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α-completely positive maps of group systems and Krein module representations

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ABSTRACT

In this paper, we study (covariant) α-completely positive maps on group systems. We first introduce a notion of α-completely positive maps of groups into (locally) C^*-algebras and show that bounded α-completely positive maps on discrete groups induce α-completely positive linear maps on group C^*-algebras. We establish the (covariant) KSGNS type representation theorem for (covariant) α-completely positive maps of group systems into locally C^*-algebras. These constructions provide a projective covariant J-representation of a group system into a locally C^*-algebra.

1. Introduction

An indefinite inner product space is a vector space equipped with a symmetric or Hermitian bilinear form $[v, w]$ for which $[v, v]$ can be positive, negative or zero. In the most important special case, it is a Hilbert space equipped with the indefinite inner product given by $[v, w] = (Jv, w)$ where J is a bounded, invertible and Hermitian operator [2]. Such a space was first mathematically studied by Pontryagin for studying a mechanics problem. It is known that in massless quantum field theory the state space will be a space with indefinite metric. Since the positivity is lack in a local quantum field theory, the GNS-construction on an indefinite inner product is of increasing interest in the general (axiomatic) quantum field theory [1,3]. In particular, in the gauge quantum field theory, the locality is in conflict with positivity and then from the axiomatic point of view, it is better to keep the locality condition and to give up the positivity condition which leads to the modification of the axiom of positivity.

Motivated by these physical facts, Heo–Hong–Ji [5] introduced a notion of α-completely positive linear maps between C^*-algebras as a natural generalization of completely positive maps between C^*-algebras, and Heo–Ji [6] proved the Radon–Nikodým type theorem for the class of α-completely positive maps and constructed a covariant representation associated to a covariant α-completely positive maps. Here, positivity is inherent in Hermitian maps in terms of the map α. The α-complete positivity provides a positive definite inner product associated to the indefinite one, and the interplay between these two is indeed the characteristic feature of Krein spaces among all indefinite inner product spaces.

In this paper, we first introduce a notion of a (covariant) α-completely positive map of a topological group into a (locally) C^*-algebra, which is a counterpart of a (covariant) α-completely positive linear map between (locally) C^*-algebras [5–8]. We construct a (covariant) KSGNS (Kasparov–Stinespring–Gelfand–Naimark–Segal) type representation of a group on a Krein module over a (locally) C^*-algebra, which is associated to a (covariant) α-completely positive map of a group (system) [8].

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We now give a brief overview of the organization of the paper. In Section 2, we define a notion of an α-completely positive map of a topological group with an involution into a (locally) C^*-algebra. We prove a KSGNS type theorem for an α-completely positive map of a group into a C^*-algebra and construct an α-completely positive linear map of $C(\mathbb{T}^n)$, which is induced by an α-completely positive map of $\mathbb{Z}^n$. More generally, we show that bounded α-completely positive maps on discrete groups induce α-completely positive linear maps on group C^*-algebras.

Section 3 contains a covariant version of the KSGNS type theorem for a covariant α-completely positive map of a group under some equivariance assumption of the map α and an action of group systems. In Section 4, we briefly review definitions and properties of Hilbert modules over locally C^*-algebras and we establish the KSGNS representation theorem for an α-completely positive map of a topological group into a locally C^*-algebra. Finally, in Section 5 we study a (projective) covariant α-completely positive map of a (semi-)group system into a locally C^*-algebra and construct the associated (projective) covariant J-representations.

2. α-completely positive maps on groups

A *locally C^*-algebra* is a complete Hausdorff (complex) topological $*$-algebra of which the topology is determined by the collection of all continuous C^*-seminorms on it. It was first systematically studied by Inoue [9] as a generalization of a C^*-algebra, and then Phillips [12] studied locally C^*-algebras that are needed for representable K-theory. It was known that locally C^*-algebras are useful for the study of non-commutative algebraic topology and quantum field theory, etc. We refer [9,12] and its references for examples of locally C^*-algebras.

We now introduce a notion of an α-completely positive map on a topological group with an involution which is a counterpart of α-completely positive linear maps between (locally) C^*-algebras [5–7].

Definition 2.1. Let G be a topological group with an involution α, that is, $\alpha^2 = \mathrm{id}_G$, $\alpha(g)^{-1} = \alpha(g^{-1})$ and $\alpha(e) = e$ where e is a unit element of G and let $\mathfrak{A}$ be a (locally) C^*-algebra. A map $\phi : G \to \mathfrak{A}$ is called *α-completely positive* if

(i) $\phi\big(\alpha(g_1g_2)\big) = \phi\big(\alpha(g_1)\alpha(g_2)\big) = \phi(g_1g_2)$ for all $g_1, g_2 \in G$,

(ii) for all $g_1, \ldots, g_n \in G$, the operator matrix $\left[\phi(\alpha(g_i^{-1})g_j)\right]_{i,j=1}^n$ is positive,

(iii) there exist a constant $K > 0$ such that

$$\left[\phi(g_i)^*\phi(g_j)\right]_{i,j=1}^n \le K\left[\phi\big(\alpha(g_i)^{-1}g_j\big)\right]_{i,j=1}^n$$

for every $g_1, \ldots, g_n \in G$.

(iv) there exist a constant $M(g) > 0$ such that

$$\left[\phi(\alpha(gg_i)^{-1}\, gg_j)\right]_{i,j=1}^n \le M(g)\left[\phi(\alpha(g_i)^{-1}g_j)\right]_{i,j=1}^n$$

for every $g, g_1, \ldots, g_n \in G$.

It follows from (ii) in Definition 2.1 that $\phi(\alpha(g^{-1}))^* = \phi(g)$ for all $g \in G$. Let $\mathcal{B}$ be a C^*-algebra and let X, Y be Hilbert $\mathcal{B}$-modules. We denote by $\mathcal{L}_{\mathcal{B}}(X, Y)$ the set of all right $\mathcal{B}$-module maps $T : X \to Y$ for which there is an operator $T^* : Y \to X$, called the *adjoint* of T, such that $\langle Tx, y\rangle_Y = \langle x, T^*y\rangle_X (x \in X,\ y \in Y)$. It follows from the uniform boundedness theorem that each operator T in $\mathcal{L}_{\mathcal{B}}(X, Y)$ is bounded. We write $\mathcal{L}_B(X)$ for $\mathcal{L}_B(X, X)$, which becomes a C^*-algebra with the operator norm. For a detailed information on Hilbert C^*-modules, we refer to [11].

Let J be a (fundamental) symmetry on a Hilbert $\mathcal{B}$-module X, i.e., $J = J^* = J^{-1}$. Then we define a $\mathcal{B}$-valued indefinite inner product $\langle\cdot,\cdot\rangle_J$ by

$$\langle x, y\rangle_J = \langle x, Jy\rangle, \quad (x, y \in X).$$

In this case, the pair (X, J) is called a *Krein $\mathcal{B}$-module*. For each $T \in \mathcal{L}_{\mathcal{B}}(X)$, there exists an operator $T^J \in \mathcal{L}_{\mathcal{B}}(X)$ such that

$$\langle T(x), y\rangle_J = \langle x, T^J(y)\rangle_J, \quad (x, y \in X).$$

The operator T^J is called the *J-adjoint* of T and we can see that $T^J = JT^*J$. We denote by $\mathcal{U}_J(X)$ the set of all J-unitary operators in $\mathcal{L}_{\mathcal{B}}(X)$, i.e. $T^JT = TT^J = I$. For more detailed study for indefinite inner product spaces, we refer to [2].

A *representation* of G on X means a homomorphism $\pi : G \to \mathcal{L}_{\mathcal{B}}(X)$. A *unitary representation* π of G on X is a representation of G on X such that $\pi(g)^* = \pi(g^{-1})$ for every $g \in G$. A representation $\pi : G \to \mathcal{L}_{\mathcal{B}}(X)$ is called a *J-unitary representation* of G on a Krein $\mathcal{B}$-module (X, J) if

$$\pi(g^{-1}) = \pi(g)^J \equiv J\pi(g)^*J \quad \text{for all } g \in G.$$

Note that even if $\pi : G \to \mathcal{U}_J(X)$ is a J-unitary representation, the operator $\pi(g)$ in general is not a unitary for all $g \in G$. However, if $\pi(g)$ commutes with J, then $\pi(g)$ becomes a unitary operator.

Throughout this paper, $\mathcal{B}, X$ and G denote a (locally) C^*-algebra, a Hilbert $\mathcal{B}$-module with a $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle$ and a (topological) group, respectively, unless otherwise specified.

Let $\mathbb{C}[G, X]$ be the set of finitely supported functions from a group G into X. Then $\mathbb{C}[G, X]$ becomes a right $\mathcal{B}$-module with the natural operations

$$(f_1 + f_2)(g) = f_1(g) + f_2(g), \qquad (\lambda \cdot f)(g) = \lambda f(g), \quad \text{and} \quad (f \cdot b)(g) = f(g)b$$

for all $g \in G$, $\lambda \in \mathbb{C}$ and $b \in \mathcal{B}$. We can identify an element $f \in \mathbb{C}[G, X]$ with $\sum_g \delta_g \otimes f(g)$ where $\delta_g(g')$ is 1 if $g = g'$ or 0 otherwise. Thus, we will use the identification $\mathbb{C}[G, X] = \mathbb{C}[G] \otimes X$ where $\mathbb{C}[G]$ denotes a group algebra.

In the following theorem, we will give a KSGNS type representation on a Krein C^*-module associated to an α-completely positive map on a group.

Theorem 2.2. *If $\phi : G \to \mathcal{L}_{\mathcal{B}}(X)$ is an α-completely positive map, then there exist a Krein $\mathcal{B}$-module (Y_ϕ, J_ϕ), a J_ϕ-unitary representation $\pi_\phi : G \to \mathcal{L}_{\mathcal{B}}(Y_\phi)$ and an adjointable operator $V_\phi : X \to Y_\phi$ such that*

(i) *$\phi(g) = V_\phi^* \pi_\phi(g) V_\phi$ for all $g \in G$;*
(ii) *the set $[\pi_\phi(G) V_\phi(X)]$ is dense in Y_ϕ;*
(iii) *$V_\phi^* \pi_\phi(g)^* \pi_\phi(g') V_\phi = V_\phi^* \pi_\phi(\alpha(g^{-1})g') V_\phi$ for all $g, g' \in G$.*

If, in addition, $\phi(e) = id_X$, then V_ϕ is an isometry.

Proof. The proof is standard, but we give a proof for readers' convenience and the proof of Theorem 3.2. We first define a $\mathcal{B}$-valued sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[G, X]$ by

$$\langle f_1, f_2 \rangle = \sum_{g, g'} \left\langle f_1(g), \phi(\alpha(g^{-1})g') f_2(g') \right\rangle_X \tag{2.1}$$

where $f_1 = \sum_g \delta_g \otimes f_1(g)$ and $f_2 = \sum_{g'} \delta_{g'} \otimes f_2(g')$ are in $\mathbb{C}[G, X]$. Then the form $\langle \cdot, \cdot \rangle$ is positive semi-definite since $\langle f, f \rangle \geq 0$ for all $f \in \mathbb{C}[G, X]$. By the Cauchy–Schwarz inequality, the space $\mathcal{N}_\phi = \{f \in \mathbb{C}[G, X] : \langle f, f \rangle = 0\}$ becomes a $\mathcal{B}$-submodule of $\mathbb{C}[G, X]$. Hence, the $\mathcal{B}$-valued sesquilinear form (2.1) induces the $\mathcal{B}$-valued inner product on the quotient $\mathcal{B}$-module $\mathbb{C}[G, X]/\mathcal{N}_\phi$. Let Y_ϕ be the completion of the quotient space $\mathbb{C}[G, X]/\mathcal{N}_\phi$ with respect to the norm induced by the inner product.

We see that the involution α on G induces the involution J on $\mathbb{C}[G, X]$ given by

$$J(f) = J\left(\sum_g \delta_g \otimes f(g)\right) = \sum_g \delta_{\alpha(g)} \otimes f(g). \tag{2.2}$$

Now we define an indefinite inner product $[\cdot, \cdot]$ on the quotient space $\mathbb{C}[G, X]/\mathcal{N}_\phi$ by

$$\left[f_1 + \mathcal{N}_\phi, f_2 + \mathcal{N}_\phi\right] = \sum_{g, g'} \left\langle f(g), \phi(g^{-1}g') f(g') \right\rangle_X .$$

Since the indefinite inner product $[\cdot, \cdot]$ is separately continuous, by the continuity, it can be extended to the whole space Y_ϕ. We denote by J_ϕ the extension of J to Y_ϕ. Moreover, it is obvious that

$$\left[f_1 + \mathcal{N}_\phi, f_2 + \mathcal{N}_\phi\right] = \left\langle f_1 + \mathcal{N}_\phi, J(f_2 + \mathcal{N}_\phi) \right\rangle.$$

From the properties of α-completely positive maps, we see that the operator J_ϕ is a fundamental symmetry on Y_ϕ, that is, $J_\phi = J_\phi^* = J_\phi^{-1}$. Hence, the pair (Y_ϕ, J_ϕ) is a Krein $\mathcal{B}$-module.

For each $x \in X$, we define a map $V_\phi(x) : G \to X$ by $\left(V_\phi x\right)(g) = x \cdot \delta_e(g)$ where e denotes the unit element of G. By identifying $\mathbb{C}[G, X]/\mathcal{N}_\phi$ with $\mathbb{C}[G] \otimes X/\mathcal{N}_\phi$, we can regard $V_\phi x$ as $\delta_e \otimes x + \mathcal{N}_\phi$. We easily see that V_ϕ is isometric whenever $\phi(e) = \mathrm{id}_X$. Let $f \in \mathbb{C}[G, X]$ and $x \in X$. Then we have that

$$\left\langle x, V_\phi^*(f + \mathcal{N}_\phi) \right\rangle_X = \sum_g \left\langle x, \phi(\alpha(e)g) f(g) \right\rangle_X = \sum_g \left\langle x, \phi(g) f(g) \right\rangle_X,$$

so that $V_\phi^*(f + \mathcal{N}_\phi) = \sum_g \phi(g) f(g)$. Since $\|V_\phi^*(f + \mathcal{N}_\phi)\|^2 \leq K \|f + \mathcal{N}_\phi\|^2$ for any $f \in \mathbb{C}[G, X]$, V_ϕ^* is bounded. Thus, the operator V_ϕ^* can be extended to the whole space Y_ϕ.

For each $g \in G$, we define a linear operator $\pi_\phi(g) : \mathbb{C}[G, X]/\mathcal{N}_\phi \to \mathbb{C}[G, X]/\mathcal{N}_\phi$ by

$$\pi_\phi(g)(f + \mathcal{N}_\phi) = \sum_{g'} \delta_{gg'} \otimes f(g') + \mathcal{N}_\phi \quad (f \in \mathbb{C}[G, X]).$$

Clearly, we have that $[\pi_\phi(g) f](g') = f(g^{-1}g')$ and $\pi_\phi(gg') = \pi_\phi(g)\pi_\phi(g')$ for all $g, g' \in G$. By (iv) in Definition 2.1, we obtain that for each $g \in G$

$$\begin{aligned} \left\| \pi_\phi(g)(f + \mathcal{N}_\phi) \right\|^2 &\leq M(g) \left\| \sum_{g', g''} \left\langle f(g'), \phi\left(\alpha(g')^{-1} g''\right) f(g'') \right\rangle_X \right\| \\ &= M(g) \left\| f + \mathcal{N}_\phi \right\|^2. \end{aligned}$$

Therefore, $\pi_\phi(g)$ can be extended to Y_ϕ as a bounded linear operator, which we still denote by the same notation $\pi_\phi(g)$. On the other hand, we obtain that for any $f_1, f_2 \in \mathbb{C}[G, X]$

$$\begin{aligned}\left\langle \pi_\phi(g)(f_1 + \mathcal{N}_\phi), f_2 + \mathcal{N}_\phi \right\rangle &= \sum_{g', g''} \left\langle f_1(g'), \phi\left(\alpha(g')^{-1}\alpha(g^{-1}\alpha(g''))\right) f_2(g'') \right\rangle \\ &= \left\langle f_1 + \mathcal{N}_\phi, J_\phi \pi_\phi(g^{-1}) J_\phi (f_2 + \mathcal{N}_\phi) \right\rangle.\end{aligned}$$

Thus, we have that $\pi_\phi(g)^* = J_\phi \pi_\phi(g^{-1}) J_\phi$, which implies that π_ϕ is a J_ϕ-unitary representation of G on Y_ϕ.

Moreover, for any $g, g' \in G$ and $x \in X$ we have that

$$\begin{aligned} V_\phi^* \pi_\phi(g)^* \pi(g') V_\phi(x) &= V_\phi^* J_\phi \pi_\phi(g^{-1})(\delta_{\alpha(g')} \otimes x) \\ &= \phi(g^{-1}\alpha(g'))x = \phi(\alpha(g^{-1})g')x \\ &= V_\phi^* \pi_\phi(\alpha(g^{-1})g') V_\phi(x), \end{aligned}$$

so that $V_\phi^* \pi_\phi(g)^* \pi_\phi(g') V_\phi = V_\phi^* \pi_\phi(\alpha(g^{-1})g') V_\phi$. □

We call the quadruple $(\pi_\phi, V_\phi, Y_\phi, J_\phi)$ in Theorem 2.2 *the minimal KSGNS dilation* of an α-completely positive map ϕ of G into $\mathcal{L}_{\mathcal{B}}(X)$.

Remark 2.3. Let $\mathcal{B}$ be a C^*-algebra. In Theorem 2.2, if $\phi : G \to \mathcal{B}$ is an α-completely positive map, then we get a GNS type representation theorem as that in [4] for completely positive maps on a topological group. More precisely, there exist a Krein $\mathcal{B}$-module (X_ϕ, J_ϕ), a J_ϕ-unitary representation $\pi_\phi : G \to \mathcal{L}_{\mathcal{B}}(X_\phi)$ and a vector $x_\phi \in X_\phi$ such that the set $\{\pi_\phi(g)(x_\phi \cdot b) : g \in G,\ b \in \mathcal{B}\}$ is total in X_ϕ and $\phi(g) = \langle x_\phi, \pi_\phi(g) x_\phi \rangle$ for any $g \in G$. □

The notion of α-completely positive linear maps between C^*-algebras was introduced in [5] and was systematically studied in [5–8]. We should mention that, in general, it is not easy to prove the α-complete positivity of linear maps between C^*-algebras.

Let $\mathbb{Z}^n$ be the Cartesian product of n copies of the integers with an involution α. If a map $\psi : \mathbb{Z}^n \to \mathcal{L}_{\mathcal{B}}(X)$ is α-completely positive, by Theorem 2.2, there exists the minimal KSGNS dilation $(\pi_\psi, V_\psi, Y_\psi, J_\psi)$ of ψ. Let $\mathbb{T}^n$ be the Cartesian product of n copies of the unit circle and let z_i be the ith coordinate function on $\mathbb{T}^n$. For any $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, we set $z^I = z_1^{i_1} \cdots z_n^{i_n}$. Let $\sigma : C(\mathbb{T}^n) \to \mathcal{L}_{\mathcal{B}}(Y_\psi)$ be a homomorphism given by

$$\sigma(z_i) = \pi_\psi(e_i), \quad (i = 1, \ldots, n) \tag{2.3}$$

where e_i is the n-tuple that is 1 in the ith entry and 0 in the remaining entries. Then we see that $\sigma(z^I) = \pi_\psi(I)$ for each $I \in \mathbb{Z}^n$ and that α naturally induces the linear involution $\widetilde{\alpha}$ on $C(\mathbb{T}^n)$ given by $\widetilde{\alpha}(z^I) = z^{\alpha(I)}$.

Proposition 2.4. *If a map $\psi : \mathbb{Z}^n \to \mathcal{L}_{\mathcal{B}}(X)$ is α-completely positive, then there is an $\widetilde{\alpha}$-completely positive linear map $\widetilde{\psi}$ from $C(\mathbb{T}^n)$ into $\mathcal{L}_{\mathcal{B}}(X)$ such that $\widetilde{\psi}(z^I) = \psi(I)$ for every $I \in \mathbb{Z}^n$.*

Proof. Let $(\pi_\psi, V_\psi, Y_\psi, J_\psi)$ be the minimal KSGNS dilation of ψ as in Theorem 2.2 and let σ be defined as in (2.3). We define a map $\widetilde{\psi}$ of $C(\mathbb{T}^n)$ into $\mathcal{L}_{\mathcal{B}}(X)$ by

$$\widetilde{\psi}(f) = V_\psi^* \sigma(f) V_\psi \quad (f \in C(\mathbb{T}^n)). \tag{2.4}$$

It is obvious from the definition that $\widetilde{\psi}(z^I) = \psi(I)$. To prove the $\widetilde{\alpha}$-complete positivity of the map $\widetilde{\psi}$, it is enough to consider monomials of form $z^I (I \in \mathbb{Z}^n)$ due to the linearity and continuity. For any $I, I' \in \mathbb{Z}^n$, we have that

$$\widetilde{\psi}\left(\widetilde{\alpha}(z^I) \cdot \widetilde{\alpha}(z^{I'})\right) = V_\psi^* \sigma\left(z^{\alpha(I) + \alpha(I')}\right) V_\psi = \begin{cases} \psi\left(\alpha(I + I')\right) = \widetilde{\psi}\left(\widetilde{\alpha}(z^I \cdot z^{I'})\right), \\ \psi\left(I + I'\right) = \widetilde{\psi}\left(z^I \cdot z^{I'}\right). \end{cases}$$

From the linearity and continuity, we obtain that for all $f_1, f_2 \in C(\mathbb{T}^n)$

$$\widetilde{\psi}\left(\widetilde{\alpha}(f_1) \cdot \widetilde{\alpha}(f_2)\right) = \widetilde{\psi}\left(\widetilde{\alpha}(f_1 \cdot f_2)\right) = \widetilde{\psi}(f_1 \cdot f_2).$$

For any $m \in \mathbb{N}$, take monomials $z^I, z^{I_1}, \ldots, z^{I_m} \in C(\mathbb{T}^n)$ and elements $x_1, \ldots, x_m \in X$. Then we have that

$$\left\langle x_i, \widetilde{\psi}(\widetilde{\alpha}(z^{I_i})^* z^{I_j}) x_j \right\rangle = \sum_{i,j=1}^m \left\langle x_i, \psi(\alpha(-I_i) + I_j) x_j \right\rangle \geq 0$$

where the inequality follows from the condition (ii) in Definition 2.1. We also have that

$$\sum_{i,j=1}^{m}\left\langle x_i,\widetilde{\psi}(\widetilde{\alpha}(z^I z^{I_i})^* z^I z^{I_j})x_j\right\rangle=\sum_{i,j=1}^{m}\left\langle x_i,V_\psi^*\pi_\psi(\alpha(-I_i-I)+I+I_j)V_\psi x_j\right\rangle$$
$$\le M(I)\sum_{i,j=1}^{m}\left\langle x_i,\widetilde{\psi}(\widetilde{\alpha}(z^{I_i})^* z^{I_j})x_j\right\rangle$$

where the inequality follows from (iv) in Definition 2.1. Hence, $\widetilde{\psi}$ is an $\widetilde{\alpha}$-completely positive linear map from $C(\mathbb{T}^n)$ into $\mathcal{L}_{\mathcal{B}}(X)$. □

Let G be a locally compact group with a left Haar measure μ and we denote by $L^1(G)$ the space of absolutely integrable functions with respect to μ. Then $L^1(G)$ becomes a $*$-algebra with operations of convolution and involution;

$$f_1 * f_2(g)=\int_G f_1(h)f_2(h^{-1}g)d\mu(h)\quad\text{and}\quad f^*(g)=\Delta(g)^{-1}\overline{f(g^{-1})}$$

where Δ is a modular function of G. The algebra $L^1(G)$ is unital if G is discrete. The group C^*-algebra $C^*(G)$ is the closure of the universal representation π_u of G where π_u is the direct sum of all irreducible representations (up to unitary equivalence) of G. Equivalently, the group C^*-algebra $C^*(G)$ is the closure of $L^1(G)$ with respect to the norm

$$\|f\|=\sup\left\{\|\pi(f)\|:\pi\text{ is a }*\text{-representation of }L^1(G)\right\}.$$

We now consider a discrete group Γ with an involution α. The involution α on Γ induces a linear involution $\widetilde{\alpha}:\mathbb{C}[\Gamma]\to\mathbb{C}[\Gamma]$ given by $\widetilde{\alpha}\left(\sum_{i=1}^n s_i\,\delta_{g_i}\right)=\sum_{i=1}^n s_i\,\delta_{\alpha(g_i)}$. Clearly, we have that $\widetilde{\alpha}^2=\mathrm{id}_{\mathbb{C}[\Gamma]}$ and $\widetilde{\alpha}(\delta_e)=\delta_e$, which implies that $\widetilde{\alpha}$ is an involution on a group algebra $\mathbb{C}[\Gamma]$. Suppose that $\phi:\Gamma\to\mathcal{L}_{\mathcal{B}}(X)$ is α-completely positive. Now, we define a linear map $\Phi:\mathbb{C}[\Gamma]\to\mathcal{L}_{\mathcal{B}}(X)$ by

$$\Phi\left(\sum_{i=1}^n s_i\,\delta_{g_i}\right)=\sum_{i=1}^n s_i\phi(g_i).\tag{2.5}$$

Since $\widetilde{\alpha}$ is isometric and $\widetilde{\alpha}(f)^*=\widetilde{\alpha}(f^*)$, the map $\widetilde{\alpha}$ extends by continuity to a linear Hermitian involution on a group C^*-algebra $C^*(\Gamma)$. In the remaining part of this section, we consider an α-completely positive linear map on a group C^*-algebra, which is induced by an α-completely positive map on a group.

Theorem 2.5. *If $\phi:\Gamma\to\mathcal{L}_{\mathcal{B}}(X)$ is bounded and α-completely positive, the linear map $\Phi:\mathbb{C}[\Gamma]\to\mathcal{L}_{\mathcal{B}}(X)$ given by (2.5) extends to an $\widetilde{\alpha}$-completely positive linear map on the group C^*-algebra $C^*(\Gamma)$.*

Proof. Let $f_1=\sum_{i=1}^n s_i\,\delta_{g_i}$ and $f_2=\sum_{i=j}^m t_j\,\delta_{h_j}$ be in $\mathbb{C}[\Gamma]$. Then we have that

$$\Phi\left(\widetilde{\alpha}(f_1)*\widetilde{\alpha}(f_2)\right)=\begin{cases}\sum_{i,j}\bar{s}_i t_j\phi\left(\alpha(g_i h_j)\right)\\ \sum_{i,j}\bar{s}_i t_j\phi(g_i h_j)\end{cases}=\begin{cases}\Phi\left(\widetilde{\alpha}(f_1*f_2)\right)\\ \Phi\left(f_1*f_2\right).\end{cases}$$

We also have that $\Phi(f_1)^*=\sum_i\bar{s}_i\phi\left(\alpha(g_i^{-1})\right)=\Phi\left(\widetilde{\alpha}(f_1^*)\right)=\Phi(f_1^*)$. Thus, Φ is a linear Hermitian map on $\mathbb{C}[\Gamma]$.

Let $x_1,\ldots,x_n\in X$ and let $f_1,\ldots,f_n\in\mathbb{C}[\Gamma]$ with $f_i=\sum_k s_{ik}\delta_{g_{ik}}$. Then we have that

$$\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(f_i)^**\widetilde{\alpha}(f_j)\right)x_j\right\rangle=\sum_{i,j}\sum_{k,l}\bar{s_{ik}}s_{jl}\left\langle x_i,\phi\left(\alpha(g_{ik})^{-1}g_{jl}\right)x_j\right\rangle\ge 0$$

where the inequality follows from the α-complete positivity of ϕ. We also obtain that

$$\sum_{i,j}\left\langle x_i,\Phi(f_i)^*\Phi(f_j)x_j\right\rangle=\sum_{i,j}\sum_{k,l}\bar{s_{ik}}s_{jl}\left\langle x_i,\phi(g_{ik})^*\phi(g_{jl})x_j\right\rangle$$
$$\le K\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(f_i)^**f_j\right)x_j\right\rangle.$$

Moreover, we have that

$$\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(\delta_g*f_i)^**(\delta_g*f_j)\right)x_j\right\rangle=\sum_{i,j}\sum_{k,l}\bar{s_{ik}}s_{jl}\left\langle x_i,\phi\left(\alpha(gg_{ik})^{-1}gg_{jl}\right)x_j\right\rangle$$
$$\le M(g)\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(f_i)^**f_j\right)x_j\right\rangle$$

where the inequality follows from (iv) in Definition 2.1. Let $f=\sum_k s_k\delta_{g_k}$ be in $\mathbb{C}[\Gamma]$. Then we have that

$$\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(f*f_i)^**(f*f_j)\right)x_j\right\rangle\le\sum_{i,j}\sum_k 2|s_k|^2\left\langle x_i,\Phi\left(\widetilde{\alpha}(\delta_{g_k}*f_i)^**(\delta_{g_k}*f_j)\right)x_j\right\rangle$$
$$\le M(f)\sum_{i,j}\left\langle x_i,\Phi\left(\widetilde{\alpha}(f_i)^**(f_j)\right)x_j\right\rangle,$$

where $M(f)=2\max_k\{M(g_k)\}\sum_k|s_k|^2$. It also follows from the boundedness of ϕ that $\|\Phi(f)\|=\|\sum_k s_k\phi(g_k)\|\le\|\phi\|\cdot\|f\|_1$ where $\|\cdot\|_1$ is the L^1-norm on $\mathbb{C}[\Gamma]$. Hence, Φ is an $\widetilde{\alpha}$-completely positive and bounded linear map on $\mathbb{C}[\Gamma]$ into $\mathcal{L}_{\mathcal{B}}(X)$, so that Φ extends to an $\widetilde{\alpha}$-completely positive linear map on $C^*(\Gamma)$. □

3. Covariant α-completely positive maps on group systems

Let G,H be topological groups. We denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G endowed with the pointwise convergence topology. The action θ of H on G means a continuous homomorphism of H into $\mathrm{Aut}(G)$. We will refer to this triple (G,H,θ) as a *group system*. For example, let $H=SL_2(\mathbb{R})$ and let G be the 3-dimensional Heisenberg group. It is known that H acts as automorphisms on G.

Definition 3.1. Let (G,H,θ) be a group system with an involution α on G and let U be a strongly continuous unitary representation of H into $\mathcal{U}(X)$ where $\mathcal{U}(X)$ is a group of all unitary elements in $\mathcal{L}_{\mathcal{B}}(X)$.

(1) A map $\phi:G\to\mathcal{L}_{\mathcal{B}}(X)$ is called (θ,U)-*covariant* if

$$\phi(\theta_h(g))=U_h\phi(g)U_h^*\quad\text{for all } g\in G \text{ and } h\in H.$$

(2) Two maps θ and α are *equivariant* if $\theta_h\circ\alpha=\alpha\circ\theta_h$ for all $h\in H$.

The following theorem is a covariant version of Theorem 2.2.

Theorem 3.2. *Let (G,H,θ) be a group system with a continuous involution α on G and let U be a strongly continuous unitary representation of H into $\mathcal{U}(X)$. Suppose that $\phi:G\to\mathcal{L}_{\mathcal{B}}(X)$ is a (θ,U)-covariant and continuous α-completely positive map and that θ and α are equivariant. Then there exist*

(a) *a Krein $\mathcal{B}$-module (Y_ϕ,J_ϕ)*
(b) *a J_ϕ-representation $\pi_\phi:G\to\mathcal{L}_{\mathcal{B}}(Y_\phi)$,*
(c) *an adjointable operator $V_\phi:X\to Y_\phi$,*
(d) *a strongly continuous unitary representation $\widetilde{\theta}:H\to\mathcal{L}_{\mathcal{B}}(Y_\phi)$,*

such that

(i) *$\phi(g)=V_\phi^*\pi_\phi(g)V_\phi$ for all $g\in G$,*
(ii) *the set $[\pi_\phi(G)V_\phi(X)]$ is dense in Y_ϕ,*
(iii) *$V_\phi^*\pi_\phi(g)^*\pi_\phi(g')V_\phi=V_\phi^*\pi_\phi(\alpha(g^{-1})g')V_\phi$ for all $g,g'\in G$,*
(iv) *$\widetilde{\theta}_h\pi_\phi(g)\widetilde{\theta}_h^*=\widetilde{\theta}_h\pi_\phi(g)\widetilde{\theta}_h^{J_\phi}=\pi_\phi(\theta_h(g))$ for all $g\in G$ and $h\in H$,*
(v) *$\widetilde{\theta}_hV_\phi=V_\phi U_h$ for all $g\in G$ and $h\in H$.*

Proof. Here, we use the same notations as in the proof of Theorem 2.2. Since the map $\phi:G\to\mathcal{L}_{\mathcal{B}}(X)$ is α-completely positive, there exist a Krein $\mathcal{B}$-module (Y_ϕ,J_ϕ), a J_ϕ-unitary representation $\pi_\phi:G\to\mathcal{L}_{\mathcal{B}}(Y_\phi)$ and an adjointable operator $V_\phi:X\to Y_\phi$ satisfying properties (i)–(iii). Hence, it is sufficient to construct a unitary representation $\widetilde{\theta}$ of H on Y_ϕ satisfying properties (iv) and (v).

Suppose that two maps θ and α are equivariant. For each $h\in H$, we define a linear map $\widetilde{\theta}_h:\mathbb{C}[G,X]\to\mathbb{C}[G,X]$ by

$$\widetilde{\theta}_h\left(\sum_g\delta_g\otimes f(g)\right)=\sum_g\delta_{\theta_h(g)}\otimes U_hf(g).$$

Let $f_1=\sum_g\delta_g\otimes f_1(g)$ and $f_2=\sum_{g'}\delta_{g'}\otimes f_2(g')$ be elements in $\mathbb{C}[G,X]$. We obtain from the covariance property of ϕ that

$$\left\langle\widetilde{\theta}_h(f_1),\widetilde{\theta}_h(f_2)\right\rangle=\sum_{g,g'}\left\langle U_hf_1(g),\phi\left(\alpha(\theta_h(g)^{-1})\theta_h(g')\right)U_hf_2(g')\right\rangle$$
$$=\sum_{g,g'}\left\langle f_1(g),\phi\left(\alpha(g)^{-1}g'\right)f_2(g')\right\rangle$$
$$=\left\langle\sum_g\delta_g\otimes f_1(g),\sum_{g'}\delta_{g'}\otimes f_2(g')\right\rangle.$$

Since $\mathcal{N}_\phi$ is invariant under the linear map $\widetilde{\theta}_h$, by passing to the quotient, we get an isometric linear map on $\mathbb{C}[G, X]/\mathcal{N}_\phi$, again denoted by $\widetilde{\theta}_h$. Then we have that

$$\left\langle \widetilde{\theta}_h\left(\sum_g \delta_g \otimes f_1(g) + \mathcal{N}_\phi\right), \sum_{g'} \delta_{g'} \otimes f_2(g') + \mathcal{N}_\phi \right\rangle = \sum_{g,g'} \left\langle f_1(g), \phi(\alpha(g)^{-1}\theta_{h^{-1}}(g'))U_h^* f_2(g') \right\rangle$$
$$= \left\langle \sum_g \delta_g \otimes f_1(g) + \mathcal{N}_\phi, \sum_{g'} \delta_{\theta_{h^{-1}}(g')} \otimes U_{h^{-1}} f_2(g') + \mathcal{N}_\phi \right\rangle$$

which implies that $\widetilde{\theta}_h^{\,*} = \widetilde{\theta}_{h^{-1}}$. Since $\widetilde{\theta}$ is clearly a group homomorphism, we have that $\widetilde{\theta}$ is a unitary representation.

To show the continuity of $\widetilde{\theta}$, it suffices to consider simple tensors of the form $\delta_g \otimes \xi + \mathcal{N}_\phi$. For a net $\{h_\iota\}$ in H converging to h, we have that

$$\begin{aligned}&\left\langle \widetilde{\theta}_{h_\iota}(\delta_g \otimes \xi + \mathcal{N}_\phi) - \widetilde{\theta}_h(\delta_g \otimes \xi + \mathcal{N}_\phi), \delta_{g'} \otimes \eta + \mathcal{N}_\phi \right\rangle \\ &\quad = \left\langle U_{h_\iota}\xi, [\phi(\alpha(\theta_{h_\iota}(g^{-1}))g') - \phi(\alpha(\theta_h(g^{-1}))g')]\eta \right\rangle + \left\langle (U_{h_\iota} - U_h)\xi, \phi(\alpha(\theta_h(g^{-1}))g')\eta \right\rangle \to 0\end{aligned}$$

since U is a unitary representation and ϕ and α are continuous. Moreover, we can obtain that $\widetilde{\theta}_h \pi(g) \widetilde{\theta}_h^{\,*} = \pi(\theta_h(g))$ for all $g \in G$ and $h \in H$. However, each $\widetilde{\theta}_h$ commutes with J_ϕ, so that $\widetilde{\theta}_h \pi(g) \widetilde{\theta}_h^{\,*} = \widetilde{\theta}_h \pi(g) \widetilde{\theta}_h^{\,J_\phi}$ for every $h \in H$. Furthermore, we have that

$$\widetilde{\theta}_h V_\phi(x) = \delta_{\theta_h(e)} \otimes U_h x + \mathcal{N}_\phi = V_\phi U_h(x), \quad (x \in X),$$

which implies that $\widetilde{\theta}_h V_\phi = V_\phi U_h$ for all $h \in H$. This completes the proof. □

Remark 3.3. Let $\mathcal{B}$ be a unital C^*-algebra and u a strongly continuous unitary representation of H into $\mathcal{U}(\mathcal{B})$ where $\mathcal{U}(\mathcal{B})$ is the group of unitary elements in $\mathcal{B}$. In Theorem 3.2, if $\phi : G \to \mathcal{B}$ is a (θ, u)-covariant and continuous α-completely positive map, then we can get a covariant version of Remark 2.3, which is also similar to Theorem 3.2 in [4].

More precisely, there are a Krein $\mathcal{B}$-module (X_ϕ, J_ϕ) with a generating vector $x_\phi \in X_\phi$, a J_ϕ-representation $\pi_\phi : G \to \mathcal{L}_{\mathcal{B}}(X_\phi)$, an adjointable operator $w_\phi : \mathcal{B} \to X_\phi$ and a strongly continuous unitary representation $\widetilde{\theta} : H \to \mathcal{L}_{\mathcal{B}}(X_\phi)$ such that

(i) $\phi(g) = \langle x_\phi, \pi(g)x_\phi \rangle$ for all $g \in G$,
(ii) the set $\{\pi(g)(x_\phi \cdot b) : g \in G,\ b \in \mathcal{B}\}$ is total in X_ϕ,
(iii) $\widetilde{\theta}_h \pi_\phi(g) \widetilde{\theta}_h^{\,*} = \pi_\phi(\theta_h(g))$ for all $g \in G$ and $h \in H$,
(iv) $\widetilde{\theta}_h w_\phi = w_\phi u_h$ for all $g \in G$ and $h \in H$,
(v) $w_\phi^* \pi_\phi(g) w_\phi = m_{\phi(g)}$ for all $g \in G$,

where m is the left multiplication operator on $\mathcal{B}$. If, in addition, $\phi(e) = \mathrm{id}_{\mathcal{B}}$, then w_ϕ can be chosen an isometry.

4. α-completely positive maps of groups into locally C^*-algebras

We recall that a *locally C^*-algebra* is a complete Hausdorff (complex) topological $*$-algebra of which the topology is determined by the collection of all continuous C^*-seminorms on it. We denote by $S(\mathcal{A})$ the set of all continuous C^*-seminorms on a locally C^*-algebra $\mathcal{A}$. For each $p \in S(\mathcal{A})$, the kernel $\ker(p) = \{a \in \mathcal{A} : p(a) = 0\}$ becomes a closed ideal in $\mathcal{A}$. Then $\mathcal{A}_p = \mathcal{A}/\ker(p)$ is a C^*-algebra with the norm induced by p. We denote by $\mathbf{q}_p$ the canonical map from $\mathcal{A}$ onto $\mathcal{A}_p$ and by $a_p = \mathbf{q}_p(a)$ the image of a in $\mathcal{A}_p$. Since $S(\mathcal{A})$ can be considered as a directed set with the order $p \geq q$ if $p(a) \geq q(a)(a \in \mathcal{A})$, for any $p \geq q$ in $S(\mathcal{A})$ there is a canonical surjective map $\mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q$ such that $\mathbf{q}_{pq}(a_p) = a_q$ for all $a_p \in \mathcal{A}_p$. Then the set $\{\mathcal{A}_p, \mathbf{q}_{pq} : \mathcal{A}_p \to \mathcal{A}_q, p \geq q\}$ becomes an inverse system of C^*-algebras and the inverse limit $\varprojlim_p \mathcal{A}_p$ is a locally C^*-algebra which is isomorphic to $\mathcal{A}$ [12].

Let $M_n(\mathcal{A})$ denote the $*$-algebra of all $n \times n$ matrices over $\mathcal{A}$ with the usual algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of $\mathcal{A}$. Then $M_n(\mathcal{A})$ is a locally C^*-algebra and it is isomorphic to $\varprojlim_p M_n(\mathcal{A}_p)$, where p runs through $S(\mathcal{A})$. The topology on the locally C^*-algebra $M_n(\mathcal{A})$ is determined by the family of C^*-seminorms $\{p_n : p \in S(\mathcal{A})\}$, where $p_n([a_{ij}]) = \left\| [\mathbf{q}_p(a_{ij})] \right\|_{M_n(\mathcal{A})}$.

Definition 4.1. Let $\mathcal{A}$ be a locally C^*-algebra, and let $\mathcal{E}$ be a (complex) vector space which is a right $\mathcal{A}$-module, compatibly with the algebra structure. Then $\mathcal{E}$ is called a *pre-Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ which is linear in the second variable and satisfies the following properties:

(i) $\langle \xi, \xi \rangle \geq 0$, and the equality holds only if $\xi = 0$,
(ii) $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,
(iii) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$.

We say that $\mathcal{E}$ is a *Hilbert $\mathcal{A}$-module* if $\mathcal{E}$ is complete with respect to the seminorms $\|\xi\|_p = p(\langle \xi, \xi \rangle)^{1/2}$ for $p \in S(\mathcal{A})$.

In the remaining part of sections, $\mathcal{A}$ and $\mathcal{E}$ denote a locally C^*-algebra and a Hilbert $\mathcal{A}$-module, respectively, unless specified otherwise.

For any $p \in S(\mathcal{A})$, let $\mathcal{N}_p = \{\xi \in \mathcal{E} : p(\langle \xi, \xi \rangle) = 0\}$. We write $\mathcal{E}_p$ for the Hilbert $\mathcal{A}_p$-module $\mathcal{E}/\mathcal{N}_p$ with

$$(\xi + \mathcal{N}_p)\mathbf{q}_p(a) = \xi a + \mathcal{N}_p \quad \text{and} \quad \langle \xi + \mathcal{N}_p, \eta + \mathcal{N}_p \rangle = \mathbf{q}_p(\langle \xi, \eta \rangle).$$

We denote by $\mathbf{Q}_p$ the canonical map from $\mathcal{E}$ onto $\mathcal{E}_p$ and ξ_p denotes the image $\mathbf{Q}_p(\xi)$. For any $p \geq q$ in $S(\mathcal{A})$, there is a canonical surjective map $\mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q$ such that $\mathbf{Q}_{pq}(\xi_p) = \xi_q$ for all $\xi_p \in \mathcal{E}_p$. Then $\{\mathcal{E}_p, \mathbf{Q}_{pq} : \mathcal{E}_p \to \mathcal{E}_q, p \geq q\}$ is an inverse system of Hilbert C^*-modules in the sense that

$$\mathbf{Q}_{pq}(\xi_p a_p) = \mathbf{Q}_{pq}(\xi_p)\mathbf{q}_{pq}(a_p) \quad \text{for } \xi_p \in \mathcal{E}_p, a_p \in \mathcal{A}_p,$$
$$\langle \mathbf{Q}_{pq}(\xi_p), \mathbf{Q}_{pq}(\eta_p) \rangle = \mathbf{q}_{pq}(\langle \xi_p, \eta_p \rangle) \quad \text{for } \xi_p, \eta_p \in \mathcal{E}_p,$$
$$\mathbf{Q}_{qr} \circ \mathbf{Q}_{pq} = \mathbf{Q}_{pr} \quad \text{for } p \geq q \geq r.$$

Then the inverse limit $\varprojlim_{p \in S(\mathcal{A})} \mathcal{E}_p$ is a Hilbert $\mathcal{A}$-module with

$$(\xi_p)_{p\in S(\mathcal{A})}(a_p)_{p\in S(\mathcal{A})} = (\xi_p a_p)_{p\in S(\mathcal{A})} \quad \text{and} \quad \langle (\xi_p)_{p\in S(\mathcal{A})}, (\eta_p)_{p\in S(\mathcal{A})} \rangle = (\langle \xi_p, \eta_p \rangle)_{p\in S(\mathcal{A})}$$

and it is isomorphic to the Hilbert $\mathcal{A}$-module $\mathcal{E}$.

Let $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ be the set of all adjointable operators on $\mathcal{E}$. The *strict topology* on $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is defined by the family of seminorms $\{\|\cdot\|_{p,\xi} : p \in S(\mathcal{A}),\ \xi \in \mathcal{E}\}$, where

$$\|T\|_{p,\xi} = \|T\xi\|_p + \|T^*\xi\|_p.$$

Since $T(\mathcal{N}_p) \subseteq \mathcal{N}_p$ for each $p \in S(\mathcal{A})$ and $T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E})$, we can define a linear map $(\mathbf{q}_p)_* : \mathcal{L}_{\mathcal{A}}(\mathcal{E}) \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p)$ by

$$[(\mathbf{q}_p)_*(T)](\mathbf{Q}_p(\xi)) = \mathbf{Q}_p(T(\xi)), \quad (T \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}),\ \xi \in \mathcal{E}). \tag{4.1}$$

We denote by T_p the operator $(\mathbf{q}_p)_*(T)$. The topology on $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is given by the family of seminorms $\{\tilde{p}\}_{p\in S(\mathcal{A})}$ where $\tilde{p}(T) = \|T_p\|$. Then $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ becomes a locally C^*-algebra. The connecting maps of the inverse system $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p) : p \in S(\mathcal{A})\}$ are denoted by $(\mathbf{q}_{pq})_* : \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p) \to \mathcal{L}_{\mathcal{A}_q}(\mathcal{E}_q)$ and the connecting maps are defined as follows:

$$[(\mathbf{q}_{pq})_*(T_p)](\mathbf{Q}_p(\xi)) = \mathbf{Q}_{pq}(T_p(\mathbf{Q}_p(\xi))) \quad \text{for } p \geq q.$$

Then the family $\{\mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p), (\mathbf{q}_{pq})_*, p \geq q\}$ is an inverse system of C^*-algebras and the inverse limit $\varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p)$ is isomorphic to $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$. We also refer to [12] for inverse limits of Hilbert C^*-modules and Banach spaces.

In the following theorem, we give a Krein representation associated with an α-completely positive map of a topological group into a locally C^*-algebra, which is a generalization of Theorem 2.2.

Theorem 4.2. *If $\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is an α-completely positive map, then there exist a Krein $\mathcal{A}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-unitary representation $\pi_\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$ and an adjointable operator $V_\rho : \mathcal{E} \to \mathcal{F}_\rho$ such that*

(i) *$\rho(g) = V_\rho^* \pi_\rho(g) V$ for all $g \in G$,*
(ii) *the set $[\pi_\rho(G)V_\rho(\mathcal{E})]$ is dense in $\mathcal{F}_\rho$,*
(iii) *$V_\rho^* \pi_\rho(g)^* \pi_\rho(g') V_\rho = V_\rho^* \pi_\rho(\alpha(g^{-1})g') V_\rho$ for all $g, g' \in G$.*

Proof. For each $p \in S(\mathcal{A})$, we consider the linear map $\rho_p : G \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p)$ given by

$$\rho_p = (\mathbf{q}_p)_* \circ \rho,$$

where $(\mathbf{q}_p)_* : \mathcal{L}_{\mathcal{A}}(\mathcal{E}) \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p)$ is defined as in (4.1).

We claim that the map ρ_p is α-completely positive. Indeed, it is easy to see that

$$\rho_p(gg') = \rho_p(\alpha(g)\alpha(g')) = \rho_p(\alpha(gg')) \quad \text{for all } g, g' \in G.$$

Let $g_1, \ldots, g_n \in G$ and $\xi_{p,1}, \ldots, \xi_{p,n} \in \mathcal{E}_p$. Then we have that

$$\begin{aligned}\sum_{i,j=1}^{n} \langle \xi_{p,i}, \rho_p(\alpha(g_i^{-1})g_j)\xi_{p,j} \rangle &= \sum_{i,j=1}^{n} \langle \mathbf{Q}_p(\xi_i), (\mathbf{q}_p)_*(\rho(\alpha(g_i^{-1})g_j))\mathbf{Q}_p(\xi_j) \rangle \\ &= \sum_{i,j=1}^{n} \mathbf{q}_p(\langle \xi_i, \rho(\alpha(g_i^{-1})g_j)\xi_j \rangle) \geq 0\end{aligned}$$

and that

$$[\rho_p(g_i)^* \rho_p(g_j)] = [(\mathbf{q}_p)_*(\rho(g_i)^* \rho(g_j))] \leq K[\rho_p(\alpha(g_i)^{-1}g_j)]$$

where the constant K is in the condition (iii) of Definition 2.1.

For $g, g_1, \ldots, g_n \in G$, we obtain that

$$\left[\rho_p\big(\alpha(gg_i)^{-1}gg_j\big)\right] \le M(g)\left[(\mathbf{q}_p)_*\big(\rho(\alpha(g_i)^{-1}g_j)\big)\right] = M(g)\left[\rho_p\big(\alpha(g_i)^{-1}g_j\big)\right]$$

where the inequality follows from the condition (iv) in Definition 2.1. By Theorem 2.2, there exist a Krein $\mathcal{A}_p$-module $(\mathcal{F}_p, J_p)$, a J_p-representation $\pi_p : G \to \mathcal{L}_{\mathcal{A}_p}(\mathcal{F}_p)$ and an adjointable operator $V_p : \mathcal{E}_p \to \mathcal{F}_p$ such that

(i) $\rho_p(g) = V_p^* \pi_p(g) V_p$ for all $g \in G$,
(ii) the set $[\pi_p(G)V_p(\mathcal{E}_p)]$ is dense in $\mathcal{F}_p$,
(iii) $V_p^*\pi_p(g)^*\pi_p(g')V_p = V_p^*\pi_p(\alpha(g^{-1})g')V_p$ for all $g, g' \in G$.

In construction of the Krein C^*-module $(\mathcal{F}_p, J_p)$, we know that $\mathcal{F}_p$ is the completion of the quotient space $\mathbb{C}[G] \otimes \mathcal{E}_p/\mathcal{N}_p$ where $\mathcal{N}_p = \{x \in \mathbb{C}[G] \otimes \mathcal{E}_p : \langle x, x\rangle_p = 0\}$. We define a linear map $\Psi_{pr} : \mathbb{C}[G] \otimes \mathcal{E}_p \to \mathbb{C}[G] \otimes \mathcal{E}_r$ by

$$\Psi_{pr}(\delta_g \otimes \xi_p) = \delta_g \otimes \mathbf{Q}_{pr}(\xi_p) = \delta_g \otimes \xi_r.$$

For every $g, g' \in G$ and $\xi_p, \eta_p \in \mathcal{E}_p$ we have that

$$\begin{aligned}\big\langle \Psi_{pr}(\delta_g \otimes \xi_p), \Psi_{pr}(\delta_{g'} \otimes \eta_p)\big\rangle &= \big\langle \mathbf{Q}_{pr}(\xi_p), \rho_r(\alpha(g^{-1})g')\mathbf{Q}_{pr}(\eta_p)\big\rangle \\ &= \mathbf{q}_{pr}\big(\langle \delta_g \otimes \xi_p, \delta_{g'} \otimes \eta_p\rangle\big).\end{aligned}$$

Hence, the map Ψ_{pr} induces a linear map from $\mathbb{C}[G] \otimes \mathcal{E}_p/\mathcal{N}_p$ into $\mathbb{C}[G] \otimes \mathcal{E}_r/\mathcal{N}_r$ that can be extended to a linear map, still denoted by Ψ_{pr}, from $\mathcal{F}_p$ into $\mathcal{F}_r$. Therefore, the set

$$\{\mathcal{F}_p, \mathcal{A}_p, \Psi_{pr} : \mathcal{F}_p \to \mathcal{F}_r,\ p \ge r\}$$

is an inverse system of Hilbert C^*-modules.

From the proof of Theorem 4.6 in [10], we obtain the following isomorphisms

$$\mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = \varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p) \quad \text{and} \quad \mathcal{L}_{\mathcal{A}}(\mathcal{F}) = \varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{F}_p).$$

Since $\Psi_{pr} \circ V_p = V_r \circ \mathbf{Q}_{pr}$ holds for $p, r \in S(\mathcal{A})$ with $p \ge r$, we have that

$$\big(V_p\big)_{p \in S(\mathcal{A})} \in \varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{E}_p, \mathcal{F}_p).$$

Moreover, we have that $\Psi_{pr} \circ \pi_p(g) = \pi_r(g) \circ \Psi_{pr}$ for all $g \in G$, so that

$$\big(\pi_p(g)\big)_{p \in S(\mathcal{A})} \in \varprojlim_p \mathcal{L}_{\mathcal{A}_p}(\mathcal{F}_p).$$

The map $\pi_\rho : G \to \varprojlim_p \mathcal{L}_{\mathcal{B}_p}(\mathcal{F}_p)$ given by $\pi_\rho(g) = \big(\pi_p(g)\big)_{p \in S(\mathcal{A})}$ is a representation of G on $\mathcal{F}_\rho$. From the equality $\rho_p(a) = V_p^*\pi_p(a)V_p$, we obtain that

$$\rho(g) = V_\rho^*\pi_\rho(g)V_\rho \quad \text{where } V_\rho = \big(V_p\big)_{p \in S(\mathcal{A})}.$$

From the relation $V_{\rho_p}^*\pi_{\rho_p}(g)^*\pi_{\rho_p}(g')V_{\rho_p} = V_{\rho_p}^*\pi_{\rho_p}(\alpha(g^{-1})g')V_{\rho_p}$, it follows that

$$V_\rho^*\pi_\rho(g)^*\pi_\rho(g')V_\rho = \big(V_{\rho_p}^*\pi_{\rho_p}(\alpha(g^{-1})g')V_{\rho_p}\big)_{p \in S(\mathcal{A})} = V_\rho^*\pi_\rho(\alpha(g^{-1})g')V_\rho.$$

Since π_{ρ_p} is a J_p-unitary representation of G, we also have that

$$\pi_\rho(g^{-1}) = \big(\pi_{\rho_p}(g^{-1})\big)_{p \in S(\mathcal{A})} = \big(\pi_{\rho_p}(g)^{J_p}\big)_{p \in S(\mathcal{A})} = \pi_\rho(g)^{J_\rho},$$

which implies that π_ρ is a J_ρ-unitary representation of G on $\mathcal{F}_\rho$. Finally, it follows from the density of $\pi_p(G)\big[V_p(\mathcal{E}_p)\big]$ in $\mathcal{F}_p$ that the set $\pi_\rho(G)\big[V_\rho(\mathcal{E})\big]$ is dense in $\mathcal{F}_\rho$. $\square$

5. Covariant α-completely positive maps of groups into locally C^*-algebras

In this section we establish a (projective) covariant representation theorem for α-completely positive maps of groups equipped with (semi-)group actions into locally C^*-algebras.

Let (G, H, θ) be a group system and let $(\mathcal{E}, J)$ be a Krein $\mathcal{A}$-module. An operator $v \in \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is called a *J-unitary* if $v^J v = vv^J = \mathrm{id}_{\mathcal{E}}$ where $v^J = Jv^*J$. A (*J*-)*unitary representation* u of H on $\mathcal{E}$ is a map from H into $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$ such that each u_h is a (*J*-)unitary, $u_{hh'} = u_h u_{h'}$ and the map $h \mapsto u_h(\xi)$ is continuous for every $\xi \in \mathcal{E}$.

Definition 5.1. Let $(\mathcal{E}, J)$ be a Krein $\mathcal{A}$-module over a locally C^*-algebra.

(1) A linear map $\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is called (θ, u)-*covariant* if $\rho(\theta_h(g)) = u_h \rho(g) u_h^*$ for all $g \in G$ and $h \in H$ where u is a unitary representation of H on $\mathcal{E}$.
(2) A *covariant J-representation* π of G on $(\mathcal{E}, J)$ is a pair (π, υ) satisfying

$$\pi(\theta_h(g)) = \upsilon_h \pi(g) \upsilon_h^J \quad \text{for all } g \in G \text{ and } h \in H,$$

where π is a J-representation of G on $(\mathcal{E}, J)$ and υ is a J-unitary representation of H on $(\mathcal{E}, J)$.

In the following theorem, we give a covariant representation of a group system which is associated with a covariant α-completely positive map of a group into a locally C^*-algebra. This may be regarded as a generalization of Theorem 3.2.

Theorem 5.2. *Let (G, H, θ) be a group system with an involution α on G and let $\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ be α-completely positive. If ρ is (θ, u)-covariant and if the maps α and θ are equivariant in the sense that $\alpha \circ \theta_h = \theta_h \circ \alpha$ for all $h \in H$, then there exist*

(a) *a Krein $\mathcal{A}$-module $(\mathcal{F}_\rho, J_\rho)$*
(b) *a covariant J_ρ-unitary representation π_ρ of G into $\mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$,*
(c) *an adjointable operator $V_\rho : \mathcal{E} \to \mathcal{F}_\rho$,*
(d) *a unitary representation $\upsilon : H \to \mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$,*

such that

(i) *$\rho(g) = V_\rho^* \pi_\rho(g) V$ for all $g \in G$,*
(ii) *the set $[\pi_\rho(G) V_\rho(\mathcal{E})]$ is dense in $\mathcal{F}_\rho$,*
(iii) *$V_\rho^* \pi_\rho(g)^* \pi_\rho(g') V_\rho = V_\rho^* \pi_\rho(\alpha(g^{-1}) g') V_\rho$ for all $g, g' \in G$,*
(iv) *$\upsilon_h V_\rho = V_\rho u_h$ for all $g \in G$ and $h \in H$.*

Proof. We will follow the notations in the proof of Theorem 4.2. Since ρ is strict continuous and α-completely positive, there exist a Krein $\mathcal{A}$-module $(\mathcal{F}_\rho, J_\rho)$, a J_ρ-unitary representation $\pi_\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$ and an adjointable operator $V_\rho \in \mathcal{L}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}_\rho)$ such that properties (i)–(iii) hold. Since $\mathcal{F}_\rho = \varprojlim_p \mathcal{F}_p$ where $\mathcal{F}_p$ is the completion of $\mathbb{C}[G] \otimes \mathcal{E}_p / \mathcal{N}_p$, we may assume that $\mathcal{F}_\rho$ is the completion of $\mathbb{C}[G] \otimes \mathcal{E}_p / \mathcal{N}_p$. For each $h \in H$, we define a linear map $\upsilon_h : \mathcal{F}_\rho \to \mathcal{F}_\rho$ by

$$\upsilon_h\big(\delta_g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big) = \delta_{\theta_h(g)} \otimes \mathbf{Q}_p\big(u_h(\xi)\big) + \mathcal{N}_p, \quad (g \in G,\ \xi \in \mathcal{E}).$$

Suppose that α and θ are equivariant, that is, $\alpha \circ \theta_h = \theta_h \circ \alpha$ for all $h \in H$. Let $g, g' \in G$, $\xi, \eta \in \mathcal{E}$ and $h \in H$. Then we obtain that

$$\Big\langle \upsilon_h\big(\delta_g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big), \delta_{g'} \otimes \mathbf{Q}_p(\eta) + \mathcal{N}_p \Big\rangle = \mathbf{q}_p\left(\big\langle u_h(\xi), \rho\big(\theta_h(\alpha(g)^{-1}) g'\big)\eta \big\rangle\right)$$
$$= \big\langle \delta_g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p, \delta_{\theta_{h^{-1}}(g')} \otimes \mathbf{Q}_p\big(u_h^*(\eta)\big) + \mathcal{N}_p \big\rangle,$$

which implies that $\upsilon_h^* = \upsilon_{h^{-1}}$. We also have that

$$\upsilon_h \pi_\rho(g) \upsilon_h^* \big(\delta_{g'} \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big) = \delta_{\theta_h(g \theta_{h^{-1}}(g'))} \otimes \mathbf{Q}_p\big(u_h u_h^*(\xi)\big) + \mathcal{N}_p$$
$$= \pi_\rho\big(\theta_h(g)\big)\big(\delta_{g'} \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big).$$

By linearity and continuity of $\pi_\rho(g)$ and υ_h, we have that $\upsilon_h \pi_\rho(g) \upsilon_h^* = \pi_\rho\big(\theta_h(g)\big)$ for all $g \in G$ and $h \in H$. Moreover, we see from equivariance of α and θ that J_ρ commutes with υ_h for each $h \in H$, so that $\upsilon_h^* = \upsilon_h^{J_\rho}$. Thus, we obtain that $\upsilon_h \upsilon_h^{J_\rho} = \upsilon_h^{J_\rho} \upsilon_h = \mathrm{id}_{\mathcal{F}_\rho}$ and that

$$\upsilon_h \pi_\rho(g) \upsilon_h^* = \upsilon_h \pi_\rho(g) \upsilon_h^{J_\rho} \quad \text{for all } g \in G \text{ and } h \in H.$$

Finally, we have that for any $h \in H$ and $\xi \in \mathcal{E}$

$$\upsilon_h V_p\big(\mathbf{Q}_p(\xi)\big) = \delta_e \otimes \mathbf{Q}_p\big(u_h(\xi)\big) + \mathcal{N}_p = V_p\left(\mathbf{Q}_p\big(u_h(\xi)\big)\right),$$

which means that $\upsilon_h V_\rho = V_\rho u_h$. This completes the proof. □

Let $\mathcal{S}$ be a unital semigroup. We denote by τ an action of $\mathcal{S}$ on a topological group G, which means that $\tau_s(\tau_t(g)) = \tau_{st}(g)$ and $\tau_e(g) = g$ for all $s, t \in \mathcal{S}$, where e is a unit element of $\mathcal{S}$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. A *multiplier on* $\mathcal{S}$ is a function $\sigma : \mathcal{S} \times \mathcal{S} \to \mathbb{T}$ satisfying the equations

$$\sigma(r, s)\sigma(rs, t) = \sigma(r, st)\sigma(s, t) \quad \text{and} \quad \sigma(s, e) = \sigma(e, s) = 1$$

for all $r, s, t \in \mathcal{S}$. A *projective isometric σ-representation* of $\mathcal{S}$ on $\mathcal{E}$ is a map $w : \mathcal{S} \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ which has the following properties;

(a) w_s is an isometry for each $s \in \mathcal{S}$,
(b) $w_{st} = \sigma(s, t) w_s w_t$ for all $s, t \in \mathcal{S}$.

Definition 5.3. Let w be a projective isometric σ-representation of $\mathcal{S}$ on $\mathcal{E}$. A linear map $\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is called *projective* (τ, w)*- covariant* if

$$\rho(\tau_s(g))w_s = w_s\rho(g) \quad \text{for all } s \in \mathcal{S} \text{ and } g \in G.$$

Let $\mathcal{S}$ be a left-cancellative discrete semigroup with a unit and let σ be a multiplier on $\mathcal{S}$. We denote by $\widetilde{\mathcal{E}}$ the Hilbert $\mathcal{A}$-module of all square summable $\mathcal{E}$-valued functions defined on $\mathcal{S}$ with the obvious operations and an $\mathcal{A}$-valued inner product. Let $\mathcal{S}$ act on G by τ as an automorphism of G in the sense that $s \mapsto \tau_s$ is a homomorphism of $\mathcal{S}$ into the automorphism group $\mathrm{Aut}(G)$.

Suppose that α and τ are equivariant in the sense that $\alpha \circ \tau_s = \tau_s \circ \alpha$ for all $s \in \mathcal{S}$. We claim that if ρ is an α-completely positive map of G into $\mathcal{L}_{\mathcal{A}}(\mathcal{E})$, then the map $\widetilde{\rho} : G \to \mathcal{L}_{\mathcal{A}}(\widetilde{\mathcal{E}})$ defined by

$$\big[\widetilde{\rho}(g)f\big](s) = \rho\big(\tau_s(g)\big)\big(f(s)\big), \quad (g \in G, f \in \widetilde{\mathcal{E}}, s \in \mathcal{S}) \tag{5.1}$$

is also α-completely positive.

Indeed, it is not hard to see that

$$\widetilde{\rho}(\alpha(g_1)\alpha(g_2)) = \widetilde{\rho}(\alpha(g_1g_2)) = \widetilde{\rho}(g_1g_2) \quad \text{for every } g_1, g_2 \in G.$$

Let $g_1, \dots, g_n \in G$ and $f_1, \dots, f_n \in \widetilde{\mathcal{E}}$. Then we have that

$$\sum_{i,j=1}^{n}\big\langle f_i, \widetilde{\rho}(\alpha(g_i)^{-1}g_j)f_j\big\rangle = \sum_{s\in\mathcal{S}}\sum_{i,j=1}^{n}\big\langle f_i(s), \rho(\alpha(\tau_s(a_i)^{-1})\tau_s(a_j))f_j(s)\big\rangle \geq 0,$$

where the equality follows from the equivariance of α and τ. We also obtain that

$$\begin{aligned}\sum_{i,j=1}^{n}\langle f_i, \widetilde{\rho}(g_i)^*\widetilde{\rho}(g_j)f_j\rangle &\leq K'\sum_{s\in\mathcal{S}}\sum_{i,j=1}^{n}\big\langle f_i(s), \rho(\alpha(\tau_s(g_i)^{-1})\tau_s(g_j))\big(f_j(s)\big)\big\rangle \\ &= K'\sum_{i,j=1}^{n}\big\langle f_i, \widetilde{\rho}(\alpha(g_i)^{-1}g_j)f_j\big\rangle,\end{aligned}$$

which means the condition (iii) in Definition 2.1. Similarly, we obtain that for some $M(g)$

$$\big[\widetilde{\rho}\big(\alpha(gg_i)^{-1}gg_j\big)\big] \leq M(g)\,\big[\widetilde{\rho}\big(\alpha(g_i)^{-1}g_j\big)\big].$$

Thus, $\widetilde{\rho}$ is α-completely positive.

For example, if we define a map W_s $(s \in \mathcal{S})$ on $\widetilde{\mathcal{E}}$ by

$$\big[W_sf\big](r) = \begin{cases}\overline{\sigma(s,t)}f(t), & \text{if } r = st \text{ for some } t \in \mathcal{S},\\ 0, & \text{if } r \notin s\,\mathcal{S},\end{cases}$$

then W is a projective σ-isometric representation of $\mathcal{S}$ on $\widetilde{\mathcal{E}}$ and the map $\widetilde{\rho}$ defined by (5.1) is projective (τ, W)-covariant (see Example 3.4 in [8]). Hence, if ρ is α-completely positive, then $\widetilde{\rho}$ is projective (τ, W)-covariant and α-completely positive.

Let G, $\mathcal{A}$ and $\mathcal{E}$ be as above and let a left-cancellative semigroup $\mathcal{S}$ act on G by τ. Suppose that $\rho : G \to \mathcal{L}_{\mathcal{A}}(\mathcal{E})$ is α-completely positive and that w is a projective isometric σ-representation of $\mathcal{S}$ on $\mathcal{E}$.

Theorem 5.4. *If ρ is projective (τ, w)-covariant and if α and τ are equivariant, then there exist a quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ as in Theorem 4.2 and a projective isometric σ-representation $v : \mathcal{S} \to \mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$ such that π_ρ is projective (τ, v)-covariant.*

Proof. Since ρ is α-completely positive, then there exists a Krein quadruple $(\mathcal{F}_\rho, J_\rho, \pi_\rho, V_\rho)$ as in Theorem 4.2. We may assume that $\mathcal{F}_\rho$ is the completion of $\mathbb{C}[G] \otimes \mathcal{E}_p/\mathcal{N}_p$. For each $s \in \mathcal{S}$, we define a linear map v_s in $\mathcal{L}_{\mathcal{A}}(\mathcal{F}_\rho)$ by

$$v_s\big(g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big) = \tau_s(g) \otimes \mathbf{Q}_p\big(w_s(\xi)\big) + \mathcal{N}_p, \quad (g \in G,\ \xi \in \mathcal{E}).$$

We claim that v is a projective isometric σ-representation, For $g_1, g_2 \in G, \xi, \eta \in \mathcal{E}$ and $s, t \in \mathcal{S}$, we have that

$$\begin{aligned}\big\langle v_s\big(g_1 \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big), v_s\big(g_2 \otimes \mathbf{Q}_p(\eta) + \mathcal{N}_p\big)\big\rangle &= \big\langle \rho_p\big(\tau_s(\alpha(g_1)^{-1}g_2)\big)\mathbf{Q}_p\big(w_s(\xi)\big), \mathbf{Q}_p\big(w_s(\eta)\big)\big\rangle \\ &= \big\langle g_1 \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p, g_2 \otimes \mathbf{Q}_p(\eta) + \mathcal{N}_p\big\rangle,\end{aligned}$$

where the second equality follows from the equivariance of α and τ.

Moreover, we obtain that

$$\begin{aligned}v_{st}\big(g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big) &= \sigma(s,t)\tau_s\big(\tau_t(g)\big) \otimes \mathbf{Q}_p\big(w_s\big(w_t(\xi)\big)\big) + \mathcal{N}_p \\ &= \sigma(s,t)v_sv_t\big(g \otimes \mathbf{Q}_p(\xi) + \mathcal{N}_p\big),\end{aligned}$$

which implies that v is a projective isometric σ-representation. It follows immediately from the definition of v that π is projective (τ, v)-covariant and that $v_sV_\rho = V_\rho w_s$ for any element $s \in \mathcal{S}$. $\square$

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