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FUNDAMENTAL STABILITIES OF THE NONIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In the current work, the intuitionistic fuzzy version of Hyers-Ulam stability for a nonic functional equation by applying a fixed point method is investigated. This way shows that some fixed points of a suitable operator can be a nonic mapping.

1. Introduction

In [25], Ulam proposed the general Ulam stability problem: "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In [14], Hyers gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. On the other hand, Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [10] and for the quadratic functional equation [9] (for more applications of this method, see [3], [4], [6], [7], [8], [11] and [27]).

In 1965, Zadeh [28] introduced the notion of fuzzy sets which is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. After that, fuzzy theory has become very active area of research and a lot of developments have been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. In fact, a large number of research papers have appeared by using the concept of fuzzy set and numbers and also fuzzification of many classical theories has been made. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park in [22]. Then, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces by improving the separation condition and strengthening some conditions in the definition of [24]. Many authors

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have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces (see [1], [2], [13], [16]). Also, the generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been studied by a number of the authors (see [5], [15], [20], [21] and [26])

In this paper, we consider the nonic functional equation which was introduced in [18] as follows:

$$\begin{aligned} &f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)\\ &+126f(x+y)-126f(x)+84f(x-y)\\ &-36f(x-2y)+9f(x-3y)-f(x-4y)=362880f(y). \end{aligned} \tag{1}$$

It is easy to check that the function $f(x)=ax^9$ is a solution of the functional equation (1). Indeed, general solution of the equation (1) was found in [18]. In this paper, we study some stability results concerning the functional equation (1) in the setting of intuitionistic fuzzy normed spaces. In fact, we show that the nonic functional equation (1) can be stable (Corollary 2.4).

2. Intuitionistic fuzzy stability of (1)

In this section, we firstly restate the usual terminology, notations and conventions of the theory of intuitionistic fuzzy normed space, as in [17], [19], [20], [21] and [23]. Then, we prove the generalized Ulam-Hyers stability of the equation (1) in intuitionistic fuzzy normed spaces, based on the fixed point Theorem 2.2.

Let $\leq_L$ be an order relation on the set $L=\{(x_1,x_2):(x_1,x_2)\in[0,1]^2, x_1+x_2\leq 1\}$ defined by

$$(x_1,x_2)\leq_L (y_1,y_2) \Longleftrightarrow x_1\leq y_1, y_2\leq x_2$$

for all $(x_1,x_2),(y_1,y_2)\in L$. It is easy to check that the pair $(L,\leq_L)$ is a complete lattice (see also [19] and [23]). We denote the units of L by $0_L=(0,1)$ and $1_L=(1,0)$.

Definition 1. Let U be a non-empty set called the universe. An L-fuzzy set in U is defined as a mapping $\mathcal{F}:U\longrightarrow L$. For each u in U, $\mathcal{F}(u)$ represents the degree (in L) to which u is an element of $\mathcal{F}$. An intuitionistic fuzzy set $\mathcal{F}_{\mu,\nu}$ in a universal set U is an object $\mathcal{F}_{\mu,\nu}=\{(\mu_{\mathcal{F}}(u),\nu_{\mathcal{F}}(u)):u\in U\}$, where $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ belong to $[0,1]$ for all $u\in U$ with $\mu_{\mathcal{F}}(u)+\nu_{\mathcal{F}}(u)\leq 1$. The numbers $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{F}_{\mu,\nu}$.

Definition 2. A triangular norm (t-norm) on L is a mapping $\mathcal{T}:L\times L\longrightarrow L$ satisfying the following conditions:

(i) $\mathcal{T}(x,1_L)=x$ (boundary condition) $(x\in L)$;
(ii) $\mathcal{T}(x,y)=\mathcal{T}(y,x)$ (commutativity) $(x,y\in L)$;
(iii) $\mathcal{T}(x,\mathcal{T}(y,z))=\mathcal{T}(\mathcal{T}(x,y),z)$ (associativity) $(x,y,z\in L)$;

(iv) $x_1 \leq_L y_1$ and $x_2 \leq_L y_2 \Longrightarrow \mathcal{T}(x_1, x_2) \leq_L \mathcal{T}(y_1, y_2)$ (monotonicity) $(x_1, x_2, y_1, y_2 \in L)$.

A t-norm $\mathcal{T}$ on L is said to be *continuous* if, for any $x, y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively, then

$$\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$, $\mathcal{T}(x, y) = (x_1 y_1, \min\{x_2 + y_2, 1\})$ and $\mathcal{M}(x, y) = (\min\{x_1, y_1\}, \max\{x_2, y_2\})$ are continuous t-norm [26].

Here, we define a sequence $\mathcal{T}^n$, recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n\left(x^{(1)}, x^{(1)}, \ldots, x^{(n+1)}\right) = \mathcal{T}\left(\mathcal{T}^{n-1}\left(x^{(1)}, x^{(1)}, \ldots, x^{(n)}\right), x^{(n+1)}\right)$$

for all $n \geq 2$ and $x^{(j)} \in L$.

Definition 3. A negator on L is a decreasing mapping $\mathfrak{N} : L \longrightarrow L$ satisfying $\mathfrak{N}(0_L) = 1_L$ and $\mathfrak{N}(1_L) = 0_L$. If $\mathfrak{N}(\mathfrak{N}(x)) = x$ for all $x \in L$, then $\mathfrak{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $\mathcal{N} : L \longrightarrow L$ satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. The standard negator on $[0, 1]$ is defined by $\mathcal{N}_s(x) = 1 - x$ for all $x \in [0, 1]$.

The following definitions of an intuitionistic fuzzy normed space is taken from [20].

Definition 4. Let $\mathcal{L} = (L, \leq_L)$. Let X be a vector space, $\mathcal{T}$ be a continuous t-norm on L and $\mathcal{P}$ be an L-fuzzy set on $X \times (0, \infty)$ satisfying the following conditions:

(i) $0 <_L \mathcal{P}(x, t)$;
(ii) $\mathcal{P}(x, t) = 1_L$ if and only if $x = 0$;
(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(iv) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
(v) The map $\mathcal{P}(x, \cdot) : (0, \infty) \longrightarrow L$ is continuous;
(vi) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_L$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_L$;

for all $x, y \in X$ and all $t, s > 0$. Then the triple $(X, \mathcal{P}, \mathcal{T})$ is called an L-fuzzy normed space. In this case $\mathcal{P}$ is called $\mathcal{L}$-fuzzy norm (briefly, L-fuzzy norm). If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set, then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an intuitionistic fuzzy normed space (briefly, IFN-space). In this case, $\mathcal{P}_{\mu,\nu}$ is called an intuitionistic fuzzy norm on X (Some example of IFN-spaces are provided in [26] and [27]).

Note that, if $\mathcal{P}$ is an L-fuzzy norm on X, then the following statements hold:

(i) $\mathcal{P}(x, t)$ is nondecreasing with respect to t for all $x \in X$;
(ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in X$ and $t > 0$.

Example 2.1 ([26]). Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(x, y) = (x_1 y_1, \min\{x_2 + y_2, 1\})$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$ and μ, ν be membership and non-membership degree, respectively, of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t),\nu(x,t)) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right) \qquad (t \in \mathbb{R}^+).$$

Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFN-space.

Definition 5. Let $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an IFN-space.

(1) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *convergent* to a point x if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_L$ as $n \to \infty$ for all $t > 0$;

(2) A sequence $\{x_n\}$ in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for every $t > 0$ and $0 < \epsilon < 1$, there exists a positive integer N such that $(N_s(\epsilon), \epsilon) \leq_L \mathcal{P}_{\mu,\nu}(x_n - x_m, t)$ for all $m, n > N$, where N_s is the standard negator;

(3) $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is convergent to a point in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

We need the following theorem which a result in fixed point theory [12]. This result plays a fundamental role to arrive our purpose in this paper.

Theorem 2.2 (The fixed point alternative theorem)**.** *Let (Δ, d) be a complete generalized metric space and $\mathcal{J} : \Delta \longrightarrow \Delta$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Delta$, either $d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

(i) *$d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) < \infty$ for all $n \geq n_0$;*

(ii) *the sequence $\{\mathcal{J}^n\alpha\}$ is convergent to a fixed point β^* of $\mathcal{J}$;*

(iii) *β^* is the unique fixed point of $\mathcal{J}$ in the set $\Delta_1 = \{\beta \in \Delta : d(\mathcal{J}^{n_0}\alpha, \beta) < \infty\}$;*

(iv) *$d(\beta, \beta^*) \leq \frac{1}{1-L} d(\beta, \mathcal{J}\beta)$ for all $\beta \in \Delta_1$.*

In the sequel, we use the difference operator for the given mapping $f : X \longrightarrow Y$ as follows:

$$\begin{aligned} D_n f(x,y) := {} & f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) \\ & + 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) \\ & + 9f(x-3y) - f(x-4y) - 362880f(y) \end{aligned}$$

for all $x, y \in X$.

From now on, we assume that all t-norms are as

$$\mathcal{T}(x,y) = (\min\{x_1, y_1\}, \max\{x_2, y_2\})$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in L$. In the upcoming theorem which is our aim in this paper, we prove the generalized Ulam-Hyers stability of the equation (1) in intuitionistic fuzzy normed spaces.

Theorem 2.3. *Let $l \in \{1, -1\}$ be fixed and let α be a real number with $\alpha \neq 512$. Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Suppose that $\phi : X \times X \longrightarrow Z$ is a mapping such that*

$$\mathcal{P}'_{\mu,\nu}(\alpha^l \phi(x, y), t) \leq_L \mathcal{P}'_{\mu,\nu}(\phi(2^l x, 2^l y), t) \tag{2}$$

for all $x \in X$ and $t > 0$. If $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is a complete intuitionistic fuzzy normed space and $f : X \longrightarrow Y$ is a mapping such that

$$\mathcal{P}'_{\mu,\nu}(\phi(x, y), t) \leq_L \mathcal{P}_{\mu,\nu}(D_n f(x, y), t) \tag{3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique nonticc mapping $N : X \longrightarrow Y$ such that

$$\Lambda(x, |512 - \alpha| t) \leq_L \mathcal{P}_{\mu,\nu}(N(x) - f(x), t) \tag{4}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
\Lambda(x, t) := \mathcal{T}^{37}(&\mathcal{P}'_{\mu,\nu}(\phi(5x, x), 945t), \mathcal{P}'_{\mu,\nu}(\phi(0, 2x), 945t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 8x), 342921600t), \mathcal{P}'_{\mu,\nu}(\phi(8x, -8x), 342921600t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 6x), 381024000t), \mathcal{P}'_{\mu,\nu}(\phi(6x, -6x), 381024000t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 4x), 9525600t), \mathcal{P}'_{\mu,\nu}(\phi(4x, -4x), 9525600t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 2x), 4082400t), \mathcal{P}'_{\mu,\nu}(\phi(2x, -2x), 4082400t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 0), 5443200t), \mathcal{P}'_{\mu,\nu}(\phi(4x, x), 2520t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 0), 342921600t), \mathcal{P}'_{\mu,\nu}(\phi(0, x), \frac{1371686400}{37}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(x, -x), \frac{1371686400}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(3x, x), \frac{7560}{37}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 0), \frac{609638400}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(0, x), \frac{76204800}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(x, -x), \frac{76204800}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(0, 2x), \frac{685843200}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x, -2x), \frac{685843200}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(2x, x), \frac{15120}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 0), \frac{38102400}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(0, x), 564480t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(x, -x), 564480t), \mathcal{P}'_{\mu,\nu}(\phi(0, 2x), 2257920t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x, -2x), 2257920t), \mathcal{P}'_{\mu,\nu}(\phi(0, 3x), 20321280t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x, -3x), 20321280t), \mathcal{P}'_{\mu,\nu}(\phi(x, x), 112t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0, 0), 241920t), \mathcal{P}'_{\mu,\nu}(\phi(0, x), 576t), \mathcal{P}'_{\mu,\nu}(\phi(0, 4x), 26127360t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(4x, -4x), 26127360t), \mathcal{P}'_{\mu,\nu}(\phi(0, 3x), 2903040t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x, -3x), 2903040t), \mathcal{P}'_{\mu,\nu}(\phi(0, 2x), 725760t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x, -2x), 725760t), \mathcal{P}'_{\mu,\nu}(\phi(0, x), \frac{2177280}{7}t),
\end{aligned}$$

$$\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{2177280}{7}t),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{2903040}{7}t)). \tag{5}$$

Proof. For the cases $l=1$ and $l=-1$, we consider $\alpha<512$ and $\alpha>512$, respectively. Putting $x=y=0$ in (3), we have

$$\mathcal{P}'_{\mu,\nu}(\phi(0,0),362880t)\leq_L \mathcal{P}_{\mu,\nu}(f(0),t) \tag{6}$$

for all $t>0$. Replacing (x,y) by $(0,x)$ in (3), we get

$$\begin{aligned}\mathcal{P}'_{\mu,\nu}(\phi(0,x),t)\leq_L \mathcal{P}_{\mu,\nu}(&f(5x)-9f(4x)+36f(3x)-84f(2x)+126f(x)\\ &-126f(0)+84f(-x)-36f(-2x)+9f(-3x)\\ &-f(-4x)-362880f(x),t)\end{aligned} \tag{7}$$

for all $x\in X$ and $t>0$. Interchanging (x,y) into $(x,-x)$ in (3), we obtain

$$\begin{aligned}\mathcal{P}'_{\mu,\nu}(\phi(x,-x),t)\leq_L \mathcal{P}_{\mu,\nu}(&f(-4x)-9f(-3x)+36f(-2x)-84f(-x)\\ &+126f(0)-126f(x)+84f(2x)-36f(3x)\\ &+9f(4x)-f(5x)-362880f(-x),t)\end{aligned} \tag{8}$$

for all $x\in X$ and $t>0$. By (7) and (8), we obtain

$$\mathcal{T}(\mathcal{P}'_{\mu,\nu}(\phi(0,x),181440t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),181440t))\leq_L \mathcal{P}_{\mu,\nu}(f(x)+f(-x),t) \tag{9}$$

for all $x\in X$ and $t>0$. Substituting (x,y) by $(5x,x)$ in (3), we get

$$\begin{aligned}\mathcal{P}'_{\mu,\nu}(\phi(5x,x),t)\leq_L \mathcal{P}_{\mu,\nu}(&f(10x)-9f(9x)+36f(8x)-84f(7x)\\ &+126f(6x)-126f(5x)+84f(4x)\\ &-36f(3x)+9f(2x)-362881f(x),t)\end{aligned} \tag{10}$$

for all $x\in X$ and $t>0$. Again, by replacing (x,y) to $(0,2x)$ in (3), we arrive at

$$\begin{aligned}\mathcal{P}'_{\mu,\nu}(\phi(0,2x),t)\leq_L \mathcal{P}_{\mu,\nu}(&f(10x)-9f(8x)+36f(6x)-84f(4x)\\ &+126f(2x)-126f(0)+84f(-2x)-36f(-4x)\\ &+9f(-6x)-8f(-8x)-362880f(2x),t)\end{aligned} \tag{11}$$

for all $x\in X$ and $t>0$. It follows from (10) and (11) that

$$\begin{aligned}&\mathcal{T}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x),\frac{t}{2}),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{t}{2}))\\ &\leq_L \mathcal{P}_{\mu,\nu}(-9f(9x)+45f(8x)-84f(7x)+90f(6x)-126f(5x)\\ &\qquad +168f(4x)-36f(3x)+362763f(2x)-362881f(x)+126f(0)\\ &\qquad -84f(-2x)+36f(-4x)-9f(-6x)+f(-8x),t)\end{aligned} \tag{12}$$

for all $x\in X$ and $t>0$. By (6), (9) and (12), we find

$$\begin{aligned}&\mathcal{T}^9(\mathcal{P}'_{\mu,\nu}(\phi(5x,x),\frac{t}{12}),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{t}{12}),\mathcal{P}'_{\mu,\nu}(\phi(0,8x),30240t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x),30240t),\mathcal{P}'_{\mu,\nu}(\phi(0,6x),3360t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(6x,-6x),3360t),\mathcal{P}'_{\mu,\nu}(\phi(0,4x),840t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),840t),\end{aligned}$$

$$\mathcal{P}'_{\mu,\nu}(\phi(0,2x),360t),\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),360t),\mathcal{P}'_{\mu,\nu}(\phi(0,0),480t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(-9f(9x)+44f(8x)-84f(7x)+99f(6x)-126f(5x)$$
$$+132f(4x)-36f(3x)+362847f(2x)-362881f(x),t) \tag{13}$$

for all $x \in X$ and $t > 0$. Putting (x, y) by $(4x, x)$ in (3), we get

$$\mathcal{P}'_{\mu,\nu}(\phi(4x,x),t) \leq_L \mathcal{P}_{\mu,\nu}(f(9x)-9f(8x)+36f(7x)-84f(6x)+126f(5x)$$
$$-126f(4x)+84f(3x)-36f(2x)-362871f(x)-f(0),t) \tag{14}$$

for all $x \in X$ and $t > 0$. Using (6) and (14), we have

$$\mathcal{T}(\mathcal{P}'_{\mu,\nu}(\phi(4x,x),\frac{t}{2}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),181440t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(f(9x)-9f(8x)+36f(7x)-84f(6x)+126f(5x)-126f(4x)$$
$$+84f(3x)-36f(2x)-362871f(x),t) \tag{15}$$

for all $x \in X$ and $t > 0$. Hence

$$\mathcal{T}(\mathcal{P}'_{\mu,\nu}(\phi(4x,x),\frac{t}{18}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),20160t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(9f(9x)-81f(8x)+324f(7x)-756f(6x)+1134f(5x)$$
$$-1134f(4x)+756f(3x)-324f(2x)-3265839f(x),t) \tag{16}$$

for all $x \in X$ and $t > 0$. By (13) and (16), we deduce that

$$\mathcal{T}^{11}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x),\frac{t}{24}),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{t}{24}),\mathcal{P}'_{\mu,\nu}(\phi(0,8x),15120t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x),15120t),\mathcal{P}'_{\mu,\nu}(\phi(0,6x),1680t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(6x,-6x),16800t),\mathcal{P}'_{\mu,\nu}(\phi(0,4x),420t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),420t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,2x),180t),\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),180t),\mathcal{P}'_{\mu,\nu}(\phi(0,0),240t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(4x,x),\frac{t}{36}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),15080t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(-37f(8x)+240f(7x)-657f(6x)+1008f(5x)$$
$$-1002f(4x)+720f(3x)-362523f(2x)-3628720f(x),t) \tag{17}$$

for all $x \in X$ and $t > 0$. Putting (x, y) by $(3x, x)$ in (3), we get

$$\mathcal{P}'_{\mu,\nu}(\phi(3x,x),t) \leq_L \mathcal{P}_{\mu,\nu}(f(8x)-9f(7x)+36f(6x)-84f(5x)+126f(4x)$$
$$-126f(3x)+84f(2x)-362916f(x)+9f(0)-f(-x),t) \tag{18}$$

for all $x \in X$ and $t > 0$. Applying (6), (9) and (18), we get

$$\mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\phi(0,x),60480t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),60480t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(3x,x),\frac{t}{3}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),13440t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(f(8x)-9f(7x)+36f(6x)-84f(5x)+126f(4x)$$
$$-126f(3x)+84f(2x)-362915f(x),t) \tag{19}$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned}
&\mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{60480}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{60480}{37}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x,x), \frac{t}{111}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{13440}{37}t))\\
&\leq_L \mathcal{P}_{\mu,\nu}(37f(8x) - 333f(7x) + 1332f(6x) - 3108f(5x)\\
&\quad + 4662f(4x) - 4662f(3x) + 1332f(2x) - 13427855f(x), t) \qquad (20)
\end{aligned}$$

for all $x \in X$ and $t > 0$. It follows from (17) and (20) that

$$\begin{aligned}
&\mathcal{T}^{13}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x), \frac{t}{48}), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{t}{48}), \mathcal{P}'_{\mu,\nu}(\phi(0,8x), 7560t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x), 7560t), \mathcal{P}'_{\mu,\nu}(\phi(0,6x), 8400t), \mathcal{P}'_{\mu,\nu}(\phi(6x,-6x), 8400t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0,4x), 210t), \mathcal{P}'_{\mu,\nu}(\phi(4x,-4x), 210t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), 90t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), 90t), \mathcal{P}'_{\mu,\nu}(\phi(0,0), 120t), \mathcal{P}'_{\mu,\nu}(\phi(4x,x), \frac{t}{18}),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0,0), 7540t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{30240}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{30240}{37}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x,x), \frac{t}{222}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{13440}{37}t))\\
&\leq_L \mathcal{P}_{\mu,\nu}(-93f(7x) + 675f(6x) - 2100f(5x) + 3660f(4x)\\
&\quad - 3942f(3x) + 365631f(2x) - 17056575f(x), t) \qquad (21)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Replacing (x, y) by $(2x, x)$ in (3), we obtain

$$\begin{aligned}
\mathcal{P}'_{\mu,\nu}(\phi(2x,x), t) \leq_L\ &\mathcal{P}_{\mu,\nu}(f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x)\\
&- 126f(2x) - 362796f(x) - 36f(0) + 9f(-x)\\
&- f(-2x), t) \qquad (22)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Applying (6), (9) and (22), we get

$$\begin{aligned}
&\mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\phi(0,x), 5040t), \mathcal{P}'_{\mu,\nu}(\phi(x,-x), 5040t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), 45360t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), 45360t), \mathcal{P}'_{\mu,\nu}(\phi(2x,x), \frac{t}{4}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), 2520t))\\
&\leq_L \mathcal{P}_{\mu,\nu}(f(7x) - 9f(6x) + 36f(5x) - 84f(4x)\\
&\quad + 126f(3x) - 125f(2x) - 362796f(x), t) \qquad (23)
\end{aligned}$$

for all $x \in X$ and $t > 0$. The relation (23) implies that

$$\begin{aligned}
&\mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{1680}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{1680}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{15120}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), \frac{15120}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(2x,x), \frac{t}{372}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{840}{31}t))\\
&\leq_L \mathcal{P}_{\mu,\nu}(93f(7x) - 837f(6x) + 3348f(5x) - 7812f(4x)\\
&\quad + 11718f(3x) - 11625f(2x) - 33740865f(x), t) \qquad (24)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Plugging (21) to (24), one can obtain

$$\begin{aligned}
&\mathcal{T}^{19}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x),\frac{t}{48}),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{t}{48}),\mathcal{P}'_{\mu,\nu}(\phi(0,8x),7560t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x),7560t),\mathcal{P}'_{\mu,\nu}(\phi(0,6x),8400t),\mathcal{P}'_{\mu,\nu}(\phi(6x,-6x),8400t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0,4x),210t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),210t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),90t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),90t),\mathcal{P}'_{\mu,\nu}(\phi(0,0),120t),\mathcal{P}'_{\mu,\nu}(\phi(4x,x),\frac{t}{18}),\\
&\mathcal{P}'_{\mu,\nu}(\phi(0,0),7540t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{30240}{37}t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{30240}{37}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x,x),\frac{t}{222}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{13440}{37}t)\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{1680}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{1680}{31}t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{15120}{31}t),\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),\frac{15120}{31}t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,x),\frac{t}{372}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{840}{31}t))\\
&\leq_L \mathcal{P}_{\mu,\nu}(-81f(6x)+624f(5x)-2076f(4x)\\
&\quad +3888f(3x)+177003f(2x)-25398720f(x),t) \qquad (25)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Interchanging (x, y) into (x, x) in (3), we have

$$\begin{aligned}
\mathcal{P}'_{\mu,\nu}(\phi(x,x),t) \leq_L\ &\mathcal{P}_{\mu,\nu}(f(6x)-9f(5x)+36f(4x)-84f(3x)+126f(2x)\\
&-363006f(x)+84f(0)-36f(-x)+9f(-2x)\\
&-f(-3x),t) \qquad (26)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Using (6), (9) and (26), we get

$$\begin{aligned}
&\mathcal{T}^{7}(\mathcal{P}'_{\mu,\nu}(\phi(0,x),1008t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),1008t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),4032t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),4032t),\mathcal{P}'_{\mu,\nu}(\phi(0,3x),36288t),\\
&\mathcal{P}'_{\mu,\nu}(\phi(3x,-3x),36288t),\mathcal{P}'_{\mu,\nu}(\phi(x,x),\frac{t}{5}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),432t)) \qquad (27)\\
&\leq_L \mathcal{P}_{\mu,\nu}(f(6x)-9f(5x)+36f(4x)-83f(3x)+117f(2x)-362970f(x),t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. So,

$$\begin{aligned}
&\mathcal{T}^{7}(\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{112}{9}t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{112}{9}t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{448}{9}t),\\
&\quad\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),\frac{448}{9}t),\mathcal{P}'_{\mu,\nu}(\phi(0,3x),448t),\mathcal{P}'_{\mu,\nu}(\phi(3x,-3x),4488t),\\
&\quad\mathcal{P}'_{\mu,\nu}(\phi(x,x),\frac{t}{405}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{16}{3}t)) \leq_L \mathcal{P}_{\mu,\nu}(81f(6x)-729f(5x)\\
&\quad +2916f(4x)-6723f(3x)+9477f(2x)-29400570f(x),t) \qquad (28)
\end{aligned}$$

for all $x \in X$ and $t > 0$. It follows (25) and (28) that

$$\mathcal{T}^{27}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x),\frac{t}{96}),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{t}{96}),\mathcal{P}'_{\mu,\nu}(\phi(0,8x),3780t),$$

$$\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x),3780t),\mathcal{P}'_{\mu,\nu}(\phi(0,6x),4200t),\mathcal{P}'_{\mu,\nu}(\phi(6x,-6x),4200t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,4x),105t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),105t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),45t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),45t),\mathcal{P}'_{\mu,\nu}(\phi(0,0),60t),\mathcal{P}'_{\mu,\nu}(\phi(4x,x),\frac{t}{36}),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,0),3780t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{15120}{37}t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{15120}{37}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(3x,x),\frac{t}{444}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{6720}{37}t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{840}{31}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{840}{31}t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{7560}{31}t),\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),\frac{7560}{31}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(2x,x),\frac{t}{744}),\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{420}{31}t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{56}{9}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{56}{9}t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),\frac{224}{9}t),\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),\frac{224}{9}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,3x),224t),\mathcal{P}'_{\mu,\nu}(\phi(3x,-3x),2244t),\mathcal{P}'_{\mu,\nu}(\phi(x,x),\frac{t}{810}),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{8}{3}t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(-105f(5x)+840f(4x)-2835f(3x)+186480f(2x)$$
$$-54799290f(x),t) \tag{29}$$

for all $x \in X$ and $t > 0$. Replacing (x,y) into $(0,x)$ in (3), we obtain

$$\mathcal{P}'_{\mu,\nu}(\phi(0,x),t) \leq_L \mathcal{P}_{\mu,\nu}(f(5x)-9f(4x)+36f(3x)-84f(2x)-362754f(x)$$
$$-126f(0)+84f(-x)-36f(-2x)+9f(-3x)-f(-4x),t) \tag{30}$$

for all $x \in X$ and $t > 0$. By (6), (9) and (30), we have

$$\mathcal{T}^9(\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{t}{6}),\mathcal{P}'_{\mu,\nu}(\phi(0,4x),30240t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),30240t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,3x),3360t),\mathcal{P}'_{\mu,\nu}(\phi(3x,-3x),3360t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),840t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),840t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),360t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),360t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,0),480t))$$
$$\leq_L \mathcal{P}_{\mu,\nu}(f(5x)-8f(4x)+27f(3x)$$
$$-48f(2x)-362838f(x),t) \tag{31}$$

for all $x \in X$ and $t > 0$. Thus

$$\mathcal{T}^9(\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{t}{630}),\mathcal{P}'_{\mu,\nu}(\phi(0,4x),288t),\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x),288t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,3x),32t),\mathcal{P}'_{\mu,\nu}(\phi(3x,-3x),32t),\mathcal{P}'_{\mu,\nu}(\phi(0,2x),8t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x),8t),\mathcal{P}'_{\mu,\nu}(\phi(0,x),\frac{24}{7}t),\mathcal{P}'_{\mu,\nu}(\phi(x,-x),\frac{24}{7}t),$$
$$\mathcal{P}'_{\mu,\nu}(\phi(0,0),\frac{32}{7}t))$$

$$\begin{aligned}&\leq_L \mathcal{P}_{\mu,\nu}(105f(5x) - 840f(4x) + 2835f(3x)\\ &\quad - 5040f(2x) - 38097990f(x), t)\end{aligned} \tag{32}$$

for all $x \in X$ and $t > 0$. By (29) and (32), we have

$$\begin{aligned}&\mathcal{T}^{37}(\mathcal{P}'_{\mu,\nu}(\phi(5x,x), \frac{t}{192}), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{t}{192}), \mathcal{P}'_{\mu,\nu}(\phi(0,8x), 1890t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(8x,-8x), 1890t), \mathcal{P}'_{\mu,\nu}(\phi(0,6x), 2100t), \mathcal{P}'_{\mu,\nu}(\phi(6x,-6x), 2100t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(0,4x), \frac{105}{2}t), \mathcal{P}'_{\mu,\nu}(\phi(4x,-4x), \frac{105}{2}t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{45}{2}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), \frac{45}{2}t), \mathcal{P}'_{\mu,\nu}(\phi(0,0), 30t), \mathcal{P}'_{\mu,\nu}(\phi(4x,x), \frac{t}{72}),\\ &\mathcal{P}'_{\mu,\nu}(\phi(0,0), 1890t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{7560}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{7560}{37}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(3x,x), \frac{t}{888}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{3360}{37}t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{420}{31}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{420}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{3780}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), \frac{3780}{31}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(2x,x), \frac{t}{372}), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{210}{31}t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{28}{9}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{28}{9}t), \mathcal{P}'_{\mu,\nu}(\phi(0,2x), \frac{112}{9}t), \mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), \frac{112}{9}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(0,3x), 112t), \mathcal{P}'_{\mu,\nu}(\phi(3x,-3x), 112t), \mathcal{P}'_{\mu,\nu}(\phi(x,x), \frac{t}{1620}),\\ &\mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{4}{3}t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{t}{315}), \mathcal{P}'_{\mu,\nu}(\phi(0,4x), 144t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(4x,-4x), 144t), \mathcal{P}'_{\mu,\nu}(\phi(0,3x), 16t), \mathcal{P}'_{\mu,\nu}(\phi(3x,-3x), 16t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(0,2x), 4t), \mathcal{P}'_{\mu,\nu}(\phi(2x,-2x), 4t), \mathcal{P}'_{\mu,\nu}(\phi(0,x), \frac{12}{7}t),\\ &\mathcal{P}'_{\mu,\nu}(\phi(x,-x), \frac{12}{7}t), \mathcal{P}'_{\mu,\nu}(\phi(0,0), \frac{16}{7}t)) \qquad (33)\\ &\leq_L \mathcal{P}_{\mu,\nu}(181440f(2x) - 92897280f(x), t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus

$$\Lambda(x,t) \leq_L \mathcal{P}_{\mu,\nu}(f(2x) - 2^9 f(x), t), \tag{34}$$

where $\Lambda(x,t)$ is defined in (5). Thus

$$\Lambda\left(2^{\frac{1}{2}(l-1)}x, 2^{\frac{9}{2}(l+1)}t\right) \leq_L \mathcal{P}_{\mu,\nu}(2^{-9l} f(2^l x) - f(x), t) \tag{35}$$

for all $x \in X$ and $t > 0$. We consider the set $\Omega = \{h : X \longrightarrow Y\}$ and introduce the generalized metric on X as follows:

$$\begin{aligned}d(h_1, h_2) := \inf\{C \in (0,\infty) : \Lambda\left(2^{\frac{1}{2}(l-1)}x, t\right)\\ \leq_L \mathcal{P}_{\mu,\nu}(h_1(x) - h_2(x), Ct),\ \forall x \in X, t > 0\}.\end{aligned}$$

if there exist such constant C, and $d(h_1, h_2) = \infty$, otherwise. It is easy to check that d is a complete metric (see also [11]). Define the mapping $\mathcal{J} : \Omega \longrightarrow \Omega$ by $\mathcal{J}h(x) = 2^{-9l}h(2^l x)$ for all $x \in X$. Given $h_1, h_2 \in \Omega$ and ϵ be an arbitrary constant with $d(h_1, h_2) < \epsilon$. Then

$$\Lambda\left(2^{\frac{1}{2}(l-1)}x, t\right) \leq_L \mathcal{P}_{\mu,\nu}(h_1(x) - h_2(x), \epsilon t)$$

for all $x \in X$ and $t > 0$. So,

$$\begin{aligned}\Lambda\left(2^{\frac{1}{2}(l-1)}x, \frac{2^{9l}}{\alpha^l}t\right) &\leq_L \mathcal{P}_{\mu,\nu}(h_1(2^l x) - h_2(2^l x), 2^{9l}\epsilon t)\\ &= \mathcal{P}_{\mu,\nu}(\mathcal{J}h_1(x) - \mathcal{J}h_2(x), \epsilon t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Hence, $d(\mathcal{J}h_1, \mathcal{J}h_2) \leq \frac{\alpha^l}{2^{9l}} d(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. Thus $\mathcal{J}$ is a strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha^l}{2^{9l}}$. It follow from (35) that $d(f, \mathcal{J}f) \leq 2^{\frac{-9}{2}(l+1)}$. By Theorem 2.2, there exists a mapping $N : X \longrightarrow Y$ satisfying:

(1) N is a unique fixed point of $\mathcal{J}$ in the set $\Omega_1 = \{h \in \omega : d(f, h) < \infty\}$, which is satisfied

$$N(2^l x) = 2^{9l}N(x) \tag{36}$$

for all $x \in X$. In other words, there exists a $C > 0$ with

$$\Lambda\left(2^{\frac{1}{2}(l-1)}x, t\right) \leq_L \mathcal{P}_{\mu,\nu}(f(x) - N(x), Ct)$$

for all $x \in X$ and $t > 0$.

(2) $d(\mathcal{J}^n f, N) \to 0$ as $n \to \infty$. This implies that

$$N(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} 2^{-9ln} f(2^{ln}x)$$

for all $x \in X$.

(3) For every $f \in \Omega$, we have $d(f, N) \leq \frac{1}{1-\frac{\alpha^l}{2^{9l}}} d(\mathcal{J}f, f)$. Since, $d(\mathcal{J}f, f) \leq 2^{\frac{-9}{2}(l+1)}$, we have $d(f, N) \leq \frac{\alpha^{-\frac{1}{2}(l-1)}}{|512-\alpha|}$. The last inequality shows that

$$\Lambda(x, t) \leq_L \mathcal{P}_{\mu,\nu}\left(f(x) - N(x), \frac{\alpha^{-\frac{1}{2}(l-1)}}{|512 - \alpha|}t\right) \tag{37}$$

for all $x \in X$ and $t > 0$. it follows from the relations (2) and (37) that the inequality (4) holds. Replacing $2^{ln}x$ and $2^{ln}y$ by x and y in (3), respectively, we get

$$\begin{aligned}\mathcal{P}'_{\mu,\nu}\left(\phi(x, y), \frac{2^{9n}}{\alpha^n}t\right) &\leq_L \mathcal{P}'_{\mu,\nu}(D_n f(2^{ln}x, 2^{ln}y), 2^{9n}t)\\ &\leq_L \mathcal{P}_{\mu,\nu}\left(2^{-9ln}D_n f(2^{ln}x, 2^{ln}y), t\right)\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Letting n tends to infinity, we see that N is a nonic mapping. □

The following corollary is a direct consequence of Theorem 2.3 concerning the stability of (1).

Corollary 2.4. *Let λ be a nonnegative real number with $\lambda \neq 9$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \longrightarrow Y$ is a mapping such that*

$$\mathcal{P}'_{\mu,\nu}((\|x\|^\lambda + \|y\|^\lambda)z_0, t) \leq_L \mathcal{P}_{\mu,\nu}(D_n f(x,y), t)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique nonic mapping $N : X \longrightarrow Y$ such that

$$\mathcal{P}'_{\mu,\nu}(\|x\|^\lambda z_0, \min\{\frac{945}{5^\lambda+1}, \frac{342921600}{8^\lambda}, \frac{381024000}{6^\lambda}, \frac{1371688400}{37}, \frac{685843200}{31\cdot 2^\lambda}, \frac{20321280}{3^\lambda}, \frac{26127360}{4^\lambda}\}|512 - 2^\lambda|t) \leq_L \mathcal{P}_{\mu,\nu}(N(x) - f(x), t)$$

for all $x \in X$ and $t > 0$.

Proof. Setting $\phi(x,y) := (\|x\|^\lambda + \|y\|^\lambda)z_0$ and applying Theorem 2.3, we get the desired result. □

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