z}$ of rank 2. We call an element of $\mathbb{Z} * \mathbb{Z}$ *primitive* if it is a member of a generating pair of $\mathbb{Z} * \mathbb{Z}$. Primitive elements of $\mathbb{Z} * \mathbb{Z}$ have been well understood. For example, given a generating pair $\{y, z\}$ of $\mathbb{Z}*\mathbb{Z}$, a cyclically reduced form of any primitive element w can be written as a product of terms each of the form $y^\epsilon z^n$ or $y^\epsilon z^{n+1}$, or else a product of terms each of the form $z^\epsilon y^n$ or $z^\epsilon y^{n+1}$, for some $\epsilon \in \{1, -1\}$ and some $n \in \mathbb{Z}$. Consequently, no cyclically reduced form of w in terms of y and z can contain y and y^{-1} (and z and z^{-1}) simultaneously. Furthermore, we have an explicit characterization of primitive elements containing only positive powers of y and z as follows, which is given in Osborne–Zieschang [14].

Lemma 3.1. Suppose that w consists of exactly m z's and n y's where $1 \leq m \leq n$. Then w is primitive if and only if $(m, n) = 1$ and w has the following cyclically reduced form

$$w = w(m, n) = g(1)g(1+m)g(1+2m)\cdots g(1+(m+n-1)m)$$

where the function $g : \mathbb{Z} \to \{z, y\}$ is defined by

$$g(i) = g_{m,n}(i) = \begin{cases} z & \text{if} \quad i \equiv 1, 2, \ldots, m \pmod{(m+n)} \\ y & \text{otherwise.} \end{cases}$$

□

For example, $w(3, 5) = zy^2zy^2zy$ and $w(3, 10) = zy^4zy^3zy^3$.

Let $\{z, y\}$ be a generating pair of the free group of rank 2. Given relatively prime integers p and q with $0 < q < p$, we define a sequence of $(p+1)$ elements $w_0, w_1, \ldots, w_{p-1}, w_p$ in term of z and y as follows.

Define first w_0 to be y^p. For each $j \in \{1, 2, \ldots, p\}$, let $f_j : \mathbb{Z} \to \{z, y\}$ be the function given by

$$f_j(i) = \begin{cases} z & \text{if } \; i \equiv 1, 1+q, 1+2q, \ldots, 1+(j-1)q \pmod{p} \\ y & \text{otherwise,} \end{cases}$$

and then define $w_j = f_j(1)f_j(2)\cdots f_j(p)$. Each of w_j has length p and consists of j z's and $(p-j)$ y's. In particular, $w_1 = zy^{p-1}$, $w_{p-1} = z^{p-q}yz^{q-1}$, and $w_p = z^p$. We call the sequence $w_0, w_1, \ldots, w_p$ the (p, q)-*sequence* of the pair (z, y). For example, the $(8, 3)$-sequence is given by

$$\begin{array}{lll} w_0 = yyyyyyyy & w_1 = zyyyyyyy & w_2 = zyyzyyyy \\ w_3 = zyyzyyzy & w_4 = zzyzyyzy & w_5 = zzyzzyzy \\ w_6 = zzyzzyzz & w_7 = zzzzzyzz & w_8 = zzzzzzzz. \end{array}$$

Observe that w_{p-j} is a cyclic permutation of $\overline{\psi(w_j)}$ for each j, where ψ is the automorphism exchanging z and y, and $\overline{w}$ is the reverse of w. Thus, w_j is primitive if and only if w_{p-j} is primitive. We can find all primitive elements in the sequence as follows.

Lemma 3.2. Let $w_0, w_1, \dots, w_p$ be the (p,q)-sequence of the generating pair $\{z, y\}$ with $0 < q < p$. Let q' be the unique integer satisfying $1 \leq q' \leq p/2$ with $qq' \equiv \pm 1 \pmod p$. Then w_j is primitive if and only if $j \in \{1, q', p-q', p-1\}$. □

Proof. It is clear that w_1 and w_{p-1} are primitive while w_0 and w_p are not.

Claim 1. $w_{q'}$ is primitive.

Proof of Claim 1. We write $w_{q'} = f_{q'}(1)f_{q'}(2)\cdots f_{q'}(p)$, and $w(q', p-q') = g(1)g(1+q')g(1+2q')\cdots g(1+(p-1)q')$ where $g = g_{q',p-q'}$ in the notation in Lemma 3.1. Since $f(i) = z$ if and only if $i \equiv 1 + nq \pmod p$ for some $n \in \{0, 1, \dots, q'-1\}$, it can be directly verified that

$$f_{q'}(i) = \begin{cases} g(1+(i-1)q') & \text{if } qq' \equiv 1 \pmod p \\ g(1+(i+q)q') & \text{if } qq' \equiv -1 \pmod p. \end{cases}$$

Thus, $w_{q'}$ is $w(q', p-q')$ itself if $qq' \equiv 1 \pmod p$ or is a cyclic permutation of it if $qq' \equiv -1 \pmod p$. In either cases, $w_{q'}$ is primitive.

Claim 2. If $1 < j \leq p/2$ and $j \neq q'$, then w_j is not primitive.

Proof of Claim 2. From the assumption, there is a unique integer r satisfying $2 \leq r \leq p-2$ and $qj \equiv r \pmod p$. Suppose, for contradiction, that w_j is primitive. Then, by Lemma 3.1, $(p, j) = 1$ and w_j is a cyclic permutation of $w(j, p-j)$. We write $w_j = f_j(1)f_j(2)\cdots f_j(p)$ and $w(j, p-j) = g(1)g(1+j)g(1+2j)\cdots g(1+(p-1)j)$ where $g = g_{j,p-j}$ as in Lemma 3.1. Then, there is a constant k such that $f_j(i) = g(1+(i-1+k)j)$ for all $i \in \mathbb{Z}$. In particular, $f_j(1+nq) = z = g(1+(nq+k)j)$ for each $n \in \{0, 1, \dots, j-1\}$.

From the definition of $g = g_{j,p-j}$ and the choice of the integer r, we have $1 + (nq+k)j \equiv 1 + nr + kj \equiv 1, 2, \dots, j \pmod p$. Let a_n be the unique integer satisfying $1 + nr + kj \equiv a_n$ with $a_n \in \{1, 2, \dots, j\}$ for each $n \in \{0, 1, \dots, j-1\}$. Observe that $a_n + r \equiv a_{n+1}$ for each $n \in \{0, 1, \dots, j-2\}$, and in particular, $a_0 + r \equiv a_1$. Since $1 \leq a_0 \leq j < p$ and $2 \leq r \leq p-2 < p$, we have only two possibilities: either $a_0 + r = a_1$ or $a_0 + r = a_1 + p$.

First consider the case $a_0 + r = a_1$. Then $r \leq j-1$ and $a_n < a_{n+1}$, consequently $a_0 = 1, a_1 = 2, \dots, a_{j-1} = j$, which implies $r = 1$, a contradiction. Next, if $a_0 + r = a_1 + p$, then $p+1-j \le r$ and $a_n > a_{n+1}$, thus we have $a_0 = j, a_1 = j-1, \ldots, a_{j-1} = 1$, and consequently $r = p-1$, a contradiction again.

By the claims, if $1 \le j \le p/2$, then w_j is primitive only when $j = 1$ or $j = q'$. If $p/2 \le j \le p$, due to the fact that w_{p-j} is a cyclic permutation of $\overline{\psi(w_j)}$, the only primitive elements are $w_{p-q'}$ and w_{p-1}, which completes the proof. ■

A simple closed curve in the boundary of a genus-2 handlebody W represents elements of $\pi_1(W) = \mathbb{Z} * \mathbb{Z}$. We call a pair of essential disks in W a *complete meridian system* for W if the union of the two disks cuts off W into a three-ball. Given a complete meridian system $\{D, E\}$, assign symbols x and y to the circles ∂D and ∂E, respectively. Suppose that an oriented simple closed curve l on ∂W that meets $\partial D \cup \partial E$ transversely and minimally. Then l determines a word in terms of x and y which can be read off from the the intersections of l with ∂D and ∂E (after a choice of orientations of ∂D and ∂E), and hence l represents an element of the free group $\pi_1(W) = \langle x, y \rangle$.

In this set up, the following is a simple criterion for the primitiveness of the elements represented by such simple closed curves.

Lemma 3.3. With a suitable choice of orientations of ∂D and ∂E, if a word corresponding to a simple closed curve l contains one of the pairs of terms: (1) both of xy and xy^{-1} or (2) both of $xy^n x$ and y^{n+2} for $n \ge 0$, then the element of $\pi_1(W)$ represented by l cannot be (a positive power of) a primitive element. □

Proof. Let Σ' be the four-holed sphere cut off from ∂W along $\partial D \cup \partial E$. Denote by d_+ and d_- (by e_+ and e_-, respectively) the boundary circles of Σ' that came from ∂D (from ∂E, respectively).

Suppose first that l represents an element of a form containing both xy and xy^{-1}. Then we may assume that there are two subarcs l_+ and l_- of $l \cap \Sigma'$ such that l_+ connects d_+ and e_+, and l_- connects d_+ and e_- as in Figure 2. Since $|\, l \cap d_+| = |\, l \cap d_-|$ and $|\, l \cap e_+| = |\, l \cap e_-|$, we must have two other arcs m_+ and m_- of $l \cap \Sigma'$ such that m_+ connects d_- and e_+, and m_- connects d_- and e_- (Figure 2).

Consequently, there exists no arc component of $l \cap \Sigma'$ that meets only one of d_+, d_-, e_+, and e_-. That is, any word corresponding to l contains neither $x^{\pm 1} x^{\mp 1}$ nor $y^{\pm 1} y^{\mp 1}$, and hence it is cyclically reduced. Considering all possible directions of the arcs l_+, l_-, m_+ and m_-, each word represented by l must contain both x and x^{-1} (or both y and y^{-1}), which means that l cannot represent (a positive power of) a primitive element of $\pi_1(W)$.

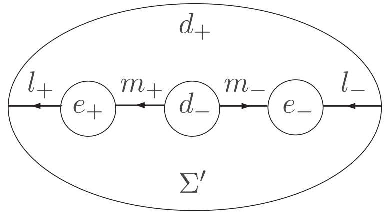


Fig. 2. The four-holed sphere Σ'.

Next, suppose that a word corresponding to l contains x^2 and y^2, which is the case of $n = 0$ in the second condition. Then there are two arcs l_+ and l_- of $l \cap \Sigma'$ such that l_+ connects d_+ and d_-, and l_- connects e_+ and e_-. By a similar argument to the above, we see again that any word corresponding to l is cyclically reduced, but contains both of x^2 and y^2. Thus l cannot represent (a positive power of) a primitive element.

Suppose that a word corresponding to l contains xy^nx and y^{n+2} for $n \geq 1$. Then there are two subarcs α and β of l which correspond to xy^nx and y^{n+2}, respectively. In particular, we may assume that α starts at d_+, intersects ∂E in n points, and ends in d_-, while β starts at e_+, intersects ∂E in its interior in n points, and ends in e_-.

Let m be the subarc of α corresponding to xy. Then m connects two circles d_+ and one of $e_\pm$, say e_+. Choose a disk E^* properly embedded in the three-ball W cut off by $D \cup E$ such that the boundary circle ∂E^* is the frontier of a regular neighborhood of $d_+ \cup m \cup e_+$ in Σ'. Then, E^* is a non-separating disk in W and forms a complete meridian system with D. Assigning the same symbol y to ∂E^*, the arc α determines $xy^{n-1}x$ while β determines y^{n+1}. Thus the conclusion follows by induction. ■

3.2 Primitive disks in a genus-2 handlebody

We recall that $(V, W; \Sigma)$ denotes a genus-2 Heegaard splitting of a lens space $L = L(p, q)$. The primitive disks in V or in W are introduced in Section 2. We call a pair of disjoint, non-isotopic primitive disks in V a *primitive pair* in V. Similarly, a triple of pairwise disjoint, non-isotopic primitive disks is a *primitive triple*. A non-separating disk E_0 properly embedded in V is called *semiprimitive* if there is a primitive disk E' in W disjoint from E_0.

Any simple closed curve on the boundary of the genus-2 handlebody W represents an element of $\pi_1(W)$ which is the free group of rank 2. We interpret primitive disks algebraically as follows, which is a direct consequence of Gordon [9].

Lemma 3.4. Let D be a non-separating disk in V. Then D is primitive if and only if ∂D represents a primitive element of $\pi_1(W)$. □

Note that no disk can be both primitive and semiprimitive since the boundary circle of a semiprimitive disk in V represents the pth power of a primitive element of $\pi_1(W)$.

Lemma 3.5. Let $\{D, E\}$ be a primitive pair of V. Then D and E have a common dual disk if and only if there is a semiprimitive disk E_0 in V disjoint from D and E. □

Proof. The necessity is clear. For sufficiency, let E' be a primitive disk in W disjoint from the semiprimitive disk E_0 in V. It is enough to show that E' is a dual disk of every primitive disk in V disjoint from E_0, since then E' would be a common dual disk of D and E.

Claim: If E is a primitive disk in V dual to E', then E is disjoint from E_0.
Proof of claim. Denote by E_0^+ and E_0^- the two disks on the boundary of the solid torus V cut off by E_0 that came from E_0. Suppose that E intersects E_0. We may assume that C is incident to E_0^+. Considering $|E \cap E_0^+| = |E \cap E_0^-|$, there is a subarc of ∂E whose two endpoints lie in ∂E_0^-, which also intersects $\partial E'$, and hence ∂E intersects $\partial E'$ at least in two points, a contradiction.

Let D be a primitive disk in V disjoint from E_0. Among all the primitive disks in V dual to E', choose one, denoted by E again, such that $|D \cap E|$ is minimal. By the claim, E is disjoint from E_0. Let E_0' be the unique semiprimitive disk in W disjoint from $E \cup E'$. Since $\{E', E_0'\}$ forms a complete meridian system of W, by assigning symbols x and y to oriented $\partial E'$ and $\partial E_0'$, respectively, any oriented simple closed curve on ∂W represents an element of the free group $\pi_1(W) = \langle x, y \rangle$ as in the previous section. In particular, we may assume that ∂E and ∂E_0 represents elements of the form x and y^p, respectively.

Denote by Σ_0 the four-holed sphere ∂V cut off by $\partial E \cup \partial E_0$. Consider Σ_0 as a two-holed annulus with two boundary circles $\partial E_0^\pm$ came from ∂E_0 and with two holes $\partial E^\pm$ came from ∂E. Then $\partial E_0'$ consists of p spanning arcs in Σ_0 which divide Σ_0 into p rectangles, and the two holes $\partial E^\pm$ are contained in a single rectangle. Notice that $\partial E'$ is an arc in the rectangle connecting the two holes (Figure 3.1).

Suppose that D is disjoint from E. Then D is a non-separating disk in V disjoint from $E \cup E_0$, and hence the boundary circle ∂D can be considered as the frontier of a regular neighborhood in Σ_0 of the union of one of the two boundary circles, one of the two holes of Σ_0, and an arc α connecting them. The arc α cannot intersect $\partial E_0'$ in

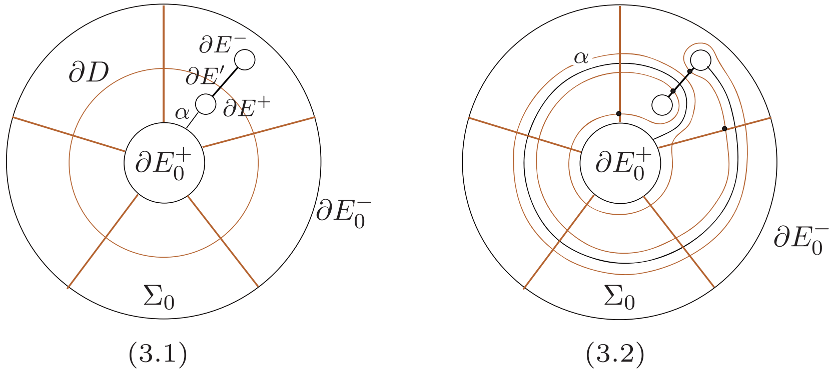


Fig. 3. The two-holed annulus Σ_0 when $p = 5$, for example.

Σ_0, otherwise an element represented by ∂D must contain both of xy and xy^{-1} (after changing orientations if necessary), which contradicts that D is primitive by Lemma 3.3 (see Figure 3.2). Thus α is disjoint from $\partial E'_0$, and consequently D intersects $\partial E'$ in a single point. That is, E' is a dual disk of D (see Figure 3.1).

Suppose next that D intersects E. Let C be an outermost subdisk of D cut off by $D \cap E$. Then one of the resulting disks from surgery on E along C is E_0 and the other, say E', is isotopic to none of E and E_0. The arc $\partial C \cap \Sigma_0$ can be considered as the frontier of a regular neighborhood of the union of a boundary circle of Σ_0 came from ∂E_0 and an arc, denoted by α_0, connecting this circle and a hole came from ∂E. By a similar argument to the above, one can show that α_0 is disjoint from $\partial E'_0$, otherwise D would not be primitive. Consequently, the boundary circle of the resulting disk E_1 from the surgery intersects $\partial E'$ in a single point, which means E_1 is primitive with the dual disk E'. But we have $|D \cap E_1| < |D \cap E|$ from the surgery construction, which contradicts the minimality of $|D \cap E|$. ■

In the proof of Lemma 3.5, if we assume that the primitive disk D also intersects E_0, then the subdisk C of D cut off by $D \cap (E \cup E_0)$ would be incident to one of E and E_0. The argument to show that the resulting disk E_1 from the surgery is primitive with the dual disk E' still holds when C is incident to E_0 and even when D is semiprimitive. This observation suggests the following lemma.

Lemma 3.6. Let E_0 be a semiprimitive disk in V and let E be a primitive disk in V disjoint from E_0. If a primitive or semiprimitive disk D in V intersects $E \cup E_0$, then one of the disks from surgery on $E \cup E_0$ along an outermost subdisk of D cut off by $D \cap (E \cup E_0)$

is either E or E_0, and the other, say E_1, is a primitive disk, which has a common dual disk with E. □

3.3 The link of the vertex of a primitive disk

Again, we have a genus-2 Heegaard splitting $(V, W; \Sigma)$ of a lens space $L = L(p, q)$ and we assume $1 \le q \le p/2$. In this section, we introduce a special subcomplex of the non-separating disk complex $\mathcal{D}(V)$, which we will call a *shell* of the vertex of a primitive disk, and then develop its several properties we need.

Let E be a primitive disk in V. Choose a dual disk E' of E, then we have unique semiprimitive disks E_0 and E'_0 in V and W, respectively, which are disjoint from $E \cup E'$. The circle $\partial E'_0$ is a $(p, \bar{q})$-curve on the boundary of the solid torus $\mathrm{cl}(V - \mathrm{Nbd}(E))$, where $\bar{q} \in \{q, p-q, q', p-q'\}$ and q' is the unique integer satisfying $1 \le q' \le p/2$ and $qq' \equiv \pm 1 \pmod{p}$. We first assume that $\partial E'_0$ is a (p, q)-curve. Assigning symbols x and y to oriented $\partial E'$ and $\partial E'_0$, respectively, as in the previous sections, any oriented simple closed curve on ∂W represents an element of the free group $\pi_1(W) = \langle x, y \rangle$. We simply denote the circles $\partial E'$ and $\partial E'_0$ by x and y, respectively. The circle y is disjoint from ∂E and intersects ∂E_0 in p points, and x is disjoint from ∂E_0 and intersects ∂E in a single point. Thus, we may assume that ∂E_0 and ∂E determine the elements of the form y^p and x, respectively.

Let Σ_0 be the four-holed sphere ∂V cut off by $\partial E \cup \partial E_0$. Denote by $e^{\pm}$ the boundary circles of Σ_0 came from ∂E and similarly $e_0^{\pm}$ came from ∂E_0. The four-holed sphere Σ_0 can be regarded as a two-holed annulus where the two boundary circles are $e_0^{\pm}$ and the two holes $e^{\pm}$. Then the circle y in Σ_0 is the union of p spanning arcs which cuts the annulus into p rectangles, and x is a single arc connecting two holes $e^{\pm}$, where $x \cup e^{\pm}$ is contained in a single rectangle (see the surface Σ_0 in Figure 4).

Any non-separating disk in V disjoint from $E \cup E_0$ and not isotopic to either of E and E_0 is determined by an arc properly embedded in Σ_0 connecting one of $e^{\pm}$ and one of $e_0^{\pm}$. That is, the boundary circle of such a disk is the frontier of a regular neighborhood of the union of the arc and the two circles connected by the arc in Σ_0. Choose such an arc α_0 so that α_0 is disjoint from y, and denote by E_1 the non-separating disk determined by α_0. Observe that there are infinitely many choices of such arcs α_0 up to isotopy, and so are the disks E_1. But the element represented by ∂E_1 has one of the forms $x^{\pm 1} y^{\pm p}$, so we may assume that ∂E_1 represents xy^p by changing the orientations if necessary.

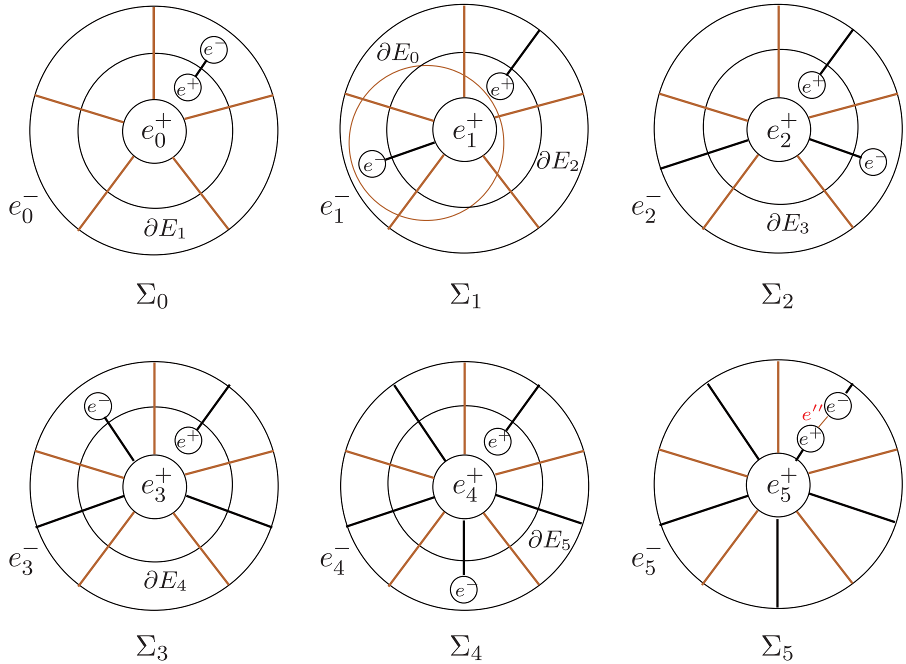


Fig. 4. The disks in a $(5, 2)$-shell in $\mathcal{D}(V)$ for $L(5, 2)$.

Next, let Σ_1 be the four-holed sphere ∂V cut off by $\partial E \cup \partial E_1$. As in the case of Σ_0, consider Σ_1 as a two-holed annulus with boundaries $e_1^{\pm}$ and with two holes $e^{\pm}$ where $e_1^{\pm}$ came from ∂E_1. Then the circle y cuts off Σ_1 into p rectangles as in the case of Σ_0, but two holes e^+ and e^- are now contained in different rectangles. In particular, we can give labels $0, 1, \ldots, p-1$ to the rectangles consecutively so that e^+ lies in the 0th rectangle while e^- in the qth rectangle. The circle x in Σ_1 is the union of two arcs connecting $e_1^{\pm}$ and $e^{\pm}$ contained in the 0th and pth rectangles, respectively.

Now consider a properly embedded arc in Σ_1 connecting one of $e^{\pm}$ and one of $e_1^{\pm}$. Choose such an arc α_1 so that α_1 is disjoint from y and parallel to none of the two arcs of $x \cap \Sigma_1$. Then α_1 determines a non-separating disk, denoted by E_2, whose boundary circle is the frontier of a regular neighborhood of the union of α_1 and the two circles connected by α_1. (If α_1 is isotopic to one of the two arcs $x \cap \Sigma_1$, then the resulting disk is E_0.) Observe that ∂E_2 represents an element of the form $xy^q xy^{p-q}$ (see the surface Σ_1 in Figure 4).

We continue this process in the same way. Then Σ_2 is the four-holed sphere ∂V cut off by $\partial E \cup \partial E_2$, and we choose an arc α_2 in Σ_2 disjoint from y and parallel to none of the arcs $x \cap \Sigma_2$, which determines the disk E_3. The boundary circle ∂E_3 represents an

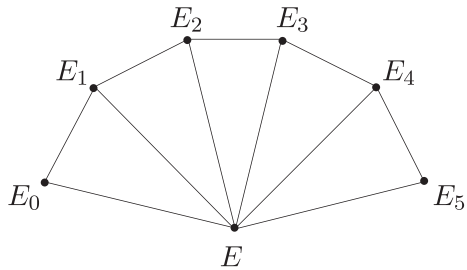


Fig. 5. A $(5, 2)$-shell.

element of the form $xy^qxy^qxy^{p-2q}$. In general, we have a non-separating disk E_j whose boundary circle lies in the four-holed sphere Σ_{j-1}. We finish the process in the pth step to have the disk E_p whose boundary circle lies in Σ_{p-1}. The disks E_{p-1} and E_p represent elements of the form $(xy)^{p-q}y(xy)^{q-1}$ and $(xy)^p$, respectively. Observe that there are infinitely many choices of the arc α_0, and so choices of the disk E_1 as we have seen, but once E_1 have been chosen, the next disks E_j for each $j \in \{1, 2, \ldots, p-1\}$ are uniquely determined.

We call the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices $E_0, E_1, \ldots, E_p$ and E a *shell* centered at the primitive disk E and denote it simply by $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$. In particular, since the circle $\partial E'_0$ is assumed to be a (p, q)-curve in the beginning of the construction, the shell $\mathcal{S}_E$ is called a (p, q)-shell. In general, given a genus-2 splitting of the lens space $L(p, q)$, we might have $(p, \bar{q})$-shell by the same construction, where $\bar{q} \in \{q, p-q, q', p-q'\}$ and q' is the unique integer satisfying $1 \le q' \le p/2$ and $qq' \equiv \pm 1 \pmod{p}$. We observe that there exist infinitely many shells centered at any primitive disk E by the choice of a dual disk E'. Further there exist infinitely many shells centered at E containing the vertex of a semiprimitive disk E_0 disjoint from E. That is, there are infinitely many choices of the primitive disks E_1 disjoint from $E \cup E_0$. On the contrary, once the disk E_1 is chosen, the shell centered at E and containing E_0 and E_1 is uniquely determined. Figure 5 illustrates a $(5, 2)$-shell in $\mathcal{D}(V)$ in the splitting of $L(5, 2)$.

Remark 3.7. For any consecutive vertices $E_j, E_{j+1},$ and E_{j+2} in a shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$, the disk E_j is disjoint from E_{j+1}, and intersects E_{j+2} in a single arc for each $j \in \{0, 1, \ldots, p-2\}$. For example, see ∂E_0, ∂E_2, and ∂E_1 $(= e_1^{\pm})$ in Σ_1 in Figure 4. In general, we have $|E_i \cap E_j| = j - i - 1$ for $0 \le i < j \le p$. This is obvious from the construction. Figure 6 illustrates intersections of E_j with E_{j+2}, E_{j+3} and E_{j+4} in the three-balls V cut off by $E \cup E_{j+1}$, $E \cup E_{j+2}$ and $E \cup E_{j+3,}$ respectively. □

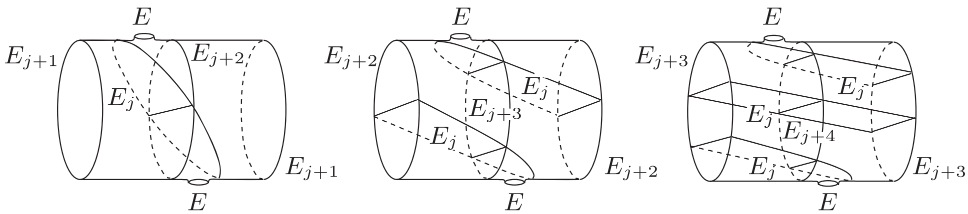


Fig. 6. Intersections of E_j with E_{j+2}, E_{j+3}, and E_{j+4}.

Lemma 3.8. Let $\mathcal{S}_E = \{E_0, E_1, \ldots, E_{p-1}, E_p\}$ be a (p,q)-shell centered at a primitive disk E in V. Then we have

(1) E_0 and E_p are semiprimitive.

(2) E_j is primitive if and only if $j \in \{1, q', p-q', p-1\}$ where q' is the unique integer satisfying $qq' \equiv \pm 1 \pmod p$ and $1 \le q' \le p/2$. □

Proof. (1) E_0 is a semiprimitive disk disjoint from E' from the construction. For the disk E_p, it is easy to find a circle e'' in Σ such that $e'' \cap \Sigma_p$ is an arc which connects the two holes e^+ and e^- and is disjoint from $x \cup y \cup e_p^+ \cup e_p^-$ (see the arc e'' in the surface Σ_5 in Figure 4). Cutting W along $E' \cup E_0'$, we have a three-ball B, and the circle e'' lies in ∂B. Thus, e'' bounds a disk E'' in W which is primitive since e'' intersects ∂E in a single point. The disk E_p is disjoint from E'' and so is semiprimitive.

(2) From the construction, each circle ∂E_j represents the element w_j in the (p,q)-sequence in Section 3.1, by the substitution of z for xy. Thus, the conclusion follows by Lemma 3.2 with Lemma 3.4. ■

Remark 3.9. We have constructed a (p,q)-shell $\mathcal{S}_E$ by assuming $\partial E_0'$ is a (p,q)-curve in the beginning of the construction. If $\mathcal{S}_E$ is a $(p,p-q)$-shell, then we have the same conclusion of Lemma 3.8. If $\mathcal{S}_E$ is a (p,q')-shell or a $(p,p-q')$-shell, the Lemma 3.8 still holds by exchanging q and q' in the conclusion. Also, we observe that a (p,q)-shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_{p-1}, E_p\}$ is identified with the $(p,p-q)$-shell $\mathcal{S}_E' = \{E_p, E_{p-1}, \ldots, E_1, E_0\}$ centered at the same E if we choose the dual disk E'' of E and then choose the primitive disk E_{p-1} disjoint from $E \cup E_p$. □

The following is a generalization of Lemma 3.6.

Lemma 3.10. Let $\mathcal{S}_E = \{E_0, E_1, \ldots, E_{p-1}, E_p\}$ be a shell centered at a primitive disk E in V, and let D be a primitive or semiprimitive disk in V. For $j \in \{1, 2, \ldots, p-1\}$,

(1) if D is disjoint from $E \cup E_j$ and is isotopic to none of E and E_j, then D is isotopic to either E_{j-1} or E_{j+1}, and
(2) if D intersects $E \cup E_j$, then one of the disks from surgery on $E \cup E_j$ along an outermost subdisk C of D cut off by $D \cap (E \cup E_j)$ is either E or E_j, and the other is either E_{j-1} or E_{j+1}. □

Proof. Suppose that D is disjoint from $E \cup E_j$. The boundary circle ∂D lies in the two-holed annulus Σ_j. Thus ∂D can be considered as the frontier of the union of one hole and one boundary circle of Σ_j, and an arc α_j connecting them. By the same argument for the proof of Lemmas 3.5 and 3.6, the arc α_j cannot intersect the arcs of $\partial E'_0 \cap \Sigma_j$ otherwise D would not be (semi)primitive. Thus, the disk D must be either E_{j-1} or E_{j+1}. (Note that if both of E_{j-1} and E_{j+1} are not primitive, then we can say that such a primitive disk D does not exist.) The second statement can be proved in the same manner by considering the arc $\partial C \cap \Sigma_j$ for the outermost subdisk C of D. ■

3.4 Primitive disks intersecting each other

The following is the main theorem of this section.

Theorem 3.11. Given a lens space $L(p,q)$, $1 \leq q \leq p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, suppose that $p \equiv \pm 1 \pmod q$. Let D and E be primitive disks in V which intersect each other transversely and minimally. Then, at least one of the two disks from surgery on E along an outermost subdisk of D cut off by $D \cap E$ is primitive. □

Proof. Let C be an outermost subdisk of D cut off by $D \cap E$. The choice of a dual disk E' of E determines a unique semiprimitive disk E_0 in V which is disjoint from $E \cup E'$. Among all the dual disks of E, choose one, denoted by E' again, so that the resulting semiprimitive disk E_0 intersects C minimally. If C is disjoint from E_0, then, by Lemma 3.6, the disk from surgery on E along C other than E_0 is primitive, having the common dual disk E' with E, and so we are done.

From now on, we assume that C intersects E_0. Then one of the disks from surgery on E_0 along an outermost subdisk C_0 of C cut off by $C \cap E_0$ is E, and the other, say E_1, is primitive having the common dual disk E' with E, by Lemma 3.6 again. Then, we have the shell $\mathcal{S}_E = \{E_0, E_1, E_2, \ldots, E_p\}$ centered at E. Let E'_0 be the unique semiprimitive disk in W disjoint from $E \cup E'$. The circle $\partial E'_0$ would be a $(p, \bar{q})$-curve on the boundary of the solid torus $\mathrm{cl}(V - \mathrm{Nbd}(E \cup E'))$ for some $\bar{q} \in \{q, q', p-q', p-q\}$, where q' satisfies $1 \leq q' \leq p/2$ and $qq' \equiv \pm 1 \pmod p$. We will consider only the case of $\bar{q} = q$. That is, $\partial E_0'$ is a (p,q)-curve and so $\mathcal{S}_E$ is a (p,q)-shell. The proof is easily adapted for the other cases.

If C intersects E_1, then one of the disks from surgery on E_1 along an outermost subdisk C_1 of C cut off by $C \cap E_1$ is E, and the other is either E_0 or E_2 by Lemma 3.10, but it is actually E_2 since we have $|C \cap E_1| < |C \cap E_0|$ from the surgery construction. In general, if C intersects each of $E_1, E_2, \ldots, E_j$, for $j \in \{1, 2, \ldots, p-1\}$, the disk from surgery on E_j by an outermost subdisk C_j of C cut off by $C \cap E_j$, other than E, is E_{j+1}, and we have $|C \cap E_{j+1}| < |C \cap E_j|$. Consequently, we see that $|C \cap E_p| < |C \cap E_0|$, but it contradicts the minimality of $|C \cap E_0|$ since E_p is also a semiprimitive disk disjoint from E. Thus, there is a disk E_j for some $j \in \{1, 2, \ldots, p-1\}$ which is disjoint from C.

Now, denote by E_j again the first disk in the sequence that is disjoint from C. Then the two disks from surgery on E along C are E_j and E_{j+1}, hence C is also disjoint from E_{j+1}. Actually they are the only disks in the sequence disjoint from C. For other disks in the sequence, it is easy to see that $|C \cap E_{j-k}| = k = |C \cap E_{j+1+k}|$ (by a similar observation to the fact that $|E_i \cap E_j| = j - i - 1$ for $0 \le i < j \le p$ in Remark 3.7). If $j \ge p/2$, then we have $|C \cap E_0| = j > p - j - 1 = |C \cap E_p|$, a contradiction for the minimality condition again. Thus, E_j is one of the disks in the first half of the sequence, that is, $1 \le j < p/2$.

Claim. The disk E_j is one of E_1, $E_{q'-1}$ or $E_{q'}$, where q' is the unique integer satisfying $1 \le q' \le p/2$ and $qq' \equiv \pm 1 \pmod{p}$.

Proof of Claim. We have assumed that $p \equiv \pm 1 \pmod{q}$ with $1 \le q \le p/2$, and so $q' = 1$ if $q = 1$, and $p = qq' + 1$ if $q = 2$, and $p = qq' \pm 1$ if $q \ge 3$. Assigning symbols x and y to oriented $\partial E'$ and $\partial E_0'$, respectively, $\partial E_{q'}$ may represent the primitive element of the form $xy^q xy^q \cdots xy^q xy^{q \pm 1}$ if $q \ge 2$ or xy^p if $q = 1$. In general, ∂E_k represents an element of the form $xy^{n_1} xy^{n_2} \cdots xy^{n_k}$ for some positive integers $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = p$ for each $k \in \{1, 2, \ldots, p\}$. Furthermore, since C is disjoint from E_j and E_{j+1}, the word determined by the arc $\partial C \cap \Sigma_j$ is of the form $y^{m_1} xy^{m_2} \cdots xy^{m_{j+1}}$ (or its reverse) when ∂E_{j+1} represents an element of the form $xy^{m_1} xy^{m_2} \cdots xy^{m_{j+1}}$.

If $2 \le j \le q' - 2$, then an element represented by ∂E_{j+1} has the form $xy^q xy^q \cdots xy^q xy^{p-jq}$, and so an element represented by ∂D contains $xy^q x$ and y^{p-jq}, which lies in the part $\partial C \cap \Sigma_j$ of ∂D. We have $q' \ge 4$ in this case, and so $q \ge 2$. Thus, $p - jq = qq' \pm 1 - jq \ge q + 2$. By Lemma 3.3, the disk D cannot be primitive, a contradiction.

Suppose that $q' < j < p/2$. First, observe that $\partial E_{q'+1}$ represents an element of the form $xy^q \cdots xy^q xy$ if $p = qq' + 1$ or $xyxy^{q-1} xy^q \cdots xy^q xy^{q-1}$ if $p = qq' - 1$. Also a word represented by ∂E_{j+1} is obtained by changing one xy^q of a word represented by ∂E_j into $xy^{q-1} xy$ or $xyxy^{q-1}$. Thus, when we write $xy^{n_1} xy^{n_2} \cdots xy^{n_{j+1}}$ a word represented by

∂E_{j+1}, at least one of $n_2, n_3, \ldots, n_j$ must be 1, and one of $n_1, n_2, \ldots, n_{j+1}$ is greater than 2. Since C is disjoint from E_j and E_{j+1}, the word corresponding to $\partial C \cap \Sigma_j$ is of the form $y^{n_1}xy^{n_2}\cdots xy^{n_{j+1}}$, which contains both of xyx and y^n for some $n > 2$. Consequently, by Lemma 3.3, the disk D cannot be primitive, a contradiction again.

From the claim, at least one of the disks from surgery on E along C is either E_1 or $E_{q'}$. The disk E_1 is primitive, and since we assumed that the circle $\partial E'_0$ is a (p,q)-curve on the boundary of the solid torus $\mathrm{cl}(V - \mathrm{Nbd}(E \cup E'))$, the disk $E_{q'}$ is also primitive by Lemma 3.8, which completes the proof. ∎

In the proof of the above theorem, we assumed $\bar{q} = q$, which implied that a resulting disk from surgery is E_1 or $E_{q'}$. The same result holds when $\bar{q} = p - q$. But if we assume $\bar{q} \in \{q', p - q'\}$, then the resulting disk will be E_1 or E_q which are primitive. Together with this observation, assuming that D is disjoint from E, and so taking the disk D instead of an outermost subdisk C in the proof of Therorem 3.11, we have the following result.

Lemma 3.12. Given a lens space $L(p,q)$, $0 < q < p$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, let $\{E, D\}$ be a primitive pair of V. Then, there exists a unique shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E containing D. That is, D is one of $E_0, E_1, \ldots, E_p$. Furthermore, if $\mathcal{S}_E$ is a (p,q)-shell, then the vertex D is one of $E_1, E_{q'}, E_{p-q'}$ or E_{p-1}, where q' is the unique integer satisfying $1 \le q' \le p/2$ and $qq' \equiv \pm 1 \pmod{p}$. □

Let D be an essential disk in V. We denote by V_D the solid torus $\mathrm{cl}(V - \mathrm{Nbd}(D))$. We remark that V_D and its exterior form a genus-1 Heegaard splitting of $L(p,q)$ if and only if D is a primitive disk in V. We refine the above lemma as follows.

Lemma 3.13. Given a lens space $L(p,q)$, $0 < q < p$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, let $\{E, D\}$ be a primitive pair of V. Let $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ and $\mathcal{S}_D = \{D_0, D_1, \ldots, D_p\}$ be the unique shells centered at E and at D containing D and E, respectively. Assume further that $\mathcal{S}_E$ is a (p,q)-shell.

(1) If $\{E, D\}$ has a common dual disk, then $\mathcal{S}_D$ is a (p,q)-shell. Further, E is D_1 or D_{p-1} and D is E_1 or E_{p-1}.

(2) If $\{E, D\}$ has no common dual disk, then $\mathcal{S}_D$ is a (p,q')-shell, where q' is the unique integer with $1 \le q' \le p/2$ and $qq' \equiv \pm 1 \pmod{p}$. Further, D is $E_{q'}$ or $E_{p-q'}$ and E is D_q or D_{p-q}. □

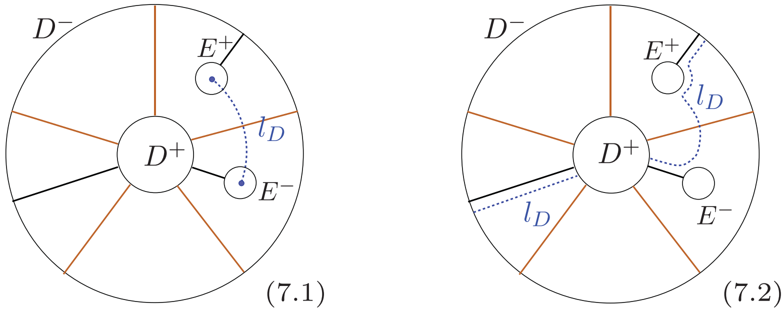


Fig. 7. The loop l_D in the case of $L(5, 2)$.

Proof. Let E' (D', respectively) be the unique dual disks of E (D, respectively) disjoint from E_0 (D_0, respectively), and let E'_0 (D'_0, respectively) be the unique semi-primitive disk in W disjoint from E (D, respectively).

(1) Suppose $\{E, D\}$ has a common dual disk. Then, V_D is isotopic to V_E in $L(p, q)$. This implies that $\partial D'_0$ is also a (p, q)-curve on ∂V_D. Hence, $\mathcal{S}_D$ is a (p, q)-shell as well. It is clear that E is D_1 or D_{p-1} and D is E_1 or E_{p-1} by Lemma 3.5.

(2) Suppose $\{E, D\}$ has no common dual disk. We note that $1 < q < p - 1$ in this case, and so $1 < q' \leq p/2$. By Lemma 3.5 and Lemma 3.12, D is one of $E_{q'}$ and $E_{p-q'}$, and E is one of D_q, $D_{q'}$, D_{p-q} and $D_{p-q'}$.

The solid torus V_D and its exterior form a genus-1 Heegaard splitting of $L(p, q)$. We will show that V_E is not isotopic to the solid torus V_D. Let E' be a dual disk of E that has minimal intersection with D. Let l_D and l_E be the core loops of the solid tori V_D and V_E, respectively. We may assume that l_D and l_E intersect E and D, respectively, once and transversely (see Figure 7.1).

We may move l_D by isotopy in $V \cup \mathrm{Nbd}(E')$ so that l_D lies in ∂V_E (see Figure 7.2). Now the two core circles l_E and l_D lie in the solid torus V_E of which D is a meridian disk. We observe that the circle l_D intersects D in q' points transversely and minimally after an isotopy, while the circle l_E intersects D in a single point. That is, we see that $[l_D] = q'[l_E]$ in $H_1(L(p, q))$ after giving a suitable orientation on each of l_D and l_E. Since $1 < q' \leq p/2$, this implies that V_D and V_E are not isotopic in $L(p, q)$. By the uniqueness of a genus-1 Heegaard surface of $L(p, q)$, V_E is actually isotopic to the solid torus which is the exterior of V_D. This implies that $\partial D'_0$ is a (p, q')-curve on ∂V_D. Thus $\mathcal{S}_D$ is a (p, q')-shell, and hence E is D_q or D_{p-q}. ■

Remark 3.14. If we assume that $\mathcal{S}_E$ is a (p,q')-shell instead of a (p,q)-shell in Lemmas 3.12 and 3.13, the conclusion is obtained by replacing q' by q and vice versa. □

4 The Structure of Primitive Disk Complexes

4.1 Contractibility theorem

The goal of this section is to find all lens spaces whose primitive disk complexes for the genus-2 splittings are connected and so contractible, Theorem 4.2. As in the previous sections, let E be a primitive disk in V with a dual disk E'. The disk E' forms a complete meridian system of W together with the semiprimitive disk E'_0 in W disjoint from $E \cup E'$. Assigning the symbols x and y to the oriented circles $\partial E'$ and $\partial E'_0$, respectively, any oriented simple closed curve, especially the boundary circle of any essential disk in V, represents an element of the free group $\pi_1(W) = \langle x, y \rangle$ in terms of x and y. Let D be a non-separating disk in V. A simple closed curve l on ∂V intersecting ∂D transversely in a single point is called a *dual circle* of D. We say that l is a *common dual circle* of two disks if it is a dual circle of each of the disks. We start with the following lemma.

Lemma 4.1. Let $\{D_1, D_2\}$ be a complete meridian system of V. Suppose that the non-separating disks D_1 and D_2 satisfy the following conditions:

(1) for each $i \in \{1,2\}$, all intersections of ∂D_i and $\partial E'$ have the same sign;
(2) for each $i \in \{1,2\}$, the circle ∂D_i represents an element w_i of the form $(xy^q)^{m_i} xy^{n_i}$, where $0 \le m_i$, $m_1 \neq m_2$ and $n_1 \neq n_2$;
(3) any subarc of $\partial E'$ with both endpoints on ∂D_1 intersects ∂D_2; and
(4) there exists a common dual circle l of D_1 and D_2 on ∂V disjoint from $\partial E'$.

Then, there exists a non-separating disk D_* in V disjoint from $D_1 \cup D_2$ satisfying the following:

(1) all intersections of ∂D_* and $\partial E'$ have the same sign;
(2) ∂D_* represents an element of the form $(xy^q)^{m_1+m_2+1} xy^{n_1+n_2-q}$;
(3) for each $i \in \{1,2\}$, any subarc of $\partial E'$ with both endpoints on ∂D_i intersects ∂D_*; and
(4) for each $i \in \{1,2\}$, there exists a common dual circle of D_i and D_* on ∂V disjoint from $\partial E'$. □

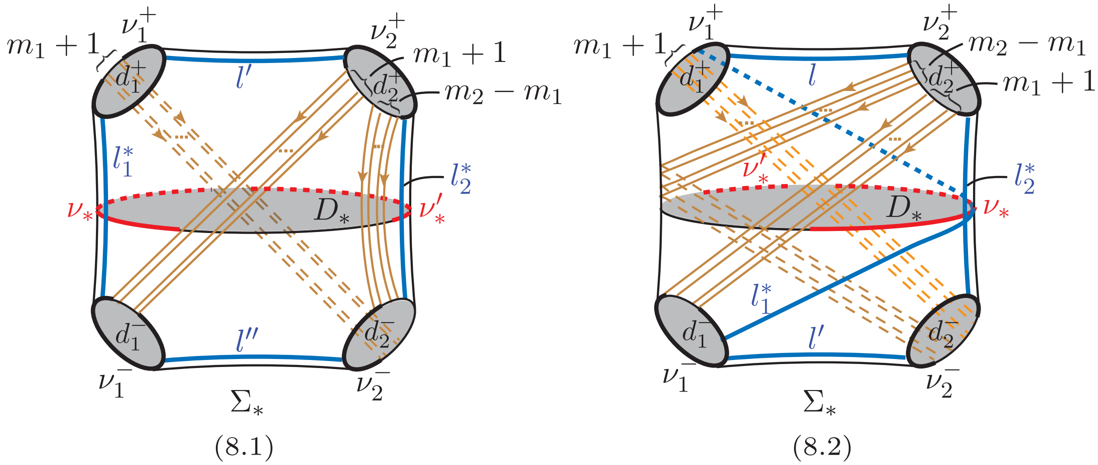


Fig. 8. The four-holed sphere Σ_*. There are two patterns of $\partial E' \cap \Sigma_*$.

Proof. We only prove the case $m_1 < m_2$. For $i \in \{1,2\}$, let ν_i be a connected subarc of ∂D_i that determines the subword y^{n_i} of w_i. Cutting off ∂V by $\partial D_1 \cup \partial D_2$, we obtain the four-holed sphere Σ_*. We denote by $d_i^{\pm}$ the boundary circles of Σ_* coming from ∂D_i, and by $\nu_i^{\pm}$ the subarc of $d_i^{\pm}$ coming from ν_i. By the assumption (2), we may assume without loss of generality that each oriented arc component $\partial E' \cap \Sigma_*$ directs from $d_{i_1}^+$ to $d_{i_2}^-$ for certain $i_1, i_2 \in \{1,2\}$. By the assumptions (3) and (4), the four-holed sphere Σ_* and the arcs $\Sigma_* \cap \partial E'$ and $\Sigma_* \cap l = l' \sqcup l''$ on Σ_* can be drawn as in one of Figures 8.1 and 8.2. In the figure, the arcs $\nu_i^{\pm}$ in $d_i^{\pm}$ are drawn in bold.

Let D_* be the horizontal disk shown in each of Figures 8.1 and 8.2. It is clear that D_* satisfies conditions (1) and (3). For each $i \in \{1,2\}$ the simple closed curve on ∂V obtained from the arc l_i^* depicted in the figure by gluing back along $d_1^{\pm}$ and $d_2^{\pm}$ is a common dual circle of D_i and D_* disjoint from E', hence the condition (4) holds. Moreover, it is easily seen that all but one component ν_* of ∂D_* cut off by $\partial E'$, shown in Figure 8, determine a word of the form y^q. Hence, it suffices to show that the arc ν_* determines a word of the form $y^{n_1+n_2-q}$. From the arcs $\nu_1^+ \cup \nu_2^+$, algebraically n_1+n_2 arcs of $\partial E_0' \cap \Sigma_*$ come down and all of them pass trough $\nu_* \cup \nu_*'$ from above, where the arc ν_*' is shown in Figure 8. Since the arc ν_*' determines a word of the form y^q, the arc ν_* determines a word of the form $y^{n_1+n_2-q}$. ■

Let (D_1, D_2) be an ordered pair of disjoint non-separating disks in V such that the (unordered) pair $\{D_1, D_2\}$ satisfies the conditions of Lemma 4.1. Then, there exists a disk D_* as in the lemma and we again obtain new ordered pairs (D_1, D_*) and (D_*, D_2)

such that both $\{D_1, D_*\}$ and $\{D_*, D_2\}$ satisfy the conditions of the lemma. We call these new pairs (D_1, D_*) and (D_*, D_2) the pairs obtained by *R-replacement* and *L-replacement*, respectively, of (D_1, D_2).

Theorem 4.2. For a lens space $L(p,q)$ with $1 \leq q \leq p/2$, the primitive disk complex $\mathcal{P}(V)$ for a genus-2 Heegaard splitting $(V, W; \Sigma)$ of $L(p,q)$ is contractible if and only if $p \equiv \pm 1 \pmod q$. □

Proof. The "if" part follows directly from Theorem 3.11 and Theorem 2.1. For the "only if" part, we will show that $\mathcal{P}(V)$ is not connected when $p \not\equiv \pm 1 \pmod q$. Suppose that $p \not\equiv \pm 1 \pmod q$. Let m and r be integers such that $p = qm + r$ with $2 \leq r \leq q - 2$. Then, there exist a natural number s and a non-negative integer t with $sr - (t+1)q = 1$. Consider the unique continued fraction expansion

$$s/(t+1) = p_0 + \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cfrac{1}{\ddots + \cfrac{1}{p_k}}}} =: [p_0; p_1, p_2, \ldots, p_k],$$

where $p_i \geq 1$ for $i \in \{0, 1, \ldots, k-1\}$ and $p_k \geq 2$.

The circle $\partial E_0'$ is a $(p, \bar{q})$-curve on the boundary of the solid torus V_E for some $\bar{q} \in \{q, q', p - q', p - q\}$, where q' satisfies $1 \leq q' \leq p/2$ and $qq' \equiv \pm 1 \pmod p$. We will consider only the case of $\bar{q} = q$, that is, $\partial E_0'$ is a (p,q)-curve on the boundary of V_E. The following argument can be easily adapted for the other cases.

Consider any (p,q)-shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ in $\mathcal{D}(V)$ centered at E. Note that the disks E_m and E_{m+1} in the sequence are not primitive since ∂E_m and ∂E_{m+1} represent elements of the form $(xy^q)^{m-1}xy^{q+r}$ and $(xy^q)^m xy^r$, respectively. Set $D_0 = E_m$ and $D_{-1} = E$. Since D_0 and D_{-1} satisfy the conditions of Lemma 4.1, we obtain a new ordered pair (D_0, D_1) by an R-replacement of (D_0, D_{-1}). The disk D_1 is not primitive since ∂D_1 represents an element of the form $(xy^q)^m xy^r$. (Actually, D_1 can be chosen to be the disk E_{m+1} in the sequence.) Applying R-replacements $(p_0 - 1)$ times more, starting at (D_0, D_1), as

$$(D_0, D_1) \to (D_0, D_2) \to \cdots \to (D_0, D_{p_0}),$$

we obtain the pair (D_0, D_{p_0}). Next, we apply L-replacements p_1 times starting at (D_0, D_{p_0}) as

$$(D_0, D_{p_0}) \to (D_{p_0+1}, D_{p_0}) \to (D_{p_0+2}, D_{p_0}) \to \cdots \to (D_{p_0+p_1}, D_{p_0})$$

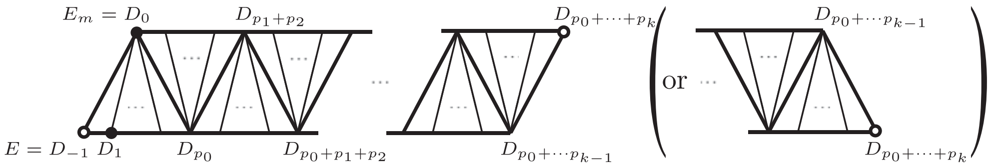


Fig. 9. The portion of $\mathcal{D}(V)$ obtained by L- and R-replacements from (D_0, D_{-1}) following the process that corresponds to the continued fraction $[p_0; p_1, p_2, \ldots, p_k]$. The vertices D_{-1} and $D_{p_0+\cdots+p_k}$ are primitive, whereas D_0 and D_1 are not primitive.

to obtain the pair $(D_{p_0+p_1}, D_{p_0})$. Continuing this process, we finally obtain either the pair $(D_{p_0+\cdots+p_k}, D_{p_0+\cdots+p_{k-1}})$ if k is odd, or $(D_{p_0+\cdots+p_{k-1}}, D_{p_0+\cdots+p_k})$ if k is even, of pairwise disjoint non-separating disks (see Figure 9).

We assign D_0 and D_{-1} the rational numbers $1/0$ and $0/1$, respectively. We inductively assign rational numbers to the disks appearing in the above process as follows. Let (D_*, D_{**}) be an ordered pair of non-separating disks appearing in the process. Assume that we have already assigned D_* and D_{**} rational numbers a_1/b_1 and a_2/b_2, respectively. Then, we assign the next disk obtained by L or R-replacement of (D_*, D_{**}) the rational number $(a_1 + a_2)/(b_1 + b_2)$.

Claim. If a disk D_j, for $-1 \leq j \leq p_0 + \cdots + p_k$, appearing in the above process is assigned a rational number a/b, then ∂D_j represents an element of the form $(xy^q)^d xy^{ar-(b-1)q}$ for some non-negative integer d.

Proof of Claim. If $j = -1$, then $a/b = 0/1$ and $\partial D_{-1} = \partial E$ represents x and we have $ar - (b-1)q = 0$. If $j = 0$, then $a/b = 1/0$ and $\partial D_0 = \partial E_m$ represents an element of the form $(xy^q)^{m-1} xy^{q+r}$ and we have $ar - (b-1)q = q + r$.

Assume that the claim is true for any D_i with i less than j and that D_j is obtained from (D_*, D_{**}). If D_* and D_{**} are assigned rational numbers a_1/b_1 and a_2/b_2, respectively, then D_j is assigned $(a_1 + a_2)/(b_1 + b_2)$ by definition. By the assumption, ∂D_* and ∂D_{**} determine elements of the forms $(xy^q)^{d_1} xy^{a_1 r-(b_1-1)q}$ and $(xy^q)^{d_2} xy^{a_2 r-(b_2-1)q}$, respectively, for some non-negative integers d_1 and d_2. By Lemma 4.1, the circle ∂D_j determines an element of the form $(xy^q)^{d_1+d_2+1} xy^{(a_1+a_2)r-(b_1+b_2-1)q}$, and hence the induction completes the proof.

Due to well-known properties of the Farey graph, see, for example, Hatcher–Thurston [10], $D_{p_0+\cdots+p_k}$ is assigned $s/(t+1)$. Therefore, by the claim, $\partial D_{p_0+\cdots+p_k}$ determines an element of the form $(xy^q)^d xy^{sr-tq}$, hence $(xy^q)^d xy^{q+1}$. This implies that $D_{p_0+\cdots+p_k}$ is primitive.

Now, we focus on the four disks D_{-1}, D_0, D_1, and $D_{p_0+\cdots+p_k}$. Since the dual complex of the disk complex $\mathcal{D}(V)$ is a tree, and the disks D_0 and D_1 are not primitive, the primitive disks D_{-1} and $D_{p_0+\cdots+p_k}$ belong to different components of $\mathcal{P}(V)$. This implies that $\mathcal{P}(V)$ is not connected. ■

4.2 The structures of primitive disk complexes

In this section, we describe the combinatorial structure of the primitive disk complex for the genus-2 Heegaard splitting of each lens space. We say simply that a primitive pair has a common dual disk if the two disks of the pair have a common dual disk.

Theorem 4.3. Given a lens space $L(p,q)$, $1 \le q \le p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, each primitive pair in V has a common dual disk if and only if $q = 1$. In this case, if $p \ge 3$, the pair has a unique common dual disk, and if $p = 2$, the pair has exactly two disjoint common dual disks, which form a primitive pair in W. □

Proof. Suppose that $q = 1$, and let $\{D, E\}$ be any primitive pair of V. By Lemma 3.12, there is a shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E, in which D is E_1 (here, we have $q' = q = 1$). By Lemma 3.5, D and E have a common dual disk.

Now, let E' be a common dual disk of D and E. Let E'_0 be the unique semiprimitive disk in W disjoint from $E \cup E'$. We recall that E'_0 is the meridian disk of the solid torus $\mathrm{cl}(W - \mathrm{Nbd}(E'))$. Then, $\partial E'_0$ intersects ∂D in p points. Cut the surface ∂W along the boundary circles $\partial E'$ and $\partial E'_0$ to obtain the four-holed sphere Σ'. In Σ', the boundary circle ∂E is a single arc connecting two boundary circles of Σ' that came from $\partial E'$. But the boundary circle ∂D in Σ' consists of $(p-1)$ arcs connecting two boundary circles that came from $\partial E'_0$ together with two arcs connecting $\partial E'$ and $\partial E'_0$ as in Figure 10.1. Observe

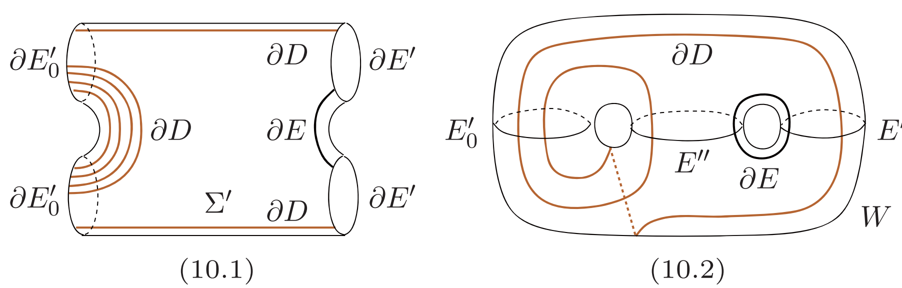


Fig. 10. (10.1) ∂E and ∂D lying in the four-holed sphere Σ' (when $p = 5$ for example). (10.2) Two common dual disks E' and E'' of D and E for $L(2,1)$.

that if there is a common dual disk of D and E other than E', then it cannot intersect $E' \cup E'_0$ otherwise it intersects ∂D or ∂E in more than one points. Thus the boundary of any common dual disk E'' of D and E other than E' is a circle inside Σ', and hence, from the figure, it is obvious that one more common dual disk E'' other than E' exists if and only if $p = 2$, and such an E'' is unique in this case (Figure 10.2).

Conversely, suppose that every primitive pair has a common dual disk. Choose any shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ in $\mathcal{D}(V)$ centered at a primitive disk E. Then one of the disks $E_{q'}$ and E_q is primitive, where q' satisfies $1 \leq q' \leq p/2$ and $qq' = \pm 1 \pmod{p}$, which forms a primitive pair with E. If $\{E, E_{q'}\}$ is a primitive pair, then it has a common dual disk, and so, by Lemma 3.5, there is a semiprimitive disk in V disjoint from E and $E_{q'}$. The only possible semiprimitive disk disjoint from E and $E_{q'}$ is $E_{q'-1}$ or $E_{q'+1}$ by Lemma 3.10, that is, $E_{q'-1} = E_0$ or $E_{q'+1} = E_p$. In any cases, we have $q = 1$ (the latter case implies $(p, q) = (2, 1)$ since we assumed $1 \leq q' \leq p/2$). The same conclusion holds in the case where $\{E, E_q\}$ is a primitive pair. ■

It is clear that any primitive disk is a member of infinitely many primitive pairs. But a primitive pair can be contained at most two primitive triples, which is shown as follows.

Theorem 4.4. Given a lens space $L(p, q)$, for $1 \leq q \leq p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$ of $L(p, q)$, there is a primitive triple in V if and only if $q = 2$ or $p = 2q + 1$. In this case, we have the following refinements.

(1) If $p = 3$, then each primitive pair is contained in a unique primitive triple.
(2) If $p = 5$, then each primitive pair having a common dual disk is contained in a unique primitive triple, and each having no common dual disk is contained in exactly two primitive triples.
(3) If $p \geq 7$, then each primitive pair having a common dual disk is contained either in a unique or in no primitive triple, and each having no common dual disk is contained in a unique primitive triple.
(4) Further, if $p = 3$, then each of the three primitive pairs in any primitive triple in V has a unique common dual disk, which form a primitive triple in W. If $p \geq 5$, then exactly one of the three primitive pairs in any primitive triple has a common dual disk, which is unique. □

Proof. Note that $L(2q+1, q)$ is homeomorphic to $L(2q+1, 2)$. We prove first the “if” part together with the refinements. Suppose that $q = 2$ or $p = 2q + 1$, and let $\{D, E\}$ be any primitive pair of V. By Lemma 3.12, there is a unique shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E containing D. We may assume that D is one of E_1, E_2 or E_q.

(1) If $p = 3$, the disk D is E_1, and so E_2 is the unique primitive disk disjoint from $E \cup E_1$ by Lemma 3.10. Thus $\{D, E\}$ is contained in the unique primitive triple $\{D, E, E_2\}$.

(2) If $p = 5$, then the disk D is either E_1 or E_2. If $\{D, E\}$ has a common dual disk, then D is E_1, and they are contained in the unique primitive triple $\{D, E, E_2\}$. If $\{D, E\}$ has no common dual disk, then D is E_2, and they are contained in exactly two primitive triples $\{D, E, E_1\}$ and $\{D, E, E_3\}$.

(3) If $p \geq 7$, then D is either E_1, E_2 or E_q. Observe that if one of E_2 and E_q is primitive, then the other is not, while E_1 is always primitive. If $\{D, E\}$ has no common dual disk, then D is E_2 or E_q. In this case, $\{D, E\}$ is contained in the unique primitive triple $\{D, E, E_1\}$ if D is E_2, or in the unique triple $\{D, E, E_{q+1}\}$ if D is E_q. Suppose next that $\{D, E\}$ has a common dual disk. Then D is E_1, and hence $\{D, E\}$ is either contained in a unique primitive triple or contained in no primitive triple, according as E_2 is primitive or not.

(4) Let $\{D, E, F\}$ be any primitive triple in V, and let $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ be the unique shell centered at E containing D. Again, we may assume that D is one of E_1, E_2 or E_q. Suppose that $p = 3$. Then, we have $D = E_1$ and $F = E_2$ in the shell $\mathcal{S}_E = \{E_0, E_1, E_2, E_3\}$. The primitive pairs $\{E, D\} = \{E, E_1\}$ and $\{E, F\} = \{E, E_2\}$ in the triple have unique common dual disks, say E' and E'', respectively, by Lemma 3.5 and Theorem 4.3. Further, $\{E', E''\}$ is a primitive pair in W (in fact, $\partial E''$ is the circle e'' in the proof of Lemma 3.8). Furthermore, exchanging the roles of D and E, there exists the unique shell $\mathcal{S}_D = \{D_0, D_1, D_2, D_3\}$ centered at D containing E. Here, we have $D = E_1$, $D_0 = E_0$, $D_1 = E$, and $D_2 = E_2 = F$. The primitive pair $\{D, D_2\} = \{D, F\}$ has a unique common dual disk, say E''', forms a primitive pair $\{E', E'''\}$ with the common dual disk E' of $\{D, E\} = \{D, D_1\}$. Finally, considering the unique shell centered at F containing E, we see that $\{E'', E'''\}$ is also a primitive pair in W. Thus, $\{E', E'', E'''\}$ is a primitive triple in W.

Next, suppose that $p \geq 5$, and let $\{D, E, F\}$ be any primitive triple of V. Suppose, for contradiction, that at least two of the primitive pairs, say $\{D, E\}$ and $\{E, F\}$, in the triple have common dual disks. Then, in the unique shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E containing D, the disk D must be E_1 by Lemma 3.5. Moreover, the disk F is E_2 by Lemma 3.10, and the disk E_3 is semiprimitive, that is, E_p by Lemma 3.5 again. Thus, we must have $p = 3$, a contradiction.

Conversely, suppose that there is a primitive triple $\{D, E, F\}$ in V. Again, we consider the unique shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E containing D. Then $\mathcal{S}_E$ is a $(p, \bar{q})$-shell for some $\bar{q} \in \{q, q', p - q', p - q\}$, where q' is the unique integer satisfying

$qq' \equiv \pm 1 \pmod{p}$ and $1 \leq q' \leq p/2$. We first consider the case $\bar{q} = q$. Then, we may assume that D is E_1 or $E_{q'}$ by Lemma 3.12. If D is E_1, then F is E_2 by Lemma 3.10, and so $q' = 2$ by Lemma 3.8. Thus $p = 2q + 1$. If D is $E_{q'}$, then F is $E_{q'-1}$ or $E_{q'+1}$ by Lemma 3.10 again. That is, $q' - 1 = 1$ or $q' + 1 = p - q'$ by Lemma 3.8 again. Thus $p = 2q + 1$ or $q = 2$. We have the same argument for the other cases, $\bar{q} \in \{q', p - q', p - q\}$. ∎

Now, we are ready to give a precise description of the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of each lens space. For convenience, we classify all the edges and two-simplices of $\mathcal{P}(V)$ as follows.

(1) An edge of $\mathcal{P}(V)$ is called an *edge of type*-0 (*type*-1, *type*-2, respectively) if a primitive pair representing the end vertices of the edge has no common dual disk (has a unique common dual disk, has exactly two common dual disks which form a primitive pair in W, respectively).

(2) A two-simplex of $\mathcal{P}(V)$ is called a *two-simplex of type*-1 (*of type*-3, respectively) if exactly one of the three primitive pairs in the primitive triple representing the three edges of the two-simplex has a unique common dual disk (if all the three pairs have unique common dual disks which form a primitive triple in W, respectively).

By Theorems 4.3 and 4.4, we see that each of the edges and two-simplices of $\mathcal{P}(V)$ is one of those types in the above. In the following theorem, we describe the combinatorial structure of $\mathcal{P}(V)$ for each of the lens spaces, which is a direct consequence of Theorems 4.2, 4.3, and 4.4.

Theorem 4.5. Given any lens space $L(p, q)$, $1 \leq q \leq p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, if $p \equiv \pm 1 \pmod{q}$, then the primitive disk complex $\mathcal{P}(V)$ is contractible and we have one of the following cases.

(1) If $q \neq 2$ and $p \neq 2q + 1$, then $\mathcal{P}(V)$ is a tree, and every vertex has infinite valency. In this case,
   i. if $p = 2$ and $q = 1$, then every edge is of type-2.
   ii. if $p \geq 4$ and $q = 1$, then every edge is of type-1.
   iii. if $q \neq 1$, then every edge is of either type-0 or type-1, and infinitely many edges of type-0 and of type-1 meet in each vertex.

(2) If $q = 2$ or $p = 2q + 1$, then $\mathcal{P}(V)$ is two-dimensional, and every vertex meets infinitely many two-simplices. In this case,

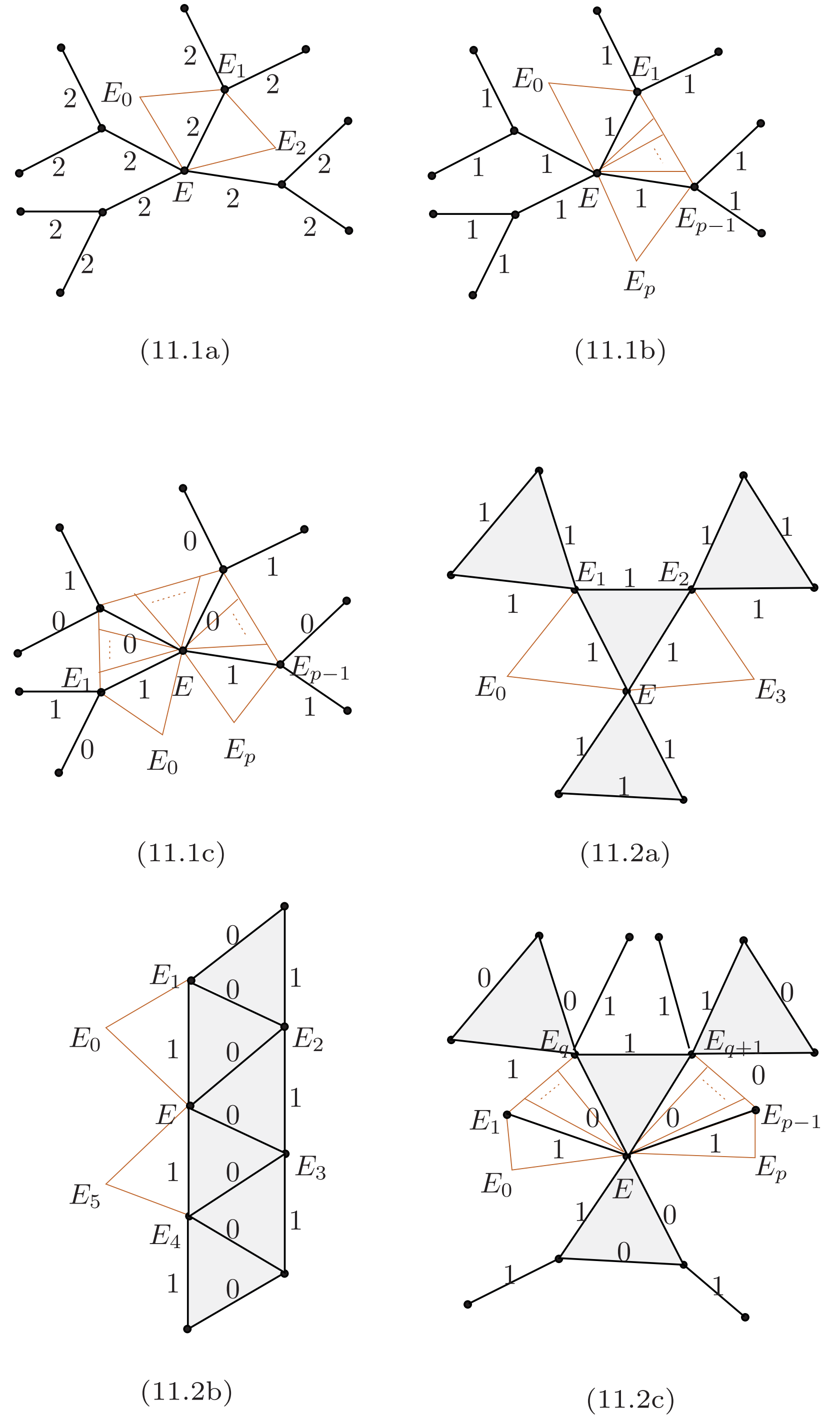


Fig. 11. A portion of each primitive disk complex $\mathcal{P}(V)$ together with the associated shells in $\mathcal{D}(V)$. Each number designates the type of the edge.

i. if $p = 3$, then every edge is of type-1, every two-simplex is of type-3, and every edge is contained in a unique two-simplex.

ii. if $p = 5$, then every edge is of either type-0 or type-1, and every two-simplex is of type-1. Every edge of type-0 is contained in exactly two two-simplices, while every edge of type-1 in a unique two-simplex.

iii. if $p \geq 7$, then every edge is of either type-0 or type-1, and every two-simplex is of type-1. Every edge of type-0 is contained in a unique two-simplex. Every edge of type-1 is contained in a unique two-simplex or in no two-simplex.

If $p \not\equiv \pm 1 \pmod q$, then $\mathcal{P}(V)$ is not connected, and it consists of infinitely many tree components. All the tree components are isomorphic to each other. Any vertex of $\mathcal{P}(V)$ has infinite valency, and further, infinitely many edges of type-0 and of type-1 meet in each vertex. □

Figure 11 illustrates a portion of each of the contractible primitive disk complexes $\mathcal{P}(V)$ classified in the above, together with its surroundings in $\mathcal{D}(V)$. We label simply E or E_j for the vertices represented by disks E or E_j. In the case of (2)-ii of the theorem (Figure 11.2b), the complex $\mathcal{P}(V)$ for $L(5, 2)$, every edge is contained a unique "band." The edges in the boundary of a band are of type-1, while the edges inside a band are of type-0. The whole figure of $\mathcal{P}(V)$ for $L(5, 2)$ can be imagined as the union of infinitely many bands such that any of two bands are disjoint from each other or intersects in a single vertex. In the case of (2)-iii of the theorem, there are two kind of shells $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ in $\mathcal{P}(V)$ centered at a primitive disk E. The first one has primitive disks E_1, E_q, E_{p-q} and E_{p-1}, while the second one has E_1, E_2, E_{p-2} and E_{p-1}. Figure 11.2c illustrates an example of the first one.

5 The Genus-2 Goeritz Groups of Lens Spaces

5.1 The primitive disks under the action of the Goeritz group

By Bonahon–Otal [3] each lens space admits a unique Heegaard surface of each genus $g \geqslant 1$ up to isotopy. Further, they showed that the two handlebodies of each genus-g Heegaard splitting are isotopic to each other when $g \geq 2$. However, the genus-1 Heegaard splitting of a lens space is somewhat more rigid in the following sense.

Lemma 5.1 (Bonahon [2]). There exists an orientation-preserving homeomorphism of $L(p,q)$ that exchanges the two solid tori of the genus-1 Heegaard splitting if and only if $q^2 \equiv 1 \pmod p$. □

Given a genus-g Heegaard splitting of a three-manifold, the *Goeritz group* of the splitting is the group of isotopy classes of orientation-preserving homeomorphisms of the manifold that preserve each of the handlebodies of the splitting setwise. By Lemma 5.1, the Goeritz group of a splitting for each lens space depends only on the genus of the splitting, and hence we say *the genus-g Goeritz group of a lens space* without mentioning a specific genus-g splitting of it. We denote by $\mathcal{G} = \mathcal{G}_{L(p,q)}$ the genus-2 Goeritz group of $L(p,q)$. We recall that $(V, W; \Sigma)$ is a genus-2 Heegaard splitting of a lens space $L(p,q)$ with $1 \le q \le p/2$. We denote by V_D the solid torus $\mathrm{cl}(V - \mathrm{Nbd}(D))$ where D is an essential non-separating disk in V.

Throughout the section, we will assume that $p \equiv \pm 1 \pmod q$, that is, the primitive disk complex $\mathcal{P}(V)$ is connected. Further we fix the following:

- A primitive disk E in V.
- A (p,q)-shell $\mathcal{S}_E = \{E_0, E_1, \ldots, E_p\}$ centered at E.
- The unique (p,q')-shell $\mathcal{S}_D = \{D_0, D_1, \ldots, D_p\}$ centered at $D = E_{q'}$ such that $E = D_q$, where q' is the unique integer satisfying $qq' \equiv \pm 1 \pmod p$ and $1 \le q' \le p/2$.

We use the above four primitive disks E, D, E_1, D_1 to describe the orbits of the action of the genus-2 Goeritz group to the set of primitive pairs. Note that if $q = 1$, then $D = E_1$ and $E = D_1$.

Lemma 5.2. If $q^2 \equiv 1 \pmod p$, the action of the Goeritz group $\mathcal{G}$ on the set of vertices of the primitive disk complex $\mathcal{P}(V)$ is transitive. If $q^2 \not\equiv 1 \pmod p$, the action of $\mathcal{G}$ on the set of vertices of $\mathcal{P}(V)$ has exactly two orbits $\mathcal{G} \cdot \{E\}$ and $\mathcal{G} \cdot \{D\}$. □

Proof. Suppose first that $q^2 \equiv 1 \pmod p$. By Lemma 5.1, there exists an orientation-preserving homeomorphism ι of $L(p,q)$ that exchanges the solid tori of a genus-1 Heegaard splitting. By the uniqueness of the genus-2 Heegaard splitting for $L(p,q)$ up to isotopy, we can assume that ι preserves V, that is, $\iota \in \mathcal{G}$. Let F be an arbitrary primitive disk in V. Then, the solid torus V_F (and $V_{\iota(F)}$) is isotopic to V_E. Thus by the uniqueness of stabilization, there exists an element $f \in \mathcal{G}$ such that $f(E) = F$ or $\iota(F)$. This implies that $\{F\} \in \mathcal{G} \cdot \{E\}$.

Next, suppose that $q^2 \not\equiv 1 \pmod{p}$. As in the proof of Lemma 3.13, V_D is isotopic to the exterior of V_E in $L(p,q)$. If there exists an element $f \in \mathcal{G}$ such that $f(D) = E$, then f maps V_D to V_E, which contradicts Lemma 5.1. ■

Lemma 5.3.

(1) If $q = 1$, the action of the Goeritz group $\mathcal{G}$ on the set of edges of the primitive disk complex $\mathcal{P}(V)$ is transitive. The two end points of the edge $\{E, D\}$ can be exchanged by the action of $\mathcal{G}$.

(2) If $q \neq 1$ and $q^2 \equiv 1 \pmod{p}$, the action of $\mathcal{G}$ on the set of edges of $\mathcal{P}(V)$ has exactly two orbits $\mathcal{G} \cdot \{E, D\}$ and $\mathcal{G} \cdot \{E, E_1\}$. The two end points of each of the edges $\{E, D\}$ and $\{E, E_1\}$ can be exchanged by the action of $\mathcal{G}$.

(3) Otherwise, the action of $\mathcal{G}$ on the set of edges of $\mathcal{P}(V)$ has exactly three orbits $\mathcal{G} \cdot \{E, D\}$, $\mathcal{G} \cdot \{E, E_1\}$, and $\mathcal{G} \cdot \{D, D_1\}$. The two end points of each of the edges $\{E, E_1\}$ and $\{D, D_1\}$ can be exchanged by the action of $\mathcal{G}$, whereas those of $\{E, D\}$ cannot. □

Proof. (1) Let $\{A, B\}$ be a primitive pair. Then by Lemma 3.12, there exists a unique shell $\mathcal{S}_B = \{B_0, B_1, B_2, \ldots, B_p\}$ centered at B containing A. Without loss of generality, we may assume that $A = B_1$. By the definition of shells, we have $\{A, B\} \in \mathcal{G} \cdot \{E, E_1\}$. Since in this case we have $q = q' = 1$, it follows from Lemma 3.13 that the two end points of the edge $\{E, E_1\}$ can be exchanged by the action of $\mathcal{G}$.

(2) In this case, we have $q = q' \neq 1$. Let $\{A, B\}$ be a primitive pair. Then by Lemma 3.12, there exists a unique shell $\mathcal{S}_B = \{B_0, B_1, B_2, \ldots, B_p\}$ centered at B containing A. Without loss of generality, we may assume that $A = B_1$ or B_q. It follows directly from the definition of shells that in the former case we have $\{A, B\} \in \mathcal{G} \cdot \{E, E_1\}$, and in the latter case we have $\{A, B\} \in \mathcal{G} \cdot \{E, D\}$. Since the primitive pair $\{E, E_1\}$ admits a common dual disk whereas the pair $\{E, D\}$ does not, we see that $\mathcal{G} \cdot \{E, D\} \cap \mathcal{G} \cdot \{E, E_1\} = \emptyset$. By Lemma 3.13, the two end points of each of the edges $\{E, D\}$ and $\{E, E_1\}$ can be exchanged by the action of $\mathcal{G}$.

(3) In this case, we have $q \neq q'$, $q > 1$ and $q' > 1$. Let $\{A, B\}$ be a primitive pair. Then by Lemma 3.12, there exists a unique shell $\mathcal{S}_B = \{B_0, B_1, B_2, \ldots, B_p\}$ centered at B containing A. Without loss of generality, we may assume that $A = B_i$, where $1 \leq i \leq p/2$. Again by the definition of shells we have:

Case 1 If $\mathcal{S}_B$ is a (p, q)-shell and $A = B_1$, then $\{A, B\} \in \mathcal{G} \cdot \{E, E_1\}$.

Case 2 If $\mathcal{S}_B$ is a (p, q')-shell and $A = B_1$, then $\{A, B\} \in \mathcal{G} \cdot \{D, D_1\}$.

Case 3 If $\mathcal{S}_B$ is a (p,q)-shell and $A = B_{q'}$, or if $\mathcal{S}_B$ is a (p,q')-shell and $A = B_q$, then $\{A,B\} \in \mathcal{G} \cdot \{E,D\}$.

By Lemma 3.13, the two end points of each of the edges $\{E,E_1\}$ and $\{D,D_1\}$ can be exchanged by an involution of $\mathcal{G}$. Since $\mathcal{G} \cdot \{E\} \cap \mathcal{G} \cdot \{D\} = \emptyset$ by Lemma 5.2, the two end points of $\{E,D\}$ cannot be exchanged. ■

5.2 Presentations of the Goertiz groups

The following is a specialized version of Bass–Serre Structure theorem, which is the key to obtain a presentation of the Goeritz group $\mathcal{G}$.

Theorem 5.4 (Serre [18])**.** Suppose that a group G acts on a tree $\mathcal{T}$ without inversion on the edges. If there exists a subtree $\mathcal{L}$ of $\mathcal{T}$ such that every vertex (every edge, respectively) of $\mathcal{T}$ is equivalent modulo G to a unique vertex (a unique edge, respectively) of $\mathcal{L}$. Then G is the free product of the isotropy groups G_v of the vertices v of $\mathcal{L}$, amalgamated along the isotropy groups G_e of the edges e of $\mathcal{L}$. □

In the following, we will denote by $\mathcal{G}_{\{A_1,A_2,\dots,A_k\}}$ the subgroup of the genus-2 Goeritz group $\mathcal{G}$ consisting of elements that preserve each of $A_1, A_2, \dots, A_k$ setwise, where each A_i will be a disk or the union of disks in V or W.

Lemma 5.5. Let A be a primitive disk in V. Then we have $\mathcal{G}_{\{A\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 \rangle$, where α is the hyperelliptic involution of both V and W, β is the half-twist along a reducing sphere, and γ exchanges two disjoint dual disks of A as described in Figure 12. □

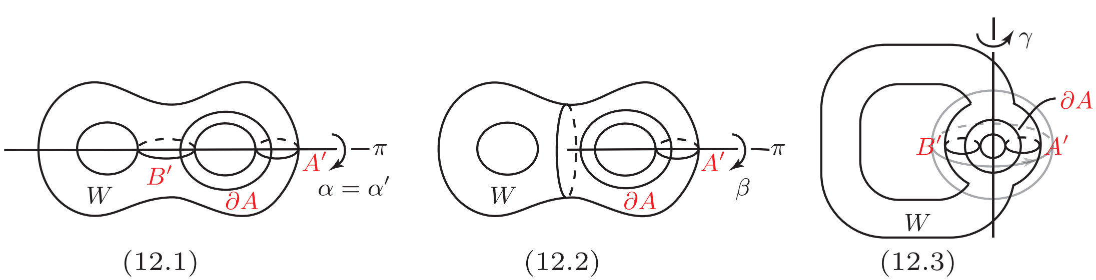


Fig. 12. Generators of $\mathcal{G}_{\{A\}}$.

Proof. Since the argument is almost the same as Lemma 5.1 of [5], we explain the outline. Let $\mathcal{P}_A$ be the full-subcomplex of $\mathcal{D}(W)$ spanned by all the dual disks of A. Then we can show that any dual disk of A in W is disjoint from the unique semiprimitive disk A'_0 disjoint from ∂A, which implies that $\mathcal{P}_A$ is one-dimensional. Further, $\mathcal{P}_A$ is a subcomplex of the disk complex for W satisfying the condition in Theorem 2.1, and hence $\mathcal{P}_A$ is a tree. Let $\mathcal{P}'_A$ be a first barycentric subdivision of $\mathcal{P}_A$. Let A' and B' be disjoint dual disks of A. The quotient of $\mathcal{P}'_A$ by the action of $\mathcal{G}$ is a single edge. It follows from Theorem 5.4 that $\mathcal{G}_{\{A\}} = \mathcal{G}_{\{A,A'\}} *_{\mathcal{G}_{\{A,A',B'\}}} \mathcal{G}_{\{A,A'\cup B'\}}$. An easy computation shows the following:

- $\mathcal{G}_{\{A,A'\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \mid - \rangle$, where α is the hyperelliptic involution of both V and W, and β is the half-twist along the reducing sphere $\partial(\mathrm{Nbd}(A \cup A'))$; see Figures 12.1 and 12.2,
- $\mathcal{G}_{\{A,A'\cup B'\}} = \langle \alpha' \mid \alpha'^2 \rangle \oplus \langle \gamma \mid \gamma^2 \rangle$, where α' is the hyperelliptic involution of both V and W, and γ exchanges A' and B'; see Figures 12.1 and 12.3,
- $\mathcal{G}_{\{A,A',B'\}} = \langle \alpha \mid \alpha^2 \rangle$, where α is the hyperelliptic involution of both V and W; see Figure 12.1.

Since the unique non-trivial element α of $\mathcal{G}_{\{A,A',B'\}}$ provides a relation $\alpha = \alpha'$ in the free product $\mathcal{G}_{\{A,A'\}} * \mathcal{G}_{\{A,A'\cup B'\}}$, we obtain the required presentation of $\mathcal{G}_{\{A\}}$. ∎

Lemma 5.6. Suppose that $p \geq 3$. Let $\{A, B\}$ be an edge of the primitive disk complex $\mathcal{P}(V)$. Then, we have $\mathcal{G}_{\{A,B\}} = \langle \alpha \mid \alpha^2 \rangle$. If the two end points of the edge $\{A, B\}$ can be exchanged by the action of $\mathcal{G}$, then we have $\mathcal{G}_{\{A\cup B\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \sigma \mid \sigma^2 \rangle$, where σ is an element of $\mathcal{G}$ exchanging A and B. Otherwise, we have $\mathcal{G}_{\{A\cup B\}} = \langle \alpha \mid \alpha^2 \rangle$. □

Proof. Let $\{A, B\}$ be an edge of $\mathcal{P}(V)$. Then by Lemma 3.12, there exists a unique shell $\mathcal{S}_B = \{B_0, B_1, \ldots, B_p\}$ centered at B containing A such that A is one of $B_0, B_1, \ldots, B_p$. Without loss of generality, we may assume that $A = B_i$, where $1 \leq i < p/2$. (We assumed $p \geq 3$.) Let f be an element of $\mathcal{G}_{\{A,B\}}$. By the uniqueness of the shell, we have $f(B_j) = B_j$ for $0 \leq j \leq p$. Let B' be the unique dual disk of B disjoint from B_0, and let B'_0 be the unique semi-primitive disk disjoint from B as in the proof of Lemma 3.12. Then again by the uniqueness of the shell, we have $f(B') = B'$ and $f(B'_0) = B'_0$. If f preserves an orientation of B, then f preserves orientations of all of B_j, B' and B'_0 since $\{B, B_{j-1}, B_j\}$ is a triple of pairwise disjoint disks cutting V into two 3-balls. Then by Alexander's trick, f is the trivial element of $\mathcal{G}$. If f reverses an orientation of B, then f reverses orientations of all of B_j, B' and B'_0. Then again by Alexander's trick, f is the hyperelliptic involution α.

If the two end points of the edge $\{A, B\}$ cannot be exchanged by the action of $\mathcal{G}$, it is clear that $\mathcal{G}_{\{A\cup B\}} = \mathcal{G}_{\{A,B\}} = \langle \alpha \mid \alpha^2 \rangle$.

Suppose that there exists an element $\sigma \in \mathcal{G}$ that exchanges the two end points of the edge $\{A, B\}$. In this case, by Lemma 3.12 there exists a unique shell $\mathcal{S}_A = \{A_0, A_1, \ldots, A_p\}$ centered at A containing B such that $B = A_i$. Using the triple $\{B, B_{i-1}, B_i\}$, we may put *compatible* orientations on B, B_{j-1} and $B_j = A$ in a sense that the orientations are coming from an orientation of V cut off by $B \cup B_{i-1} \cup B_i$. We may also put an orientation on A_{i-1} so that the triple $\{A, A_{i-1}, A_i\}$ with the pre-fixed orientations on A and $A_j = B$ are compatible. Since σ maps the shell $\mathcal{S}_B = \{B_0, B_1, \ldots, B_p\}$ to the shell $\mathcal{S}_A$, we see that $\sigma|_B : B \to A$ is orientation-preserving if and only if so is $\sigma|_A : A \to B$. This implies that $\sigma^2 = 1 \in \mathcal{G}$. Let σ_1 and σ_2 be elements of $\mathcal{G}$ that interchanges D and E. Then, $\sigma_1\sigma_2 = 1$ or α. This implies $\sigma_1 = \sigma_2$ or $\alpha\sigma_1 = \sigma_2$. Therefore, we have $\mathcal{G}_{\{A\cup B\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \sigma \mid \sigma^2 \rangle$. ■

We remark that, in the case of $p = 2$ or $q = 1$, the presentations of $\mathcal{G}_{\{A,B\}}$ and $\mathcal{G}_{\{A\cup B\}}$ have been obtained in Lemmas 5.2 and 5.3 in [5]. Using the presentations of the isotropy groups, we have the following main theorem:

Theorem 5.7. The genus-2 Goeritz group $\mathcal{G}$ of a lens space $L(p, q)$, $1 \le q \le p/2$, with $p \equiv \pm 1 \pmod q$ has the following presentations:

(1) If $q = 1$, then we have:
  - (a) $\langle \beta, \rho, \gamma \mid \rho^4, \gamma^2, (\gamma\rho)^2, \rho^2\beta\rho^2\beta^{-1} \rangle$ if $p = 2$;
  - (b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \delta, \gamma \mid \delta^3, \gamma^2, (\gamma\delta)^2 \rangle$ if $p = 3$;
  - (c) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2, \sigma^2 \rangle$ if $p \ge 4$;

(2) If $q > 1$, then we have:
  - (a) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2 \mid {\gamma_1}^2, {\gamma_2}^2 \rangle$ if $p = 5$;
  - (b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma \mid {\gamma_1}^2, {\gamma_2}^2, \sigma^2 \rangle$ if $p = 2q + 1$ and $q \geqslant 3$, or $p > 5$ and $q = 2$;
  - (c) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma_1, \sigma_2 \mid \gamma^2, {\sigma_1}^2, {\sigma_2}^2 \rangle$ if $q^2 \equiv 1 \pmod p$;
  - (d) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma_1, \sigma_2 \mid {\gamma_1}^2, {\gamma_2}^2, {\sigma_1}^2, {\sigma_2}^2 \rangle$ otherwise. □

Proof. We use the four primitive disks E, D, E_1 and D_1 defined in Section 5.1, but we use the same symbols α, β, γ and σ in Lemmas 5.5 and 5.6 for the isotropy subgroups of the disks and their unions in the above.

(1) Since this case of $q = 1$ is already described in [5], we briefly sketch the proof.

(1)a By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of $L(2, 1)$ is a tree, which is described in Figure 11.1a. Let $\mathcal{T}$ be the first barycentric

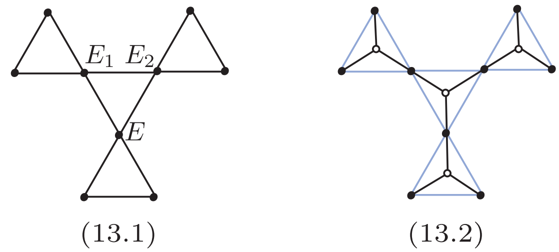


Fig. 13. (13.1) The primitive disk complex $\mathcal{P}(V)$. (13.2) The tree $\mathcal{T}$.

subdivision of $\mathcal{P}(V)$. By Lemma 5.3 the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge with distinct ends. By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{E\cup D\}} *_{\mathcal{G}_{\{E,D\}}} \mathcal{G}_{\{E\}}.$$

The presentation in (1)a is obtained by computing each of those isotropy groups.

(1)b By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of $L(3,1)$ is a two-dimensional complex, which is described in Figure 11.2a. In this case, there is a deformation retraction of $\mathcal{P}(V)$ that shrinks each two-simplex into the cone over its three vertices as shown in Figure 13. Let $\mathcal{T}$ be the resulting complex, which is a tree. By Lemma 5.3 the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge with distinct ends. By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{E\cup E_1\cup E_2\}} *_{\mathcal{G}_{\{E,E_1\cup E_2\}}} \mathcal{G}_{\{E\}}.$$

The presentation in (1)b is obtained by computing each of those isotropy groups.

(1)c By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of $L(p,1)$, $p > 3$, is a tree, which is described in Figure 11.1b. Let $\mathcal{T}$ be the first barycentric subdivision of $\mathcal{P}(V)$. By Lemma 5.3 the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge with distinct ends. By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{E\cup D\}} *_{\mathcal{G}_{\{E,D\}}} \mathcal{G}_{\{E\}}.$$

The presentation in (1)c is obtained by computing each of those isotropy groups.

(2) Suppose that $q > 1$.

(2)a By Theorem 4.5, the primitive disk complex $\mathcal{P}(V)$ for the genus-2 Heegaard splitting of $L(5,2)$ is a two-dimensional contractible complex, which is described in Figure 11.2b.

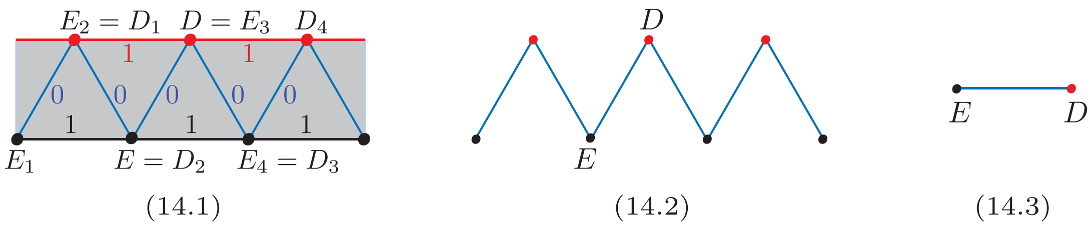


Fig. 14. (14.1) The primitive disk complex $\mathcal{P}(V)$. (14.2) The tree $\mathcal{T}$. (14.3) The quotient $\mathcal{T}/\mathcal{G}$.

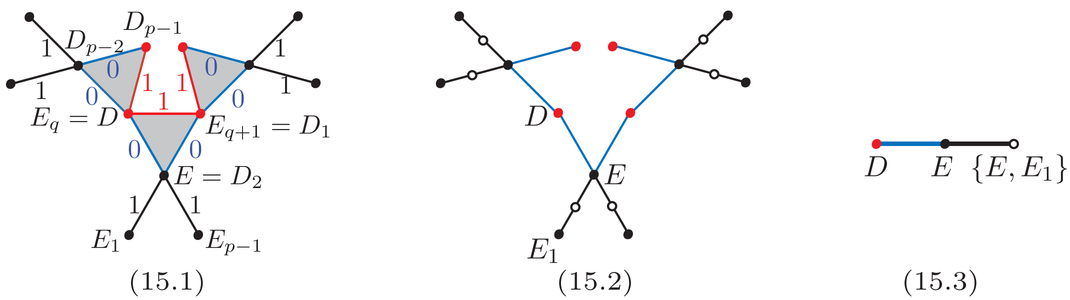


Fig. 15. (15.1) The primitive disk complex $\mathcal{P}(V)$. (15.2) The tree $\mathcal{T}'$. (15.3) The quotient $\mathcal{T}'/\mathcal{G}$.

A portion of $\mathcal{P}(V)$ containing the vertices E, D, E_1 and D_1 is illustrated in Figure 14.1. We recall that each two-simplex of $\mathcal{P}(V)$ contains exactly two edges of type-0 (both of which are elements of $\mathcal{G} \cdot \{E, E_1\}$) and one edge of type-1 (which is an element of $\mathcal{G} \cdot \{E, D\}$). We observe that the subcomplex of $\mathcal{P}(V)$ which consists only of the type-0 edges with the vertices is a tree, which we denote by $\mathcal{T}$ (see Figure 14.2). By Lemma 5.3 the Goeritz group $\mathcal{G}$ acts without inversion on the edges of $\mathcal{T}$ and the two endpoints of each edge belong to different orbits of vertices under the action of $\mathcal{G}$. Moreover, the action is transitive on the set of the edges of $\mathcal{T}$. Hence the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is a single edge, see Figure 14.3. By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{E\}} *_{\mathcal{G}_{\{E,D\}}} \mathcal{G}_{\{D\}}.$$

By Lemmas 5.5 and 5.6, we get the presentation in (2)a.

(2)b Let $L(p, q)$ be a lens space such that $p = 2q + 1$ and $q \geqslant 3$, or $p > 5$ and $q = 2$. By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ is a two-dimensional contractible complex, which is described in Figure 11.2c. A portion of $\mathcal{P}(V)$ containing the vertices E, D, E_1, and D_1 is illustrated in Figure 15.1. In this case, each two-simplex of $\mathcal{P}(V)$ contains exactly one edge of type-1 (which is an element of $\mathcal{G} \cdot \{D, D_1\}$) and two edges of type-0 (both of which are elements of $\mathcal{G} \cdot \{E, D\}$). Substituting each two-simplex of $\mathcal{P}(V)$ by the union of the two edges of type-0 with their vertices in the two-simplex, we have a subcomplex of $\mathcal{P}(V)$, which is a tree. We denote it by $\mathcal{T}$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}'$ by adding the barycenter of each of the remaining edges of type-1 (see Figure 15.2). By Lemma 5.3 the Goeritz group $\mathcal{G}$ acts without inversion on the edges of $\mathcal{T}'$ and the two endpoints of each edge belong to different orbits of vertices under the action of $\mathcal{G}$. Moreover, the complex $\mathcal{T}'$ modulo the action of $\mathcal{G}$ consists of exactly three vertices and two edges. Hence the quotient of $\mathcal{T}'$ by the action of $\mathcal{G}$ is the path graph on three vertices, that is, the tree with three vertices containing only vertices of degree 1 or 2 (see Figure 15.3). By Theorem 5.4, we have

$$\mathcal{G} = \mathcal{G}_{\{D\}} *_{\mathcal{G}_{\{E,D\}}} \mathcal{G}_{\{E\}} *_{\mathcal{G}_{\{E,E_1\}}} \mathcal{G}_{\{E\cup E_1\}}.$$

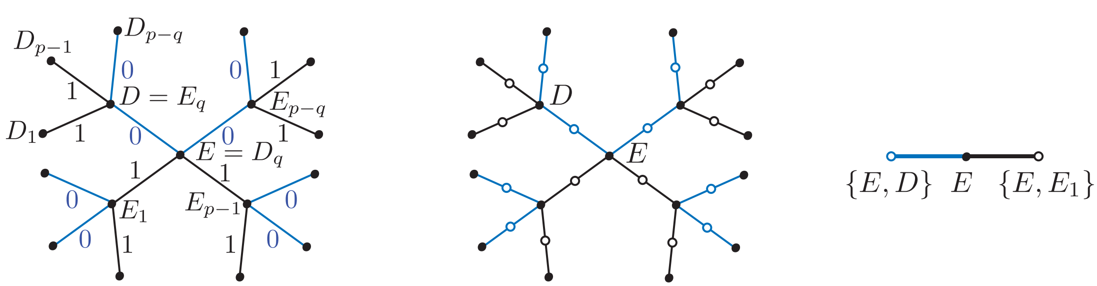
By Lemmas 5.5 and 5.6, we obtain the presentation in (2)b.

(2)c Let $L(p,q)$ be a lens space such that $q^2 \equiv 1 \pmod p$ and $q \geqslant 3$. By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ is a tree, which is described in Figure 11.1c. A portion of $\mathcal{P}(V)$ containing the vertices E, D, E_1, and D_1 is illustrated in Figure 16.1. Let $\mathcal{T}$ be the first barycentric subdivision of $\mathcal{P}(V)$ (see Figure 16.1). By Lemma 5.3 the Goeritz group $\mathcal{G}$ acts without inversion on the edges of $\mathcal{T}$ and the two endpoints of each edge belong to different orbits of vertices under the action of $\mathcal{G}$. Moreover, the complex $\mathcal{T}$ modulo the action of $\mathcal{G}$ consists of exactly three vertices and two edges. Hence, the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is the path graph on three vertices (see Figure 16.3). By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{E\cup D\}} *_{\mathcal{G}_{\{E,D\}}} \mathcal{G}_{\{E\}} *_{\mathcal{G}_{\{E,E_1\}}} \mathcal{G}_{\{E\cup E_1\}}.$$

By Lemmas 5.5 and 5.6, we obtain the presentation in (2)c.

(2)d Let $L(p,q)$ be a lens space such that $q > 1$, $p \equiv \pm 1 \pmod q$, and homeomorphic to none of the above. We assume that $p \equiv 1 \pmod q$. The argument for the case where



(16.1) (16.2) (16.3)

Fig. 16. (16.1) The primitive disk complex $\mathcal{P}(V)$. (16.2) The tree $\mathcal{T}$. (16.3) The quotient $\mathcal{T}/\mathcal{G}$.

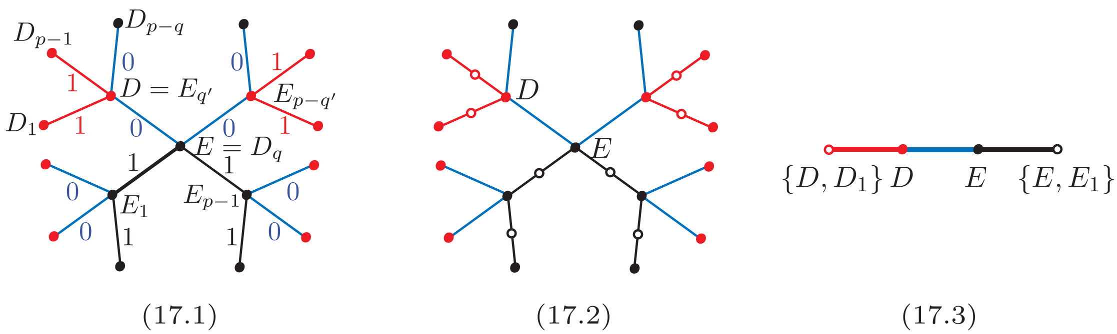


Fig. 17. (17.1) The primitive disk complex $\mathcal{P}(V)$. (17.2) The tree $\mathcal{T}$. (17.3) The quotient $\mathcal{T}/\mathcal{G}$.

$p \equiv -1 \pmod q$ is the same. By Theorem 4.5 the primitive disk complex $\mathcal{P}(V)$ is a tree, which is described in Figure 11.1c again. A portion of $\mathcal{P}(V)$ containing the vertices E, D, E_1, and D_1 is illustrated in Figure 17.1. Let $\mathcal{T}$ be the tree obtained from $\mathcal{P}(V)$ by adding the barycenter of each edge of type-1 (which is an element of $\mathcal{G} \cdot \{E, E_1\}$ or $\mathcal{G} \cdot \{D, D_1\}$) (see Figure 17.2). By Lemma 5.3 the Goeritz group $\mathcal{G}$ acts without inversion on the edges of $\mathcal{T}$ and the two endpoints of each edge belong to different orbits of vertices under the action of $\mathcal{G}$. Moreover, the complex $\mathcal{T}$ modulo the action of $\mathcal{G}$ consists of exactly four vertices and three edges. Hence, the quotient of $\mathcal{T}$ by the action of $\mathcal{G}$ is the path graph on four vertices (see Figure 17.3). By Theorem 5.4, we have:

$$\mathcal{G} = \mathcal{G}_{\{D \cup D_1\}} *_{\mathcal{G}_{\{D, D_1\}}} \mathcal{G}_{\{D\}} *_{\mathcal{G}_{\{E, D\}}} \mathcal{G}_{\{E\}} *_{\mathcal{G}_{\{E, E_1\}}} \mathcal{G}_{\{E \cup E_1\}}.$$

By Lemmas 5.5 and 5.6, we obtain the presentation in (2)d. ■

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References

[1] Akbas, E. “A presentation for the automorphisms of the 3-sphere that preserve a genus-2 Heegaard splitting.” *Pacific Journal of Mathematics* 236, no. 2 (2008): 201–22.

[2] Bonahon, F. “Difféotopies des espaces lenticulaires.” *Topology* 22, no. 3 (1983): 305–14.

[3] Bonahon, F., and J.-P. Otal. “Scindements de Heegaard des espaces lenticulaires.” *Annales Scientifiques de l'École Normale Supérieure* (4) 16, no. 3 (1983): 451–66.

[4] Cho, S. “Homeomorphisms of the 3-sphere that preserve a Heegaard splitting of genus-2.” *Proceedings of the American Mathematical Society* 136, no. 3 (2008): 1113–23.

[5] Cho, S. “Genus-2 Goeritz groups of lens spaces.” *Pacific Journal of Mathematics* 265, no. 1 (2013): 1–16.

[6] Cho, S., and Y. Koda. “The genus-2 Goeritz group of $\mathbb{S}^2 \times \mathbb{S}^1$.” *Mathematical Research Letters* 21, no. 3 (2014): 449–60.

[7] Cho, S., and Y. Koda. “Disk complexes and genus-2 Heegaard splittings for non-prime 3-manifolds.” *International Mathematics Research Notices* 2015, no. 12 (2015): 4344–71.

[8] Cho, S., Y. Koda, and A. Seo. “Arc complexes, sphere complexes and Goeritz groups.” *Michigan Mathematical Journal* (2016): preprint arXiv:1403.7832.

[9] Gordon, C. McA. “On primitive sets of loops in the boundary of a handlebody.” *Topology and its Applications* 27, no. 3 (1987): 285–99.

[10] Hatcher, A., and W. P. Thurston. “Incompressible surfaces in 2-bridge knot complements.” *Inventiones mathematicae* 79, no. 2 (1985): 225–46.

[11] Johnson, J. “Mapping class groups of medium distance Heegaard splittings.” *Proceedings of the American Mathematical Society* 138 (2010): 4529–35.

[12] Johnson, J. “Automorphisms of the three-torus preserving a genus-three Heegaard splitting.” *Pacific Journal of Mathematics* 253, no. 1 (2011): 75–94.

[13] Koda, Y. “Automorphisms of the 3-sphere that preserve spatial graphs and handlebody-knots.” *Mathematical Proceedings of the Cambridge Philosophical Society* 159 (2015): 1–22.

[14] Osborne, R. P., and H. Zieschang. “Primitives in the free group on two generators.” *Inventiones mathematicae* 63, no. 1 (1981): 17–24.

[15] Rolfsen, D. *Knots and Links*. Mathematics Lecture Series 7. Berkeley, CA: Publish or Perish, Inc., 1976.

[16] Scharlemann, M. “Automorphisms of the 3-sphere that preserve a genus-2 Heegaard splitting.” *Boletín de la Sociedad Matemática Mexicana* (3) 10, Special Issue (2004): 503–14.

[17] Scharlemann, M. “Generating the genus $g+1$ Goeritz group of a genus g handlebody.” Geometry and Topology Down Under 347–369, Contemporary Mathematics 597. Providence, RI: American Mathematical Society, 2013.

[18] Serre, J.-P. *Trees* (translated from the French by John Stillwell). Berlin/New York: Springer-Verlag, 1980.