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Arc Complexes, Sphere Complexes, and Goeritz Groups

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Abstract. We show that if a Heegaard splitting is obtained by gluing a splitting of Hempel distance at least 4 and the genus-1 splitting of $S^2 \times S^1$, then the Goeritz group of the splitting is finitely generated. To show this, we first provide a sufficient condition for a full subcomplex of the arc complex for a compact orientable surface to be contractible, which generalizes the result by Hatcher that the arc complexes are contractible. We then construct infinitely many Heegaard splittings, including the above-mentioned Heegaard splitting, for which suitably defined complexes of Haken spheres are contractible.

Introduction

Let $\Sigma_{g,n}$ be a compact connected orientable surface of genus g with n holes, where $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g \geq 1$. As an analogue of the curve complex, the *arc complex* $\mathcal{A}_{g,n}$ of $\Sigma_{g,n}$ is defined to be the simplicial complex whose vertices are isotopy classes of essential arcs in $\Sigma_{g,n}$ and whose k-simplices are collections of $k+1$ vertices represented by pairwise disjoint and nonisotopic arcs in $\Sigma_{g,n}$. Hatcher [13] proved that the complex $\mathcal{A}_{g,n}$ is contractible. See also Cho, McCullough, and Seo [8], Irmak and McCarthy [15] and Korkmaz and Papadopoulos [18] for related works on arc complexes.

In Section 1, we provide a useful sufficient condition for a full subcomplex of the arc complex to be contractible (Theorem 1.3). Since the arc complex $\mathcal{A}_{g,n}$ itself satisfies this condition, it is contractible, which gives an updated proof for Hatcher's result. Moreover, we also show that the full subcomplex $\mathcal{A}^*_{g,n}$ of $\mathcal{A}_{g,n}$, with $n \geq 2$, spanned by vertices of arcs connecting different boundary components is contractible.

A genus-g *Heegaard splitting* of a closed orientable 3-manifold M is a decomposition of the manifold into two handlebodies of the same genus g. That is, $M = V \cup W$ and $V \cap W = \partial V = \partial W = \Sigma$, where V and W are handlebodies of genus g, and Σ is their common boundary surface. We simply denote by $(V, W; \Sigma)$ the splitting and call the surface Σ the *Heegaard surface* of the splitting. It is well known that every closed orientable 3-manifold admits a genus-g Heegaard splitting for some genus $g \geq 0$. Given a genus-g Heegaard

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splitting $(V, W; \Sigma)$ with $g \geq 2$ for M, a separating sphere P embedded in M is called a *Haken sphere* if $P \cap \Sigma$ is a single essential simple closed curve in Σ. Two Haken spheres P and Q are said to be *equivalent* if $P \cap \Sigma$ is isotopic to $Q \cap \Sigma$ in Σ. When the splitting $(V, W; \Sigma)$ admits Haken spheres, we denote by $\mu = \mu(V, W; \Sigma)$ the minimal cardinality of $P \cap Q \cap \Sigma$, where P and Q vary over all pairwise nonequivalent Haken spheres for the splitting. The *sphere complex* for the splitting $(V, W; \Sigma)$ is then defined to be the simplicial complex whose vertices are equivalence classes of Haken spheres for the splitting and whose k-simplices are collections of $k+1$ vertices represented by Haken spheres $P_0, P_1, \dots, P_k$, respectively, such that the cardinality of $P_i \cap P_j \cap \Sigma$ is μ for all $0 \leq i < j \leq k$.

The structures of sphere complexes for genus-2 Heegaard splittings have been studied by several authors. If a genus-2 Heegaard splitting for a 3-manifold admits Haken spheres, then the manifold is one of S^3, $S^2 \times S^1$, lens spaces, and their connected sums. It is known that the sphere complex for the genus-2 Heegaard splitting of S^3 is connected and even contractible from Scharlemann [24], Akbas [1], and Cho [3]. Lei [19] and Lei and Zhang [20] proved that the sphere complexes are connected for genus-2 Heegaard splittings of nonprime 3-manifolds, that is, the connected sum whose summands are lens spaces or $S^2 \times S^1$, and later, Cho and Koda [7] showed that they are actually contractible.

In Section 2, we study the Heegaard splitting for a 3-manifold having a single $S^2 \times S^1$ summand in its prime decomposition. We prove that if a genus-g Heegaard splitting with $g \geq 2$ is the splitting obtained by gluing a genus-$(g-1)$ Heegaard splitting of Hempel distance at least 2 and the genus-1 Heegaard splitting of $S^2 \times S^1$, then its sphere complex is a contractible, $(4g-5)$-dimensional complex (Corollary 2.7). In fact, we show that the sphere complex is isomorphic to the full subcomplex $\mathcal{A}^*_{g-1,2}$ of the arc complex $\mathcal{A}_{g-1,2}$. As a special case, the sphere complex for the genus-2 Heegaard splitting of $S^2 \times S^1$ is a contractible, three-dimensional complex (Corollary 2.8).

For a Heegaard splitting $(V, W; \Sigma)$ for a 3-manifold, the *Goeritz group* is defined to be the group of isotopy classes of the orientation-preserving homeomorphisms of the manifold that preserve V and W setwise. We might expect that the Goeritz group would be simpler once we have more complicated Heegaard splitting in some sense. One of the important results on Goeritz group in this view point is that the Goeritz groups of Heegaard splittings of Hempel distance at least 4 are all finite groups, which is given in Johnson [16]. On the other hand, it is hard to determine whether the Goeritz group of a given Heegaard splitting of low Hempel distance is finitely generated or not. Even it remains open whether the Goeritz group of a Heegaard splitting of genus at least 3 for the 3-sphere is finitely generated or not. The Goeritz groups of genus-1 Heegaard splittings are easy to describe, and for genus-2 reducible Heegaard splittings, their Goeritz groups have been studied in [10; 24; 1; 3; 4; 5; 6; 7].

In the final section, we study the Goeritz groups of the Heegaard splittings given in Section 2. The main result is that, for a Heegaard splitting obtained by

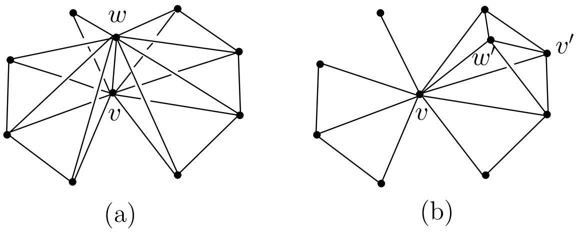

Figure 1

gluing a Heegaard splitting of Hempel distance at least 4 and the genus-1 Heegaard splitting of $S^2 \times S^1$, its Goeritz group is finitely generated (Corollary 3.4). This can be compared with the result of Johnson [17], who showed that if a Heegaard splitting is obtained by gluing a Heegaard splitting of high Hempel distance and the genus-1 Heegaard splitting of S^3, then its Goeritz group is finitely generated.

Throughout the paper, we will work in the PL category. By $\mathrm{Nbd}(X; Y)$ we denote a regular neighborhood of a subspace X of a polyhedral space Y.

1. Arc Complexes

We start with recalling a sufficient condition for contractibility of a simplicial complex, introduced in [3], which is a generalization of the proof of Theorem 5.3 in [21].

Let $\mathcal{K}$ be a simplicial complex. We say that a vertex w is *adjacent* to a vertex v of $\mathcal{K}$ if w equals v or if w is joined to v by an edge of $\mathcal{K}$. We denote by $st(v)$ the *star* of a vertex v of $\mathcal{K}$ that is the full subcomplex of $\mathcal{K}$ spanned by the vertices adjacent to v. An *adjacency pair* (X, v) in $\mathcal{K}$ is a finite multiset that consists of vertices of $st(v)$. Here the finite multiset X is a finite set $\{v_1, v_2, \ldots, v_k\}$ allowed to have $v_i = v_j$ for some $1 \leq i < j \leq k$. A *remoteness function* for a vertex v_0 of $\mathcal{K}$ is a function r from the set of vertices of $\mathcal{K}$ to $\mathbb{N} \cup \{0\}$ satisfying $r^{-1}(0) \subset st(v_0)$. A *blocking function* for a remoteness function r is a function b from the set of adjacency pairs of $\mathcal{K}$ to $\mathbb{N} \cup \{0\}$ satisfying the following properties for every adjacency pair (X, v) with $r(v) > 0$:

(1) If $b(X, v) = 0$, then there exists a vertex w of $st(v)$ such that $r(w) < r(v)$ and (X, w) is also an adjacency pair (see Figure 1(a)).
(2) If $b(X, v) > 0$, then there exist an element v' of X and a vertex w' of $st(v')$ such that
  (a) $r(w') < r(v')$,
  (b) if an element x of X is adjacent to v', then x is also adjacent to w', and
  (c) $b(X \setminus \{v'\} \cup \{w'\}, v) < b(X, v)$, where $X \setminus \{v'\} \cup \{w'\}$ is the multiset obtained by removing one instance of v' from X and adding one instance of w' to X (see Figure 1(b)).

A simplicial complex $\mathcal{K}$ is called a *flag complex* if any collection of pairwise distinct $k+1$ vertices span a k-simplex whenever any two of them span a 1-simplex.

LEMMA 1.1 ([3], Prop. 3.1). *Let $\mathcal{K}$ be a flag complex with a base vertex v_0. If there exists a remoteness function r on the set of vertices of $\mathcal{K}$ for v_0 that admits a blocking function b, then $\mathcal{K}$ is contractible.*

The idea of the proof given in [3] is to show that the homotopy groups are all trivial. That is, given any simplicial map $f : S^q \to \mathcal{K}$, $q \geq 0$, with respect to a triangulation Δ of S^q, we find a simplicial map $g : S^q \to \mathcal{K}$ with respect to a triangulation Δ' obtained from Δ by finitely many barycentric subdivisions such that g is homotopic to f and the image of g is contained in $st(v_0)$.

Now we return to the arc complex $\mathcal{A}_{g,n}$ of a compact orientable surface $\Sigma_{g,n}$ of genus g with n holes, where $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g \geq 1$. It is a standard fact that any collection of isotopy classes of essential arcs in $\Sigma_{g,n}$ can be realized by a collection of representative arcs having pairwise minimal intersection. In particular, for a collection $\{v_0, v_1, \dots, v_k\}$ of vertices of $\mathcal{A}_{g,n}$, if v_i and v_j are joined by an edge for each $0 \leq i < j \leq k$, then $\{v_0, v_1, \dots, v_k\}$ spans a k-simplex. Thus, we have the following:

LEMMA 1.2. *The arc complex $\mathcal{A}_{g,n}$ is a flag complex, and any full subcomplex of $\mathcal{A}_{g,n}$ is also a flag complex.*

Let α and α_0 be essential arcs on the surface $\Sigma_{g,n}$ that intersect each other transversely and minimally. A component β of α_0 cut off by $\alpha \cap \alpha_0$ is said to be *outermost* if $\beta \cap \alpha$ consists of a single point. We note that there exist exactly two such subarcs of α_0. The intersection $\beta \cap \alpha$ cuts α into two subarcs β' and β''. We call the two new arcs $\alpha' = \beta \cup \beta'$ and $\alpha'' = \beta \cup \beta''$ the *arcs obtained from α by surgery along β*. We observe that by a small isotopy α' and α'' are disjoint from α, and $|\alpha_0 \cap \alpha'| < |\alpha_0 \cap \alpha|$ and $|\alpha_0 \cap \alpha''| < |\alpha_0 \cap \alpha|$ since the intersection $\beta \cap \alpha$ is no longer counted.

THEOREM 1.3. *Any full subcomplex $\mathcal{A}$ of $\mathcal{A}_{g,n}$ satisfying the following property is contractible.*

Surgery Property: *Let α and α_0 be representative arcs of vertices of $\mathcal{A}$ that intersect each other transversely an minimally. If $\alpha \cap \alpha_0 \neq \emptyset$, then at least one of the two arcs obtained from α by surgery along an outermost subarc of α_0 cut off by $\alpha \cap \alpha_0$ represents a vertex of $\mathcal{A}$.*

Proof. Fix a base vertex v_0 of $\mathcal{A}$. By Lemmas 1.1 and 1.2 it suffices to find a remoteness function for v_0 that admits a blocking function. For each vertex v of $\mathcal{A}$, define $r(v)$ to be the minimal cardinality of the intersection $\alpha \cap \alpha_0$, where α and α_0 are representative arcs of v and v_0, respectively. By definition, r is a remoteness function for v_0.

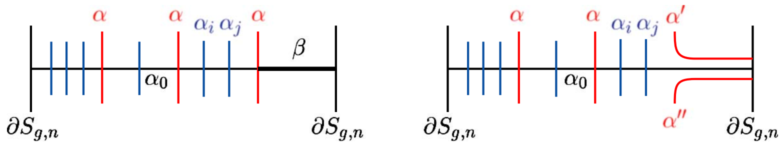

Figure 2

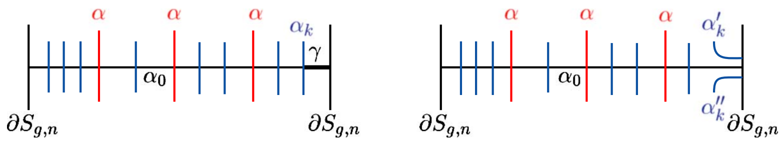

Figure 3

Let (X, v) be an adjacent pair in $\mathcal{A}$, where $r(v) > 0$ and $X = \{v_1, v_2, \dots, v_n\}$. Choose representative arcs $\alpha, \alpha_1, \alpha_2, \dots, \alpha_n$, and α_0 of $v, v_1, v_2, \dots, v_n$, and v_0, respectively, so that they have transverse and pairwise minimal intersection, and every crossing is a double point. Since $r(v) > 0$, we have $\alpha \cap \alpha_0 \neq \emptyset$. Among the two subarcs of α_0 cut off by $\alpha \cap \alpha_0$, choose one, say β, so that the cardinality of $\beta \cap (\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n)$ is minimal, and then denote this cardinality by $b_0 = b_0(\alpha, \alpha_1, \alpha_2, \dots, \alpha_n, \alpha_0)$. We define $b(X, v)$ to be the minimal number of b_0 among all such representative arcs of $v, v_1, v_2, \dots, v_n$, and v_0. In the following, we will show that b is a blocking function for r.

First, suppose that $b(X, v) = 0$. Then by an isotopy we may assume that $\beta \cap (\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) = \emptyset$. By the surgery property at least one of the two arcs obtained from α by surgery along β, say α', represents a vertex w of $\mathcal{A}$. See Figure 2. By the construction, v is adjacent to w, and (X, w) is an adjacent pair. Further, we have $r(w) \leq |\alpha_0 \cap \alpha'| < |\alpha_0 \cap \alpha| = r(v)$. Next, suppose that $b(X, v) > 0$. We may assume that $\beta \cap (\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) = b(X, v)$ by an isotopy. Let γ be the outermost subarc of α_0 cut off by $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ that is contained in β. The point $(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) \cap \gamma$ is contained in α_k for some $k \in \{1, 2, \dots, n\}$. Then again by the surgery property at least one of the arcs obtained from α_k by surgery along γ represents a vertex, say w', of $\mathcal{A}$. See Figure 3. By the construction we have $r(w') < r(v_k)$, $b(X \setminus \{v_k\} \cup \{w'\}, v) < b(X, v)$, and each element x of X adjacent to v_k is also adjacent to w'. This completes the proof. □

Let $n \geq 2$. We denote by $\mathcal{A}^*_{g,n}$ the full subcomplex of $\mathcal{A}_{g,n}$ spanned by the vertices represented by simple arcs connecting the different components of the boundary of $\Sigma_{g,n}$. It is easy to verify that the arc complex $\mathcal{A}_{g,n}$ itself and the subcomplex $\mathcal{A}^*_{g,n}$ satisfy the surgery property. Thus we have the following.

Corollary 1.4. *The complexes $\mathcal{A}_{g,n}$ and $\mathcal{A}^*_{g,n}$ are contractible.*

We end the section with the following lemma for later use.

Lemma 1.5. *The dimension of the complex* $\mathcal{A}^*_{g,2}$ *is* $4g-1$.

Proof. Let $A = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a maximal set of mutually disjoint, mutually nonisotopic simple arcs connecting the different components of the boundary of $\Sigma_{g,2}$. By contracting each of these boundary components of $\Sigma_{g,2}$ into a point we get a closed orientable surface Σ of genus g with two dots, say v^+ and v^-. On this surface, each of the arcs of A connects the two dots. Hence, A decomposes Σ into cubes with the vertex sets $\{v^+, v^-\}$. Now, the assertion follows easily from Euler characteristic considerations. □

2. Sphere Complexes

Let $(V, W; \Sigma)$ be a Heegaard splitting of genus $g \geq 2$ of a closed orientable 3-manifold M. A separating sphere P embedded in M is called a *Haken sphere* for the spitting if it intersects Σ transversely in a single essential simple closed curve. Since P is separating in M, the curve $P \cap \Sigma$ is separating in Σ. Two Haken spheres P and Q are said to be *equivalent* if $P \cap \Sigma$ and $Q \cap \Sigma$ are isotopic in Σ. We denote by $\mu = \mu(V, W; \Sigma)$ the minimal cardinality of $P \cap Q \cap \Sigma$, where P and Q vary over all pairwise nonequivalent Haken spheres for $(V, W; \Sigma)$. We note that μ is a nonnegative even number. It was shown in [23] that $\mu(V, W; \Sigma) = 4$ when the genus of the splitting is 2. When the given splitting $(V, W; \Sigma)$ admits Haken spheres, the *sphere complex* for the splitting is defined as in Introduction, which we will denote by $\mathcal{H} = \mathcal{H}(V, W; \Sigma)$.

Given a closed orientable surface Σ of genus $g \geq 1$, the *curve complex* $\mathcal{C}_g$ is defined to be the simplicial complex whose vertices are isotopy classes of simple closed curves in Σ and whose k-simplices are collections of $k+1$ vertices represented by pairwise disjoint and nonisotopic curves in Σ. It is known that the curve complex $\mathcal{C}_g$ is connected and $(3g-4)$-dimensional. When the surface is the Heegaard surface of a Heegaard splitting $(V, W; \Sigma)$ of a 3-manifold, we have the two full subcomplexes $\mathcal{D}_V$ and $\mathcal{D}_W$ of $\mathcal{C}_g$, which are spanned by the vertices of the simple closed curves bounding disks in V and W, respectively. Then we define the *Hempel distance* of the splitting to be the minimal simplicial distance in $\mathcal{C}_g$ between the two subcomplexes $\mathcal{D}_V$ and $\mathcal{D}_W$. That is, the minimal number of edges among all the paths in $\mathcal{C}_g$ from a vertex of $\mathcal{D}_V$ to a vertex of $\mathcal{D}_W$. We refer the reader to [14] for details on the Hempel distance. In the case of genus-1 Heegaard splitting for a 3-manifold, we have that the Hempel distance is 0 if the manifold is $S^2 \times S^1$ and ∞ otherwise.

Let $(V, W; \Sigma)$ be a Heegaard splitting of a closed orientable 3-manifold M. A nonseparating disk E_0 in V is called a *reducing disk* if ∂E_0 bounds a disk in W. We note that if there exists a reducing disk in M, then M has an $S^2 \times S^1$ summand for its prime decomposition, and vice versa by Waldhausen's uniqueness of Heegaard splittings of $S^2 \times S^1$ [25] and Haken's lemma [12]. Given any simple closed curve γ in Σ intersecting ∂E_0 transversely in a single point, the boundary of $\mathrm{Nbd}(\partial E_0 \cup \gamma; \Sigma)$ is a separating simple closed curve in Σ that bounds a disk in each of V and W. Thus, if the genus of the splitting is greater than 1, such a

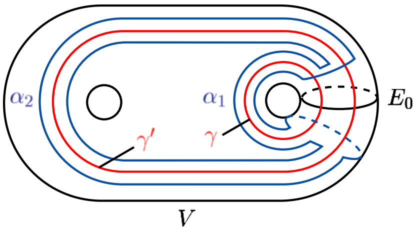

Figure 4

simple closed curve γ determines a Haken sphere $P = P(\gamma)$ for the splitting, the union of those two disks in V and W.

LEMMA 2.1. *Let $(V, W; \Sigma)$ be a genus-g Heegaard splitting of a 3-manifold M, where $g \geq 2$. Let E_0 be a reducing disk in V. Let γ and γ' be simple closed curves each of which intersects ∂E_0 transversely in a single point. Let $P = P(\gamma)$ and $P' = P(\gamma')$ be Haken spheres determined by the curves γ and γ', respectively. Then there exists an orientation-preserving homeomorphism of the manifold M onto itself that maps P to P' while preserving each of V and W setwise.*

Proof. If γ' is isotopic to γ up to Dehn twists about ∂E_0, then P and P' are equivalent, and thus there is nothing to prove. Otherwise, suppose first that γ' is disjoint from γ up to Dehn twists about ∂E_0. We may assume without loss of generality that γ' itself is disjoint from γ because Dehn twists about ∂E_0 do not change the equivalence class of P'. The boundary of $\text{Nbd}(\partial E_0; \Sigma)$ consists of two simple closed curves δ_1 and δ_2. For each $i \in \{1, 2\}$, the intersection of δ_i and $\gamma \cup \gamma'$ cuts δ_i into two arcs $\delta_{i,1}$ and $\delta_{i,2}$. We set $\alpha_i = ((\gamma \cup \gamma') \setminus \text{Nbd}(\partial E_0; \Sigma)) \cup \delta_{1,1} \cup \delta_{2,i}$. See Figure 4. We note that the union $\alpha_1 \cup \alpha_2$ bounds proper annuli A and B in V and W, respectively. Then a single Dehn twist about the annulus A, which extends to the Dehn twist about the torus $A \cup B$, is the required homeomorphism of M. Here we remark that this homeomorphism is actually a "sliding" of a foot of the 1-handle of each of V and W whose belt sphere is ∂E_0.

The general case follows from the connectivity of the arc complex as follows. The simple closed curve ∂E_0 cuts Σ into a genus-$(g-1)$ surface $\Sigma_{g-1,2}$ with two holes ∂E_0^+ and ∂E_0^- coming from ∂E_0. On the surface $\Sigma_{g-1,2}$, γ and γ' are simple arcs β and β' connecting the two holes. Since the complex $\mathcal{A}^*_{g-1,2}$ is connected by Corollary 1.4, there exists a sequence $\beta = \beta_1, \beta_2, \ldots, \beta_n = \beta'$ of mutually nonisotopic, essential arcs in $\Sigma_{g-1,2}$ connecting ∂E_0^+ and ∂E_0^- such that β_i is disjoint from β_{i+1} for each $i \in \{1, 2, \ldots, n-1\}$. Gluing ∂E_0^+ and ∂E_0^- back, this sequence gives rise to a sequence of simple closed curves $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma'$ such that γ_{i+1} is disjoint from γ_i up to Dehn twists about ∂E_0 for each $i \in \{1, 2, \ldots, n-1\}$. Then there exists an orientation-preserving homeomorphism g_i of the manifold M onto itself that maps $P(\gamma_i)$ to $P(\gamma_{i+1})$

while preserving each of V and W setwise. Then the composition $g_{n-1}g_{i-2}\cdots g_1$ is the desired homeomorphism. □

Let $(V, W; \Sigma)$ be a genus-g Heegaard splitting of a 3-manifold M with $g \geq 2$. Suppose that there exists a reducing disk E_0 in V. We denote by $\mathcal{H}_{E_0}$ the simplicial complex whose vertices are equivalence classes of Haken spheres $P = P(\gamma)$ determined by simple closed curves γ in Σ intersecting ∂E_0 transversely in a single point and whose k-simplices are collections of $k+1$ vertices represented by pairwise nonequivalent Haken spheres $P(\gamma_0), P(\gamma_1), \ldots, P(\gamma_k)$ such that the minimal cardinality of each $P(\gamma_i) \cap P(\gamma_j) \cap \Sigma$ is 4 for $0 \leq i < j \leq k$ (equivalently the arcs γ_i and γ_j are disjoint from each other). We observe that a Haken sphere P represents a vertex of $\mathcal{H}_{E_0}$ if and only if P cuts off from V a solid torus whose meridian disk is E_0. By construction, if $\mu(V, W; \Sigma) = 4$, then the complex $\mathcal{H}_{E_0}$ is a full subcomplex of the sphere complex $\mathcal{H}$ of the splitting $(V, W; \Sigma)$. The following lemma is immediate from the definition of the complex $\mathcal{H}_{E_0}$ with Corollary 1.4 and Lemma 1.5.

Lemma 2.2. *Let $(V, W; \Sigma)$ be a genus-g Heegaard splitting of a closed orientable 3-manifold M with $g \geq 2$. Let E_0 be a reducing disk in V. Then the complex $\mathcal{H}_{E_0}$ is isomorphic to the complex $\mathcal{A}^*_{g-1,2}$, and hence it is a contractible, $(4g-5)$-dimensional complex.*

Proposition 2.3. *Let $(V, W; \Sigma)$ be a genus-g Heegaard splitting of a closed orientable 3-manifold M with $g \geq 2$. Suppose that there exists a unique reducing disk E_0 in V and also that $\mu(V, W; \Sigma) > 0$. Then the sphere complex $\mathcal{H}$ for the splitting $(V, W; \Sigma)$ coincides with the complex $\mathcal{H}_{E_0}$.*

Proof. Let P be a Haken sphere for the splitting $(V, W; \Sigma)$ intersecting E_0 transversely and minimally. Suppose that $P \cap E_0 \neq \emptyset$. At least one, say M_1, of the closed 3-manifolds M_1 and M_2 obtained by cutting M along P and then capping off the resulting boundary spheres by adding 3-balls has an $S^2 \times S^1$ summand for its prime decomposition. Then as mentioned in the last paragraph before Lemma 2.1, the V part of the Heegaard splitting of M_1 naturally induced from $(V, W; \Sigma)$ contains a reducing disk, which gives rise to a reducing disk of V that is not isotopic to E_0. This contradicts the uniqueness of E_0. Therefore, any Haken sphere is disjoint from the reducing disk E_0. It suffices to show that P cuts off a solid torus from V whose meridian disks is E_0. Suppose not, that is, the component Σ' of Σ cut off by $P \cap \Sigma$ containing ∂E_0 is a compact surface of genus at least 2. Then we can choose a simple closed curve γ in Σ' intersecting ∂E_0 transversely in a single point such that the Haken sphere $Q = Q(\gamma)$ is disjoint from and is not equivalent to P. We have then $0 < \mu(V, W; \Sigma) \leq |P \cap Q \cap \Sigma| = 0$, a contradiction. □

Now we will construct (infinitely many) Heegaard splittings $(V, W; \Sigma)$ satisfying the conditions in Proposition 2.3, that is,

- there exists a unique reducing disk in V; and
- $\mu(V, W; \Sigma) > 0$.

Let $(V_1, W_1; \Sigma_1)$ and $(V_2, W_2; \Sigma_2)$ be genus-g_1 and genus-g_2 Heegaard splittings for 3-manifolds M_1 and M_2, respectively. Let B_1 and B_2 be 3-balls in M_1 and M_2 that intersect the Heegaard surfaces Σ_1 and Σ_2 in a single disk, respectively. Removing the interiors of B_1 and B_2, and identifying ∂B_1 and ∂B_2, we can construct a genus-$(g_1 + g_2)$ Heegaard splitting $(V, W; \Sigma)$ for the connected sum $M = M_1 \# M_2$ such that V and W are considered as boundary connected sums of V_1 and V_2 and of W_1 and W_2, respectively. We call the splitting $(V, W; \Sigma)$ a Heegaard splitting for M *obtained from* $(V_1, W_1; \Sigma_1)$ *and* $(V_2, W_2; \Sigma_2)$. We note that the sphere $P = \partial B_1 = \partial B_2$ is a Haken sphere for the splitting $(V, W; \Sigma)$. In the remaining of the section, we always assume the following:

- $(V_1, W_1; \Sigma_1)$ is a genus-$(g - 1)$ Heegaard splitting for a closed orientable 3-manifold M_1 with $g \geq 2$, and $(V_2, W_2; \Sigma_2)$ is the genus-1 Heegaard splitting for $S^2 \times S^1$.
- $(V, W; \Sigma)$ is a genus-g Heegaard splitting for $M = M_1 \# (S^2 \times S^1)$ obtained from $(V_1, W_1; \Sigma_1)$ and $(V_2, W_2; \Sigma_2)$ by the above construction, and $P = \partial B_1 = \partial B_2$ is the Haken sphere for the splitting $(V, W; \Sigma)$.
- E_0 and E'_0 with $\partial E_0 = \partial E'_0$ are meridian disks of the solid tori V_2 and W_2, respectively, which are reducing disks for the splitting $(V, W; \Sigma)$.

We start with the following two lemmas.

LEMMA 2.4. *Let δ be an essential simple closed curve in Σ that is disjoint from and not isotopic to ∂E_0. Suppose that δ does not cut off from Σ a torus with one hole containing ∂E_0. If δ bounds disks in V and W simultaneously, then the Hempel distance of the splitting* $(V_1, W_1; \Sigma)$ *is* 0.

From the lemma it is easy to see that if the Hempel distance of the splitting $(V_1, W_1; \Sigma_1)$ is at least 1 and if E is an essential nonseparating disk in V that is disjoint from and not isotopic to E_0, then E cannot be a reducing disk for the splitting $(V, W; \Sigma)$.

Proof of Lemma 2.4. Suppose that δ in Σ bounds disks both in V and W. We want to find an essential simple closed curve in Σ_1 that bounds disks both in V_1 and W_1.

Among the simple closed curves in Σ that intersect ∂E_0 transversely in a single point, choose one, say γ, so that γ intersects δ minimally. Then either γ is disjoint from δ or γ intersects δ in a single point. (If δ is nonseparating and $\delta \cup \partial E_0$ is separating in Σ, then we have to choose such a curve γ so that γ intersects δ in a single point. Otherwise, we can choose γ disjoint from δ.) Let $P(\gamma)$ be the Haken sphere determined by γ, that is, $P(\gamma) \cap \Sigma$ is the boundary of $\mathrm{Nbd}(\partial E_0 \cup \gamma; \Sigma)$. Applying Lemma 2.1, we may assume that the Haken sphere $P (= \partial B_1 = \partial B_2)$ equals $P(\gamma)$ and that $\mathrm{Nbd}(E_0 \cup \gamma; V)$ and $\mathrm{Nbd}(E'_0 \cup \gamma; W)$ are solid tori V_2 and W_2, respectively, with the interior of the 3-ball B_2 removed.

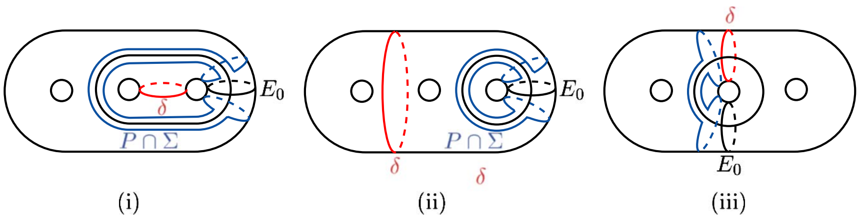

Figure 5

If γ is disjoint from δ, then by isotopy we may assume that δ lies in Σ_1 outside the disk $B_1 \cap \Sigma_1$ and that the union of the disk in V and that in W bounded by δ are disjoint from and not isotopic to P. Apparently, δ remains to be essential in Σ_1. See Figure 5(i) and (ii). Thus, the Hempel distance of $(V_1, W_1; \Sigma_1)$ is 0 in this case. If γ intersects δ in a single point, then we cannot say that δ lies in Σ_1. But by isotopy we may assume that the boundary of $\mathrm{Nbd}(\partial E_0 \cup \delta \cup \gamma; \Sigma)$, which consists of two simple closed curves, lies in Σ_1 outside the disk $B_1 \cap \Sigma_1$. Any of the two simple closed curves bound disks in V and W, which can be isotoped to be disjoint from P. Since δ is not isotopic to ∂E_0 in Σ, each of these simple closed curves is essential in Σ_1. See Figure 5(iii). Again, the Hempel distance of $(V_1, W_1; \Sigma)$ is 0. □

Lemma 2.5. *Let δ_1 and δ_2 be disjoint, nonseparating simple closed curves in Σ each of which is disjoint from and not isotopic to ∂E_0. If δ_1 bounds a disk in V and δ_2 bounds a disk in W, then the Hempel distance of the splitting $(V_1, W_1; \Sigma)$ is at most* 1.

Proof. The argument will be very similar to the proof of Lemma 2.4. We note that δ_1 is possibly isotopic to δ_2. Suppose that δ_1 and δ_2 bound disks in V and W, respectively. We want to find two disjoint, essential simple closed curves in Σ_1 such that one bounds a disk in V_1 and the other in W_1. Among the simple closed curves in Σ that intersect ∂E_0 transversely in a single point, choose one, say γ, so that γ intersects $\delta_1 \cup \delta_2$ minimally. We may assume that the Haken sphere P equals $P(\gamma)$ as in the proof of Lemma 2.4. For each $i \in \{1, 2\}$, δ_i is disjoint from γ or intersects γ in a single point, and hence we have four cases.

If each of δ_1 and δ_2 is disjoint from γ, then by isotopy we may assume that δ_1 and δ_2 lie inside Σ_1 as disjoint, essential simple closed curves, and these bound disks inside V_1 and W_1, respectively. See Figure 6(i).

If one of them, say δ_1, intersects γ in a single point and the other one δ_2 is disjoint from γ, then consider the boundary of $\mathrm{Nbd}(\partial E_0 \cup \delta_1 \cup \gamma; \Sigma)$, which consists of two simple closed curves. By isotopy we may assume that both of the two simple closed curves lie inside Σ_1 as essential simple closed curves and bound disks in V_1, whereas δ_2 is an essential simple closed curve in Σ_1 disjoint from these curves and bounding a disk in W_1. See Figure 6(ii).

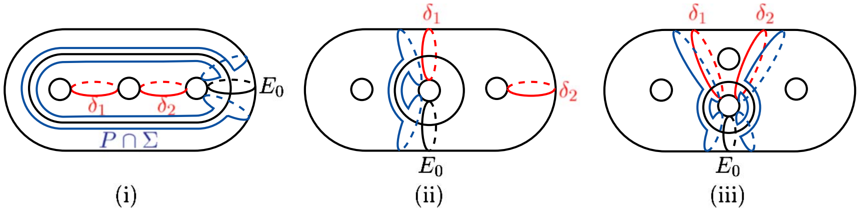

Figure 6

Finally, if each of δ_1 and δ_2 intersects γ in a single point, then consider the boundary of $\mathrm{Nbd}(\partial E_0 \cup \delta_1 \cup \delta_2 \cup \gamma; \Sigma)$, which consists of three simple closed curves. By isotopy again, we may assume that all the three curves lie in Σ_1. Among the three curves, one is a component of the boundary of $\mathrm{Nbd}(\partial E_0 \cup \delta_1 \cup \gamma; \Sigma)$, which bounds a disk in V_1, and another one is a component of the boundary of $\mathrm{Nbd}(\partial E_0 \cup \delta_2 \cup \gamma; \Sigma)$, which bounds a disk in W_1. (The third one may bound a disk neither in V_1 nor in W_1.) See Figure 6(iii). Again, these two simple closed curves are essential in Σ_1. Therefore, in any of four cases, the Hempel distance of the splitting $(V_1, W_1; \Sigma)$ is at most 1. □

Let D and E be essential disks in the handlebody V that intersect each other transversely and minimally. A subdisk Δ of D cut off by $D \cap E$ is said to be *outermost* if $\Delta \cap E$ is a single arc. For an outermost subdisk Δ of D cut off by $D \cap E$, the arc $\Delta \cap E$ cuts E into two disks, say E' and E''. We call the two disks $E_1 = E' \cup \Delta$ and $E_2 = E'' \cup \Delta$ the *disks obtained from E by surgery along Δ*. Both of E_1 and E_2 can be isotoped to be disjoint from E. By an elementary argument of the reduced homology group $H_2(V, \partial V; \mathbb{Z})$ we can check easily that at least one of E_1 and E_2 is nonseparating if E is nonseparating.

For any simple closed curves γ and δ in the surface Σ that intersect each other transversely and minimally in at least two points, we can define similarly the two simple closed curves γ_1 and γ_2 obtained from γ by surgery along an innermost subarc of δ cut off by $\gamma \cap \delta$. Here an innermost subarc, say δ', is a component of δ cut off by $\gamma \cap \delta$ that meets γ only in its endpoints and cuts γ into two arcs, say γ' and γ''. Then $\gamma_1 = \gamma' \cup \delta'$ and $\gamma_2 = \gamma'' \cup \delta'$. If the subarc δ' meets γ from the same side, then both of γ_1 and γ_2 can be isotoped to be disjoint from γ. We also see that if γ is nonseparating, then at least one of γ_1 and γ_2 is nonseparating by an elementary argument of $H_2(\Sigma; \mathbb{Z})$.

PROPOSITION 2.6. *Suppose that the Hempel distance of the splitting* $(V_1, W_1; \Sigma)$ *is at least* 2. *Then we have the following*:

(1) *E_0 is the unique reducing disk in V, and*
(2) $\mu(V, W; \Sigma) > 0$.

Proof. Statement (2) is easy to verify. In fact, if $\mu(V, W; \Sigma) = 0$, then we might find a Haken sphere for the splitting $(V_1, W_1; \Sigma_1)$, and hence the Hempel distance of $(V_1, W_1; \Sigma_1)$ would be 0, a contradiction.

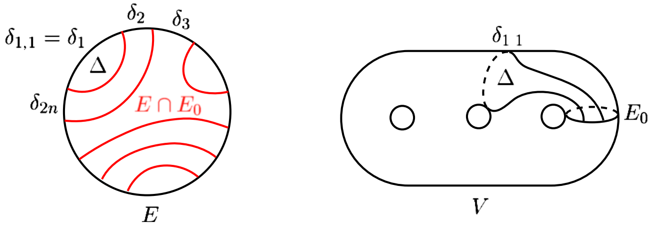

Figure 7

We prove (1). It is proved in [6] that (1) is true when $M_1 \cong S^3$ and $g = 2$. In the following, we will assume that $M_1 \ncong S^3$ and $g > 2$. Let E be an essential nonseparating disk in V that is not isotopic to E_0. We may assume that E intersects E_0 transversely and minimally. If E is disjoint from E_0, then E is not a reducing disk by Lemma 2.4. Suppose that E intersects E_0. Then ∂E_0 cuts ∂E into $2n$ ($n \geq 1$) simple arcs $\delta_1, \delta_2, \dots, \delta_{2n}$. We divide the collection of these arcs into two subcollections as

$$\{\delta_1, \delta_2, \dots, \delta_{2n}\} = \{\delta_{1,1}, \delta_{1,2}, \dots, \delta_{1,n_1}\} \sqcup \{\delta_{2,1}, \delta_{2,2}, \dots, \delta_{2,n_2}\},$$

where each of the arcs $\delta_{1,i}$ meets ∂E_0 from the same side, whereas each of $\delta_{2,j}$ does from the opposite sides. We may assume without loss of generality that there exists an outermost subdisk Δ of E cut off by $E \cap E_0$ such that $\delta_{1,1} \subset \partial\Delta$. See Figure 7.

Let $\{E_0', D_1', D_2', \dots, D_{g-1}'\}$ be a complete system of meridian disks of W, where $\partial E_0' = \partial E_0$. Fix orientations of the boundary circles $\partial E_0'$ and $\partial D_1', \partial D_2', \dots, \partial D_{g-1}'$ and assign symbols x and $y_1, y_2, \dots, y_{g-1}$ on the circles, respectively. Then any oriented simple closed curve δ in Σ intersecting the boundary circles transversely determines a word $w(\delta)$ on $\{x, y_1, y_2, \dots, y_{g-1}\}$ that can be read off from the intersections of δ with the circles. This word determines an element of the free group $\pi_1(W) = \langle x, y_1, y_2, \dots, y_{g-1}\rangle$ represented by δ. Let E_1 and E_2 be the disks obtained from E_0 by surgery along Δ. Since E_0 is nonseparating, at least one of the two, say E_1, is nonseparating. By a small isotopy we assume that E_1 is disjoint from E_0.

Claim 1. *The word $w(\delta_{1,1})$ on $\{y_1, y_2, \dots, y_{g-1}\}$ read off by the interior of the arc $\delta_{1,1}$ represents a nontrivial element of $\pi_1(W) = \langle x, y_1, y_2, \dots, y_{g-1}\rangle$.*

Proof. The disk E_1 is nonseparating, disjoint from E_0, and not isotopic to E_0, and hence, by Lemma 2.4, it is not a reducing disk. That is, ∂E_1 does not bound a disk in W. Thus, by the loop theorem $w(\partial E_1) = w(\delta_{1,1})$ determines a nontrivial element of $\pi_1(W)$. □

Claim 2. *The word $w(\delta_{1,i})$ on $\{y_1, y_2, \dots, y_{g-1}\}$ read off by the interior of the arc $\delta_{1,i}$ ($2 \leq i \leq n_1$) represents a nontrivial element of $\pi_1(W) = \langle x, y_1, y_2, \dots, y_{g-1}\rangle$.*

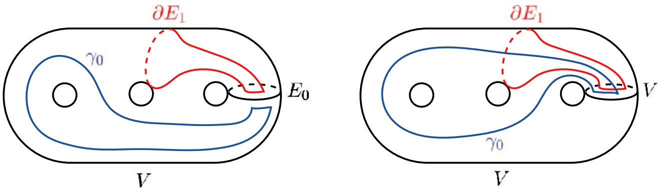

Figure 8

Proof. The arc $\delta_{1,i}$ is an innermost subarc of ∂E cut off by $\partial E \cap \partial E_0$. One of the two simple closed curves obtained from ∂E_0 by surgery along $\delta_{1,i}$ is a nonseparating curve, which we denote by γ_0. By a small isotopy we may assume that γ_0 is disjoint from ∂E_0. Further, it is easy to see that γ_0 intersects ∂E_1 transversely at most once. See Figure 8. If γ_0 does not bound a disk in W, then by the loop theorem the word $w(\gamma_0) = w(\delta_{1,i})$ determines a nontrivial element of $\pi_1(W)$, so we are done. Suppose for a contradiction that γ_0 bounds a disk in W. If γ_0 is disjoint from ∂E_1, then the Hempel distance of the splitting $(V_1, W_1; \Sigma)$ is at most 1 by Lemma 2.5, which is a contradiction. If γ_0 intersects ∂E_1 in a single point, then the boundary δ of $\mathrm{Nbd}(\gamma_0 \cup \partial E_1; \Sigma)$ is an essential, separating simple closed curve, which is disjoint from γ_0 and bound disks both in V and W. The curve δ cannot cut off from Σ a torus with one hole containing ∂E_0 since otherwise $M_1 \cong S^3$ and $g = 2$, a contradiction. Thus, by Lemma 2.4 the Hempel distance of the splitting $(V_1, W_1; \Sigma)$ is 0 by Lemma 2.5, which is also a contradiction. □

Now we can write the word $w(\partial E)$ on $\{x, y_1, y_2, \ldots, y_{g-1}\}$ as

$$x^{\epsilon_1} w(\delta_1) x^{\epsilon_2} w(\delta_2) x^{\epsilon_3} w(\delta_3) \cdots x^{\epsilon_{2n}} w(\delta_{2n}),$$

where $\epsilon_k \in \{-1, 1\}$ for $k \in \{1, 2, \ldots, 2n\}$. If $\delta_k = \delta_{1,i}$ for some $i \in \{1, 2, \ldots, n_1\}$, then $\{\epsilon_k, \epsilon_{k+1}\} = \{-1, 1\}$, but $w(\delta_k)$ is nontrivial. If $\delta_k = \delta_{2,j}$ for some $j \in \{1, 2, \ldots, n_2\}$, then $w(\delta_k)$ is possibly trivial, but $\epsilon_k = \epsilon_{k+1}$. (Here $\epsilon_{2n+1} = \epsilon_1$.) This implies that $w(\partial E)$ determines a nontrivial element of $\pi_1(W)$, and so ∂E cannot bound a disk in W. Thus, E cannot be a reducing disk. □

By Lemma 2.2, and Propositions 2.3 and 2.6 we have the main result of the section.

COROLLARY 2.7. *Let $(V_1, W_1; \Sigma_1)$ be a genus-$(g-1)$ Heegaard splitting of Hempel distance at least 2 for a closed orientable 3-manifold M_1, where $g \geq 2$, and let $(V_2, W_2; \Sigma_2)$ be the genus-1 Heegaard splitting for $S^2 \times S^1$. If $(V, W; \Sigma)$ is the splitting for $M_1 \# (S^2 \times S^1)$ obtained from $(V_1, W_1; \Sigma_1)$ and $(V_2, W_2; \Sigma_2)$, then the sphere complex $\mathcal{H}$ for the splitting $(V, W; \Sigma)$ is isomorphic to the complex $\mathcal{A}^*_{g-1,2}$, and hence it is a $(4g-5)$-dimensional contractible complex.*

Recalling that the Hempel distance of the genus-1 Heegaard splitting of S^3 or a lens space is ∞, we also have the following:

Corollary 2.8. *Let* $(V, W; \Sigma)$ *be the genus-*2 *Heegaard splitting for* $M_1\#(S^2 \times S^1)$, *where* M_1 *is* S^3 *or a lens space. Then the sphere complex* $\mathcal{H}$ *for the splitting* $(V, W; \Sigma)$ *is a three-dimensional contractible complex.*

3. Goeritz Groups

Let M be an orientable manifold. Let $X_1, X_2, \dots, X_n$, and Y be subspaces of M. We denote by

$$\mathrm{Homeo}_+(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y)$$

the group of orientation-preserving homeomorphisms of M that preserve each of the subspaces $X_1, X_2, \dots, X_n$ setwise and Y pointwise. We equip this group with the compact-open topology. Let $\mathrm{Homeo}_0(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y)$ be the connected component of $\mathrm{Homeo}_+(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y)$ containing the identity. This component is a normal subgroup, and we denote by $\mathrm{MCG}_+(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y)$ the quotient group

$$\mathrm{Homeo}_+(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y)/\,\mathrm{Homeo}_0(M, X_1, X_2, \dots, X_n \,\mathrm{rel}\, Y).$$

Let $(V, W; \Sigma)$ be a Heegaard splitting of a closed orientable 3-manifold M. We recall that the Goeritz group of the splitting $(V, W; \Sigma)$ is the group of isotopy classes of the orientation-preserving homeomorphisms of M that preserve V and W setwise. We denote by $\mathcal{G}(V, W; \Sigma)$ the Goeritz group, which is identified with the quotient group $\mathrm{MCG}_+(M, V)$. We note that there are natural injective homomorphisms $\mathrm{MCG}_+(V) \to \mathrm{MCG}_+(\Sigma)$ and $\mathrm{MCG}_+(W) \to \mathrm{MCG}_+(\Sigma)$, which can be obtained by restricting homeomorphisms of V and W to Σ, respectively. Once we regard the groups $\mathrm{MCG}_+(V)$ and $\mathrm{MCG}_+(W)$ as subgroups of $\mathrm{MCG}_+(\Sigma)$ with respect to the inclusions, $\mathcal{G}(V, W; \Sigma)$ is identified with $\mathrm{MCG}_+(V) \cap \mathrm{MCG}_+(W)$. We also note that the group $\mathcal{G}(V, W; \Sigma)$ acts on the sphere complex $\mathcal{H}$ of $(V, W; \Sigma)$ simplicially if the splitting $(V, W; \Sigma)$ admits Haken spheres.

Namazi [22] showed that if the Hempel distance of the splitting $(V, W; \Sigma)$ is sufficiently high, then $\mathcal{G}(V, W; \Sigma)$ is a finite group. Later, Johnson [16] improved this result as follows.

Theorem 3.1 (Johnson [16]). *If the Hempel distance of the splitting* $(V, W; \Sigma)$ *is at least* 4, *then the group* $\mathcal{G}(V, W; \Sigma)$ *is finite.*

For Heegaard splittings of low Hempel distance, the situation is much more complicated as mentioned in Introduction.

In this section, we are interested in the Goeritz groups of the Heegaard splittings described in Section 2. Let $(V, W; \Sigma)$ be a genus-g Heegaard splitting of a closed orientable 3-manifold M, where $g \geq 2$. Suppose that $\mu(V, W; \Sigma) > 0$ and there exists a unique reducing disk E_0 in V. Fix a Haken sphere P for the splitting $(V, W; \Sigma)$ that represents a vertex of the complex $\mathcal{H}_{E_0}$. That is, P is the Haken sphere determined by a simple closed curve in Σ intersecting ∂E_0 in a single point as in Section 2. Then the disk $P \cap V$ cuts off from V a solid torus whose meridian disk is E_0.

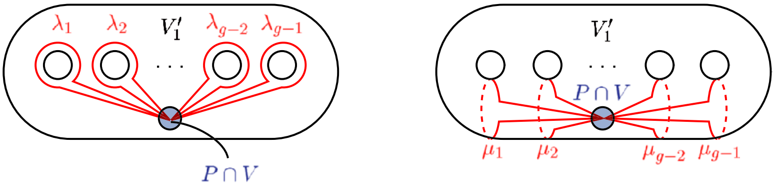

Figure 9

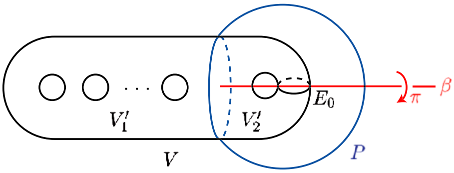

Figure 10

The handlebody V cut off by $P \cap V$ consists of two handlebodies V_1' and V_2', and similarly, W cut off by $P \cap W$ consists of W_1' and W_2'. Gluing 3-balls B_1 and B_2 on $V_1' \cup W_1'$ and $V_2' \cup W_2'$ along P, we obtain two Heegaard splittings $(V_1, W_1; \Sigma_1)$ and $(V_2, W_2; \Sigma_2)$, respectively. We may assume that $(V_2, W_2; \Sigma_2)$ is the genus-1 splitting of $S^2 \times S^1$, whereas $(V_1, W_1; \Sigma_1)$ is the genus-$(g-1)$ splitting of a 3-manifold having no $S^2 \times S^1$ summand in its prime decomposition.

Suppose that the Goeritz group $\mathcal{G}(V_1, W_1; \Sigma_1)$ is generated by finitely many elements $\omega_1, \omega_2, \ldots, \omega_m$. For each $i \in \{1, 2, \ldots, m\}$, the element ω_i has a representative homeomorphism $w_i \in \mathrm{Homeo}_+(M_1, V_1)$ satisfying $w_i|_{B_i}$ is the identity. Thus, there exists an element $\tilde{\omega}_i$ of $\mathcal{G}(V, W; \Sigma)$ represented by a homeomorphism $\tilde{w}_i \in \mathrm{Homeo}_+(M, V)$ such that $\tilde{w}_i(P) = P$, $\tilde{w}_i|_{V_1' \cup W_1'} = w_i|_{V_1' \cup W_1'}$ and $\tilde{w}_i|_{V_2' \cup W_2'}$ is the identity.

We also define the elements λ_j and μ_j for each $j \in \{1, 2, \ldots, g-1\}$ and the elements β and ϵ of $\mathcal{G}(V, W; \Sigma)$ as follows. The elements λ_j and μ_j have representative homeomorphisms l_j and m_j with $l_j|_{V_2' \cup W_2'} = m_j|_{V_2' \cup W_2'} = \mathrm{id}_{V_2' \cup W_2'}$ obtained by pushing $V_2' \cup W_2'$ so that $P \cap V$ moves along the arcs depicted in Figure 9, respectively.

The element β is defined by extending a half-Dehn twist about the disk $P \cap V$, and the element ϵ is defined by extending a Dehn twist about the unique reducing disk E_0 in V. See Figure 10. Note that all of $\tilde{\omega}_i$, λ_j, μ_j, β, and ϵ preserve the equivalence class of the Haken sphere P.

Lemma 3.2. *Under this setting, the subgroup of $\mathcal{G}(V, W; \Sigma)$ consisting of elements that preserve the equivalence class of P is generated by $\tilde{\omega}_i$, λ_j, μ_j, β, and ϵ, where $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, g-1\}$.*

Proof. Let l_j, m_j, b, and e be representative homeomorphisms of λ_j, μ_j, β, and ϵ, respectively, preserving P. We may assume that each of m_j, l_j, and b^2 fixes

$V_2' \cup W_2'$. Let φ be any element of $\mathcal{G}(V, W; \Sigma)$ that preserves the equivalence class of P, and let $f \in \mathrm{Homeo}_+(M, V)$ be one of its representatives satisfying $f(P) = P$. We will show that f is isotopic to a composition of a finite number of $\tilde{w}_i^{\pm 1}$, ${l_j}^{\pm 1}$, ${m_j}^{\pm 1}$, $b^{\pm 1}$, and $e^{\pm 1}$ up to an isotopy preserving V.

Let E_0' be an essential disk in W bounded by the unique reducing disk ∂E_0 in V. Composing f with a power of b, if necessary, and by an appropriate isotopy preserving V, we get a map $f_1 \in \mathrm{Homeo}_+(M, V)$ fixing $E_0 \cup E_0'$ and P. Moreover, by composing f_1 with a power of e, if necessary, and by an appropriate isotopy preserving V, we get a map $f_2 \in \mathrm{Homeo}_+(M, V)$ fixing $E_0 \cup E_0'$, $\partial V_2'$, and $\partial W_2'$. Note that the union of $E_0 \cup E_0'$ and $\partial V_2' \cap \partial W_2'$ cuts $V_2' \cup W_2'$ into two 3-balls. Thus, by Alexander's trick, we may assume that f_2 fixes $V_2' \cup W_2'$.

Suppose first that $g \geq 3$. Let D_1 be the disk $\Sigma_1 \cap B_1$ and choose a point p_1 in the interior of D_1. By the Birman exact sequence [2] we have the following commutative diagrams:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\Sigma_1, p_1) & \longrightarrow & \mathrm{MCG}_+(M_1, V_1, p_1) & \longrightarrow & \mathrm{MCG}_+(M_1, V_1) & \longrightarrow & 1 \\
 & & \Big\downarrow {=} & \circlearrowleft & \Big\downarrow & \circlearrowleft & \Big\downarrow & & \\
1 & \longrightarrow & \pi_1(\Sigma_1, p_1) & \xrightarrow{\text{push}} & \mathrm{MCG}_+(\Sigma_1, p_1) & \xrightarrow{\text{forget}} & \mathrm{MCG}_+(\Sigma_1) & \longrightarrow & 1
\end{array}$$

and

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{MCG}_+(M_1, V_1 \,\mathrm{rel}\, D_1) & \longrightarrow & \mathrm{MCG}_+(M_1, V_1, p_1) & \longrightarrow & 1 \\
 & & \Big\downarrow {=} & \circlearrowleft & \Big\downarrow & \circlearrowleft & \Big\downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{MCG}_+(\Sigma_1 \,\mathrm{rel}\, D_1) & \longrightarrow & \mathrm{MCG}_+(\Sigma_1, p_1) & \longrightarrow & 1.
\end{array}$$

In these diagrams, each vertical arrow is an injective homeomorphism. In the first diagram, the arrow "$\xrightarrow{\text{push}}$" implies the *pushing map,* and "$\xrightarrow{\text{forget}}$" implies the *forgetful map.* The group $\mathbb{Z}$ in the second diagram is generated by the Dehn twist about the disk D_1. See, for instance, [9; 11]. By the assumption the group $\mathrm{MCG}_+(M_1, V_1) = \mathcal{G}(V_1, W_1; \Sigma_1)$ is generated by $\omega_1, \omega_2, \ldots, \omega_m$. The image of $\pi_1(\partial V_1, p_1)$ in $\mathrm{MCG}_+(M_1, V_1, p_1)$ is the subgroup generated by the elements whose representatives correspond to $l_j|_{V_1' \cup W_1'}$ and $m_j|_{V_1' \cup W_1'}$, where $j \in \{1, 2, \ldots, g-1\}$. Moreover, a generator of $\mathbb{Z}$ in the second diagram corresponds to $b^2|_{V_1' \cup W_1'}$. Therefore, by the above diagrams and a natural identification

$$\mathrm{MCG}_+(M_1, V_1 \,\mathrm{rel}\, D_1) \cong \mathrm{MCG}_+(M, V \,\mathrm{rel}\, V_2' \cup W_2')$$

it follows that f_2 can be written as a composition of a finite number of $\tilde{w}_i$ ($i \in \{1, 2, \ldots, n\}$), ${l_j}^{\pm 1}$, ${m_j}^{\pm 1}$ ($j \in \{1, 2, \ldots, g-1\}$), and $b^{\pm 2}$ up to isotopy preserving V.

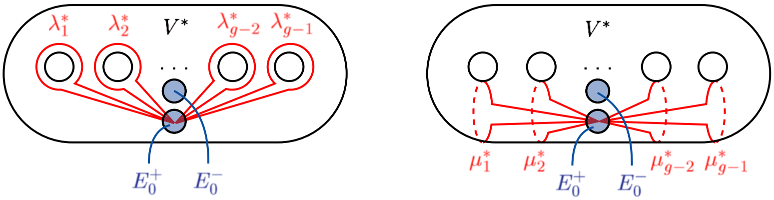

Figure 11

Suppose that $g = 2$. Then instead of the first diagram in the previous argument, we have the following simpler diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{MCG}_+(M_1, V_1, p_1) & \xrightarrow{\cong} & \mathrm{MCG}_+(M_1, V_1) & \longrightarrow & 1 \\ & & \downarrow & \circlearrowleft & \downarrow & & \\ 1 & \longrightarrow & \mathrm{MCG}_+(\Sigma_1, p_1) & \xrightarrow{\cong} & \mathrm{MCG}_+(\Sigma_1) & \longrightarrow & 1. \end{array}$$

Hence, f_2 can be written as the composition of a finite number of $\tilde{\omega}_i$ ($i \in \{1, 2, \ldots, n\}$) and $b^{\pm 2}$ up to isotopy preserving V. This completes the proof. □

In addition to the elements $\tilde{\omega}_i$, μ_j, λ_j, β, and ϵ, we define the elements λ_j^* and μ_j^* of $\mathcal{G}(V, W; \Sigma)$ for each $j \in \{1, 2, \ldots, g-1\}$ as follows. Let V^* be the handlebody V cut off by the unique reducing disk E_0. Let E_0^+ and E_0^- be disks in ∂V^* coming from E_0. The elements λ_j^* and μ_j^* for each $j \in \{1, 2, \ldots, g-1\}$ are defined by pushing E_0^+ along the arcs depicted in Figure 11. Each of these maps is realized by sliding a foot of the 1-handles $\mathrm{Nbd}(E_0; V)$ and $\mathrm{Nbd}(E_0'; W)$ of V and W, respectively, where E_0' is a disk in W bounded by ∂E_0. We observe that, for any simple arcs γ and γ' on ∂V^* connecting ∂E_0^+ and ∂E_0^-, there exists an element φ of $\mathcal{G}(V, W; \Sigma)$, which is a finite product of β, λ_j^* and μ_j^* for $j \in \{1, 2, \ldots, g-1\}$, such that φ has a representative map sending γ' to γ.

Now let ψ be any element of $\mathcal{G}(V, W; \Sigma)$. Then $\psi([P])$ is also a vertex of the complex $\mathcal{H}_{E_0}$ by Proposition 2.3. If $\psi([P]) = [P]$, then ψ is a finite product of the elements $\tilde{\omega}_i$, μ_j, λ_j, β, and ϵ by Lemma 3.2. Suppose that $\psi([P]) \neq [P]$. Then by Lemma 2.1 there exists a finite product, say φ, of β, λ_j^*, and μ_j^* such that $\varphi(\psi([P])) = [P]$. Thus, the composition $\varphi \circ \psi$ preserves the equivalence class of P, and consequently ψ is a finite product of $\tilde{\omega}_i$, λ_j, μ_j, λ_j^*, μ_j^*, β, and ϵ. We summarize this observation as follows.

THEOREM 3.3. *Let $(V, W; \Sigma)$ be the Heegaard splitting obtained from a genus-$(g-1)$ splitting $(V_1, W_1; \Sigma_1)$ for a 3-manifold and the genus-1 splitting $(V_2, W_2; \Sigma_2)$ for $S^2 \times S^1$, where $g \geq 2$. Suppose that there exists a unique reducing disk E_0 in V. If the Goeritz group of $(V_1, W_1; \Sigma_1)$ is finitely generated, then the Goeritz group of $(V, W; \Sigma)$ is also finitely generated. Moreover, under the given setting, the Goeritz group of $(V, W; \Sigma)$ is generated by $\tilde{\omega}_i$, λ_j, μ_j, λ_j^*, μ_j^*, β, and ϵ, where $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, g-1\}$.*

By Proposition 2.6 and Theorems 3.1 and 3.3 we have the following:

Corollary 3.4. *Let* $(V_1, W_1; \Sigma_1)$ *be a genus-*$(g-1)$ *Heegaard splitting of Hempel distance at least* 4 *for a closed orientable* 3*-manifold* M_1, *where* $g \geq 2$, *and let* $(V_2, W_2; \Sigma_2)$ *be the genus-*1 *Heegaard splitting for* $S^2 \times S^1$. *If* $(V, W; \Sigma)$ *is the splitting for* $M_1\#(S^2 \times S^1)$ *obtained from* $(V_1, W_1; \Sigma_1)$ *and* $(V_2, W_2; \Sigma_2)$, *then the Goeritz group of the splitting* $(V, W; \Sigma)$ *is finitely generated.*

We note that Corollary 3.4 implies, in particular, that the Goeritz group of the genus-2 Heegaard splitting for $M_1\#(S^2 \times S^1)$, where M_1 is S^3 or a lens space, is finitely generated, which is shown in [6] and [7].

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