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CUBIC AND QUARTIC ρ–FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

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Abstract. In this paper, we solve the following cubic ρ-functional inequality

$$\begin{aligned}&N(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\\&-\rho(4f(x+\frac{y}{2})+4f(x-\frac{y}{2})-f(x+y)-f(x-y)-6f(x)),t)\geqslant\frac{t}{t+\varphi(x,y)}\end{aligned}\tag{0.1}$$

and the following quartic ρ-functional inequality

$$\begin{aligned}&N(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\\&-\rho(8f(x+\frac{y}{2})+8f(x-\frac{y}{2})-2f(x+y)-2f(x-y)-12f(x)+3f(y)),t)\\&\geqslant\frac{t}{t+\varphi(x,y)}\end{aligned}\tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\rho\neq 2$.

Using the fixed point method, we prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) and the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [20] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 24, 50]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 29, 30] to investigate the Hyers-Ulam stability of cubic ρ-functional inequalities and quartic ρ-functional inequalities in fuzzy Banach spaces.

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DEFINITION 1.1. [2, 29, 30, 31] Let X be a real vector space. A function $N: X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leqslant 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N_4) $N(x+y,s+t)\geqslant\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [28, 29].

DEFINITION 1.2. [2, 29, 30, 31] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty}x_n=x$.

DEFINITION 1.3. [2, 29, 30, 31] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon>0$ and each $t>0$ there exists an $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f:X\to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0\in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f:X\to Y$ is continuous at each $x\in X$, then $f:X\to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [49] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 19, 21, 22, 25, 37, 38, 39, 43, 44, 45, 46, 47, 48]).

In [18], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [26], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leqslant \|f(x+y)\| \tag{1.3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [42]. Fechner [10] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.3). Park, Cho and Han [36] investigated the Cauchy additive functional inequality

$$\|f(x)+f(y)+f(z)\| \leqslant \|f(x+y+z)\| \tag{1.4}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.5}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.4) and (1.5) in Banach spaces.

Park [34, 35] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d: X\times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\leqslant d(x,y)+d(y,z)$ for all $x,y,z\in X$.

THEOREM 1.4. [4, 9] Let (X,d) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n\geqslant n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$;
(4) $d(y,y^*)\leqslant \frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

In 1996, G. Isac and Th. M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 28, 32, 33, 39, 40]).

In Section 2, we solve the cubic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quartic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that ρ is a fixed real number with $\rho \neq 2$.

2. Cubic ρ-functional inequality (0.1)

In this section, we solve and investigate the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces.

LEMMA 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}
&f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\
&\quad = \rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right)
\end{aligned} \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is cubic.

Proof. Letting $y = 0$ in (2.1), we get $2f(2x) - 16f(x) = 0$ and so $f(2x) = 8f(x)$ for all $x \in X$. Thus

$$\begin{aligned}
&f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\
&= \rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right) \\
&= \frac{\rho}{2}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x))
\end{aligned}$$

and so $f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) = 0$ for all $x, y \in X$, as desired. □

We prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces.

THEOREM 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leqslant \frac{L}{8}\varphi(2x,2y)$$

for all $x,y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$N\Big(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ -\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\Big) \geqslant \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

for all $x,y \in X$ *and all* $t > 0$. *Then* $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines a cubic mapping* $C : X \to Y$ *such that*

$$N\left(f(x)-C(x),t\right) \geqslant \frac{(16-16L)t}{(16-16L)t+L\varphi(x,0)} \tag{2.3}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $y = 0$ in (2.2), we get

$$N(2f(2x)-16f(x),t) \geqslant \frac{t}{t+\varphi(x,0)} \tag{2.4}$$

and so $N\left(f(x)-8f\left(\frac{x}{2}\right),\frac{t}{2}\right) \geqslant \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}$ for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \geqslant \frac{t}{t+\varphi(x,0)},\ \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [27, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right)-8h\left(\frac{x}{2}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{8}\varepsilon t\right) \\ &\geqslant \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\varphi\left(\frac{x}{2},0\right)} \geqslant \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\frac{L}{8}\varphi(x,0)} = \frac{t}{t+\varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h)$$

for all $g,h \in S$.

It follows from (2.4) that

$$N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{L}{16}t\right) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f,Jf) \leqslant \frac{L}{16}$.

By Theorem 1.4, there exists a mapping $C: X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.5}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(f,C) \leqslant \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,C) \leqslant \frac{L}{16-16L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned} &N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right)\right)\right. \\ &\left. -8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right) + 4f\left(\frac{x-\frac{y}{2}}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right)\right), 8^n t\right) \\ &\geqslant \frac{t}{t+\varphi(\frac{x}{2^n}, \frac{y}{2^n})} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right)+f\left(\frac{2x-y}{2^n}\right)-2f\left(\frac{x+y}{2^n}\right)-2f\left(\frac{x-y}{2^n}\right)-12f\left(\frac{x}{2^n}\right)\right)\right.$$
$$\left.-8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right)+4f\left(\frac{x-\frac{y}{2}}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right),t\right)$$
$$\geqslant \frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}$$

Since $\lim_{n\to\infty}\frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)$$
$$=\rho\left(4C\left(x+\frac{y}{2}\right)+4C\left(x-\frac{y}{2}\right)-C(x+y)-C(x-y)-6C(x)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $C : X \to Y$ is cubic, as desired. □

COROLLARY 2.3. *Let $\theta \geqslant 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$N(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)$$
$$-\rho(4f(x+\frac{y}{2})+4f(x-\frac{y}{2})-f(x+y)-f(x-y)-6f(x)),t)$$
$$\geqslant \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{2.6}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(x)-C(x),t) \geqslant \frac{2(2^p-8)t}{2(2^p-8)t+\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p+\|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{3-p}$, and we get the desired result. □

THEOREM 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leqslant 8L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then $C(x) := N$-$\lim_{n\to\infty}\frac{1}{8^n}f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(x)-C(x),t) \geqslant \frac{(16-16L)t}{(16-16L)t+\varphi(x,0)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.4) that

$$N\left(f(x)-\frac{1}{8}f(2x),\frac{1}{16}t\right) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J: S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$. Then $d(f,Jf) \leqslant \frac{1}{16}$. Hence

$$d(f,C) \leqslant \frac{1}{16-16L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

COROLLARY 2.5. *Let $\theta \geqslant 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f: X \to Y$ be a mapping satisfying (2.6). Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C: X \to Y$ such that*

$$N(f(x)-C(x),t) \geqslant \frac{2(8-2^p)t}{2(8-2^p)t+\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x,y \in X$. Then we can choose $L = 2^{p-3}$, and we get the desired result. □

3. Quartic ρ-functional inequality (0.2)

In this section, we solve and investigate the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

LEMMA 3.1. *Let $f: X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \\ &= \rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right) \end{aligned} \tag{3.1}$$

for all $x,y \in X$. Then $f: X \to Y$ is quartic.

Proof. Letting $y=0$ in (3.1), we get $2f(2x)-32f(x)=0$ and so $f(2x)=16f(x)$ for all $x\in X$. Thus

$$\begin{aligned}
& f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \\
& =\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right) \\
& =\frac{\rho}{2}(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y))
\end{aligned}$$

and so $f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)=0$ for all $x,y\in X$. $\square$

We prove the Hyers-Ulam stability of the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

THEOREM 3.2. *Let $\varphi: X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leqslant\frac{L}{16}\varphi(2x,2y)$$

for all $x,y\in X$. Let $f: X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}
& N\Big(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \\
& -\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right),t\Big) \\
& \qquad \geqslant\frac{t}{t+\varphi(x,y)}
\end{aligned}\tag{3.2}$$

for all $x,y\in X$ and all $t>0$. Then $Q(x):=N\text{-}\lim_{n\to\infty}16^n f\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines a quartic mapping $Q: X\to Y$ such that

$$N(f(x)-Q(x),t)\geqslant\frac{(32-32L)t}{(32-32L)t+L\varphi(x,0)}\tag{3.3}$$

for all $x\in X$ and all $t>0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Letting $y=0$ in (3.2), we get

$$N(2f(2x)-32f(x),t)=N(32f(x)-2f(2x),t)\geqslant\frac{t}{t+\varphi(x,0)}\tag{3.4}$$

for all $x\in X$. Now we consider the linear mapping $J: S\to S$ such that

$$Jg(x):=16g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x)-Jh(x),L\varepsilon t) = N\left(16g\left(\frac{x}{2}\right)-16h\left(\frac{x}{2}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{16}\varepsilon t\right)$$
$$\geqslant \frac{\frac{Lt}{16}}{\frac{Lt}{16}+\varphi\left(\frac{x}{2},0\right)} \geqslant \frac{\frac{Lt}{16}}{\frac{Lt}{16}+\frac{L}{16}\varphi(x,0)} = \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h)$$

for all $g,h \in S$.

It follows from (3.4) that

$$N\left(f(x)-16f\left(\frac{x}{2}\right),\frac{L}{32}t\right) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f,Jf) \leqslant \frac{L}{32}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{3.5}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(f(x)-Q(x),\mu t) \geqslant \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f,Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leqslant \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,Q) \leqslant \frac{L}{32-32L}.$$

This implies that the inequality (3.3) holds.

By the same method as in the proof of Theorem 2.2, it follows from (3.2) that

$$
\begin{aligned}
&Q(2x+y)+Q(2x-y)-4Q(x+y)-4Q(x-y)-24Q(x)+6Q(y)\\
&=\rho\left(8Q\left(x+\frac{y}{2}\right)+8Q\left(x-\frac{y}{2}\right)-2Q(x+y)-2Q(x-y)-12Q(x)+3Q(y)\right)
\end{aligned}
$$

for all $x,y\in X$. By Lemma 3.1, the mapping $Q:X\to Y$ is quartic. $\square$

COROLLARY 3.3. *Let $\theta\geqslant 0$ and let p be a real number with $p>4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$
\begin{aligned}
&N(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\\
&-\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right),t)\\
&\qquad\geqslant\frac{t}{t+\theta(\|x\|^p+\|y\|^p}
\end{aligned}
\tag{3.6}
$$

for all $x,y\in X$ and all $t>0$. Then $Q(x):=N\text{-}\lim_{n\to\infty}16^n f(\frac{x}{2^n})$ exists for each $x\in X$ and defines a quartic mapping $Q:X\to Y$ such that

$$
N(f(x)-Q(x),t)\geqslant\frac{2(2^p-16)t}{2(2^p-16)t+\theta\|x\|^p}
$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$. Then we can choose $L=2^{4-p}$, and we get the desired result. $\square$

THEOREM 3.4. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$
\varphi(x,y)\leqslant 16L\varphi\left(\frac{x}{2},\frac{y}{2}\right)
$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (3.2). Then $Q(x):=N\text{-}\lim_{n\to\infty}\frac{1}{16^n}f(2^n x)$ exists for each $x\in X$ and defines a quartic mapping $Q:X\to Y$ such that

$$
N(f(x)-Q(x),t)\geqslant\frac{(32-32L)t}{(32-32L)t+\varphi(x,0)}
\tag{3.7}
$$

for all $x\in X$ and all $t>0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.4) that

$$N\left(f(x)-\frac{1}{16}f(2x),\frac{1}{32}t\right)\geqslant\frac{t}{t+\varphi(x,0)}$$

for all $x\in X$ and all $t>0$. Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{16}g(2x)$$

for all $x\in X$. Then $d(f,Jf)\leqslant\frac{1}{32}$. Hence

$$d(f,Q)\leqslant\frac{1}{32-32L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

COROLLARY 3.5. *Let $\theta\geqslant 0$ and let p be a real number with $0<p<4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (3.6). Then $Q(x):=N\text{-}\lim_{n\to\infty}\frac{1}{16^n}f(2^nx)$ exists for each $x\in X$ and defines a quartic mapping $Q:X\to Y$ such that*

$$N(f(x)-Q(x),t)\geqslant\frac{2(16-2^p)t}{2(16-2^p)t+\theta\|x\|^p}$$

for all $x\in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$.
Then we can choose $L=2^{p-4}$, and we get the desired result. □

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