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Equivalence of semistability of Picard, Mann, Krasnoselskij and Ishikawa iterations

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Abstract
In this paper, we show that convergence of Picard, Mann, Krasnoselskij and Ishikawa iterations is equivalent in cone normed spaces. Also, we prove that semistability of these iterations is equivalent.

Keywords: semistability; Picard iteration; Mann iteration; Krasnoselskij iteration; Ishikawa iteration

1 Introduction

Let $(E, \|\cdot\|_E)$ be a real Banach space. A subset $P \subseteq E$ is called a *cone* in E if it satisfies the following conditions:

(i) P is closed, nonempty and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone P, by defining $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called *normal* if there exists a constant $k > 0$ such that $0 \leq x \leq y$ implies $\|x\|_E \leq k\|y\|_E$. The least positive number satisfying above is called the normal constant of P.

From now on, we suppose that E is a real Banach space, P is a cone in E and $\leq$ is a partial ordering with respect to P.

Lemma 1.1 ([1]) *Let P be a normal cone and let $\{a_n\}$ and $\{b_n\}$ be sequences in E satisfying the following inequality*:

$$a_{n+1} \leq h a_n + b_n, \tag{1}$$

where $h \in (0,1)$ and $b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Definition 1.2 ([2]) Let X be a vector space over the field F. Assume that the function $p : X \to E$ having the properties:

(i) $0 \leq p(x)$ for all x in X,
(ii) $p(x + y) \leq p(x) + p(y)$ for all x, y in X,
(iii) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in F$ and $x \in X$.

Then p is called a *cone seminorm* on X. A *cone norm* is a cone seminorm p such that

(iv) $x = 0$ if $p(x) = 0$.

We will denote a cone norm by $\|\cdot\|_c$ and $(X, \|\cdot\|_c)$ is called a *cone normed space*. Also, $d_c(x,y) = \|x-y\|_c$ defines a cone metric on X.

Definition 1.3 ([3]) Let $(X, \|\cdot\|_c)$ be a cone normed space. Then $A \subseteq X$ is called *bounded above* if there exists $c \in E$, $0 \ll c$ such that $\|x-y\|_c \preceq c$ for all $x, y \in A$.

Definition 1.4 Let $(X, \|\cdot\|_c)$ be a cone normed space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$, there exists an integer $N \geq 1$ such that for all $n \geq N$, $\|x_n - x\|_c \ll c$, then we will say $\{x_n\}$ converges to x and we write $\lim_{n\to\infty} x_n = x$.

Definition 1.5 Let $(X, \|\cdot\|_c)$ be a cone normed space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in E$ with $0 \ll c$, there exists an integer $N \geq 1$ such that for all $n, m \geq N$, $\|x_n - x_m\|_c \ll c$, then $\{x_n\}$ is said to be a *Cauchy* sequence. If every Cauchy sequence is convergent in X, then X is called a *cone Banach space*.

Lemma 1.6 ([4]) *Let (X, d_c) be a cone metric space, P be a normal cone. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if and only if* $\lim_{n\to\infty} d_c(x_n, x) = 0$.

Lemma 1.7 *Let $(X, \|\cdot\|_c)$ be a cone normed space over the real Banach space E with the cone P which is normal with the normal constant k. The mapping $N : X \to [0, \infty)$ defined by $N(x) = \|(\|x\|_c)\|_E$ satisfies the following properties*:

(i) $\|x\|_c \preceq \|y\|_c$ *implies* $N(x) \leq kN(y)$,
(ii) $N(x+y) \leq k[N(x) + N(y)]$ *for all* $x, y \in X$,
(iii) $N(\alpha x) = |\alpha| N(x)$ *for all* $\alpha \in F$ *and* $x \in X$,
(iv) $N(x-y) \leq k[N(x-z_1) + \cdots + N(x-z_n)]$ *for all* $x, y, z_1, \ldots, z_n \in X$,
(v) $x = 0$ *if and only if* $N(x) = 0$.

Moreover, let A be a bounded above subset of X, then

(vi) $\{N(x) : x \in A\}$ *is a bounded set*.

Proof The proof is obvious. □

Definition 1.8 Let $(X, \|\cdot\|_c)$ be a cone normed space over the real Banach space E with the normal cone P. The mapping N, defined in Lemma 1.7, is called a *norm type with respect to* $\|\cdot\|_c$.

Lemma 1.9 *Let $(X, \|\cdot\|_c)$ be a cone normed space over the real Banach space E with the normal cone P. Also, let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if and only if* $\lim_{n\to\infty} N(x_n - x) = 0$.

Proof Note that $\{\|x_n - x\|_c\}$ is a sequence in E and by Lemma 1.6, the proof is obvious. □

Definition 1.10 Let X be a cone normed space and $T : X \to X$ be a map for which there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b < 1/2$ and $0 < c < 1/2$. Then T is called a *Zamfirescu operator* with respect to (a, b, c) if and only if for each pair $x, y \in X$, T satisfies at least one of the following conditions:

(Z1) $\|Tx - Ty\|_c \preceq a\|x-y\|_c$,
(Z2) $\|Tx - Ty\|_c \preceq b(\|x - Tx\|_c + \|y - Ty\|_c)$,
(Z3) $\|Tx - Ty\|_c \preceq c(\|x - Ty\|_c + \|y - Tx\|_c)$.

Usually, for simplicity, T is called a Zamfirescu operator if T is Zamfirescu with respect to some triple (a, b, c) of scalers a, b and c with above restrictions. Also, T is called *f-Zamfirescu* operator if at least one of the relations (Z1), (Z2) and (Z3) hold for all $x \in X$ and for all $y \in F(T)$.

Remark 1.11 Let T be a Zamfirescu operator and $x, y \in X$ be arbitrary. Since T is Zamfirescu, at least one of the conditions (Z1), (Z2) and (Z3) is satisfied. If (Z2) holds, then

$$\begin{aligned} \|Tx - Ty\|_c &\leq b\big(\|x - Tx\|_c + \|y - Ty\|_c\big) \\ &\leq b\big(2\|x - Tx\|_c + \|y - x\|_c + \|Tx - Ty\|_c\big). \end{aligned}$$

Thus we get

$$(1 - b)\|Tx - Ty\|_c \leq b\|x - y\|_c + 2b\|x - Tx\|_c.$$

Since $0 < b < 1$, we have

$$\|Tx - Ty\|_c \leq \frac{b}{1 - b}\|x - y\|_c + \frac{2b}{1 - b}\|x - Tx\|_c.$$

Similarly, if (Z3) holds, then we obtain

$$\|Tx - Ty\|_c \leq \frac{c}{1 - c}\|x - y\|_c + \frac{2c}{1 - c}\|x - Tx\|_c.$$

Hence

$$\|Tx - Ty\|_c \leq \delta\|x - y\|_c + 2\delta\|x - Tx\|_c, \tag{2}$$

where $\delta := \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ and $0 < \delta < 1$.

Definition 1.12 Let X be a cone normed space. A self-map T of X is called a *quasi-contraction* if for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists

$$u \in C(T; x, y) \equiv \big\{\|x - y\|_c, \|x - Tx\|_c, \|y - Ty\|_c, \|y - Tx\|_c, \|x - Ty\|_c\big\}$$

such that $\|Tx - Ty\|_c \leq \lambda u$. If this inequality holds for all $x \in X$ and $y \in F(T)$, we say that T is a *f-quasi-contraction*.

Definition 1.13 Let X be a cone normed space, T be a self-map of X and $p_0 = u_0 = x_0 = v_0 \in X$. The *Picard iteration* is given by

$$p_{n+1} = Tp_n. \tag{3}$$

For a sequence of self-maps $\{T_n\}_{n \in \mathbb{N}}$, the iteration $p_{n+1} = T_n p_n$ is called the *Picard's S-iteration*.

Another two well-known iteration procedures for obtaining fixed points of T are *Mann iteration* defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n \tag{4}$$

and *Ishikawa iteration* defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{5}$$

where $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq [0,1)$. Also, the Krasnoselskij iteration is defined by

$$v_{n+1} = (1-\lambda)v_n + \lambda T v_n, \tag{6}$$

where $\lambda \in (0,1)$.

If T is a self-map of X, then by $F(T)$ we mean the set of fixed points of T. Also, $\mathbb{N}_0$ denotes the set of nonnegative integers, *i.e.*, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 1.14 ([5]) *Let (X,d_c) be a complete cone metric space and P be a normal cone. Suppose that the mapping $T: X \to X$ satisfies the contractive condition*

$$d_c(Tx,Ty) \preceq k d_c(x,y)$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant. Then T has a unique fixed point in X and for each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Lemma 1.15 ([5]) *Let (X,d_c) be a complete cone metric space and P be a normal cone. Suppose that the mapping $T: X \to X$ satisfies the contractive condition*

$$d_c(Tx,Ty) \preceq k\bigl(d_c(Tx,x) + d_c(Ty,y)\bigr)$$

for all $x,y \in X$, where $k \in [0,1/2)$ is a constant. Then T has a unique fixed point in X and for each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Lemma 1.16 ([5]) *Let (X,d_c) be a complete cone metric space and P be a normal cone. Suppose that the mapping $T: X \to X$ satisfies the contractive condition*

$$d_c(Tx,Ty) \preceq k\bigl(d_c(Tx,y) + d_c(Ty,x)\bigr)$$

for all $x,y \in X$, where $k \in [0,1/2)$ is a constant. Then T has a unique fixed point in X and for each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Lemma 1.17 ([2]) *Let T be a quasi-contraction with* $0 < \lambda < 1/2$. *Then T is a Zamfirescu operator.*

Definition 1.18 Let $(X, \|\cdot\|_c)$ be a cone normed space and $\{T_n\}_n$ be a sequence of self-maps of X with $\bigcap_n F(T_n) \neq \emptyset$. Let x_0 be a point of X and assume that $x_{n+1} = f(T_n, x_n)$ is an iteration procedure involving $\{T_n\}$, which yields a sequence $\{x_n\}$ of points from X. The iteration $x_{n+1} = f(T_n, x_n)$ is said to be $\{T_n\}$-*semistable* (or *semistable with respect to* $\{T_n\}$) if whenever $\{x_n\}$ converges to a fixed point q in $\bigcap_n F(T_n)$ and $\{y_n\}$ is a sequence in X with

$\lim_{n\to\infty} \|y_{n+1} - f(T_n, y_n)\|_c = 0$ and $\|y_n - f(T_n, y_n)\|_c = o(t_n)$ for some sequence $\{t_n\} \subseteq \mathbb{R}^+$, then $y_n \to q$.

The iteration $x_{n+1} = f(T_n, x_n)$ is said to be $\{T_n\}$-*stable* (or *stable with respect to* $\{T_n\}$) if $\{x_n\}$ converges to a fixed point q in $\bigcap_n F(T_n)$ and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} \|y_{n+1} - f(T_n, y_n)\|_c = 0$, we have $y_n \to q$.

Note that if $T_n = T$ for all n, then Definition 1.18 gives the definitions of T-semistability and T-stability respectively.

Lemma 1.19 ([2]) *Let (X, d_c) be a cone metric space, P be a normal cone and $\{T_n\}_{n\in\mathbb{N}_0}$ be a sequence of self-maps of X with $\bigcap_n F(T_n) \neq \emptyset$. Suppose that there exist nonnegative bounded sequences $\{a_n\}$, $\{b_n\}$ with* $\sup_n b_n < 1$ *such that*

$$d_c(T_n x, q) \leq a_n d_c(x, T_n x) + b_n d_c(x, q)$$

for each $n \in \mathbb{N}_0$, $x \in X$ and $q \in \bigcap_n F(T_n)$. Then the Picard's S-iteration is semistable with respect to $\{T_n\}$.

Lemma 1.20 ([2]) *Let (X, d_c) be a cone metric space, P be a normal cone and $\{T_n\}_{n\in\mathbb{N}_0}$ be a sequence of self-maps of X with $\bigcap_n F(T_n) \neq \emptyset$. If for all $n \in \mathbb{N}_0$, T_n is a f-Zamfirescu operator with respect to $(\alpha_n, \beta_n, \gamma_n)$ with* $\sup_n \gamma_n < 1/2$. *Then the Picard's S-iteration is semistable with respect to $\{T_n\}$.*

Lemma 1.21 ([2]) *Under the conditions of Lemma* 1.22 *if T_n is a Zamfirescu operator for all n, then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.*

Lemma 1.22 ([2]) *Let (X, d_c) be a cone metric space, P be a normal cone and $\{T_n\}_{n\in\mathbb{N}_0}$ be a sequence of self-maps of X with $\bigcap_n F(T_n) \neq \emptyset$. If for all $n \in \mathbb{N}_0$, T_n is a f-quasi-contraction with λ_n such that* $\sup_n \lambda_n < 1$, *then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.*

For some other sources on these topics, we refer to [6–23].

2 Main results

Theorem 2.1 *Let X be a cone normed space and P be a normal cone. Suppose that T is a Zamfirescu self-map of X and $q \in F(T)$. Then the following are equivalent*:

(i) *the Picard iteration converges to q,*
(ii) *the Mann iteration converges to q.*

Proof Let $\{\alpha_n\} \subseteq (0,1)$ be given. We prove the implication (i) $\Rightarrow$ (ii). Suppose that $\lim_{n\to\infty} p_n = q$. Now, by using (3) and (4), we have

$$\begin{aligned}
\|u_{n+1} - p_{n+1}\|_c &\leq (1-\alpha_n)\|u_n - Tp_n\|_c + \alpha_n \|Tu_n - Tp_n\|_c \\
&\leq (1-\alpha_n)\|u_n - p_n\|_c + (1-\alpha_n)\|p_n - Tp_n\|_c + \alpha_n \|Tu_n - Tp_n\|_c \\
&\leq (1-\alpha_n)\|u_n - p_n\|_c + (1-\alpha_n)\big(\|p_n - q\|_c + \|Tp_n - Tq\|_c\big) \\
&\quad + \alpha_n \|Tu_n - Tp_n\|_c. \qquad (7)
\end{aligned}$$

Using (2) with $x := p_n$, $y := u_n$, we get

$$\begin{aligned}\|Tu_n - Tp_n\|_c &\le \delta\|u_n - p_n\|_c + 2\delta\|p_n - Tp_n\|_c \\ &\le \delta\|u_n - p_n\|_c + 2\delta\big(\|p_n - q\|_c + \|Tp_n - Tq\|_c\big).\end{aligned} \tag{8}$$

Using (2) with $x := q$, $y := p_n$, we obtain

$$\|Tp_n - Tq\|_c \le \delta\|p_n - q\|_c. \tag{9}$$

Relations (7), (8) and (9) lead to

$$\|u_{n+1} - p_{n+1}\|_c \le \big(1 - (1-\delta)\alpha_n\big)\|u_n - p_n\|_c + (1 - \alpha_n + 2\delta\alpha_n)(1+\delta)\|p_n - q\|_c.$$

Set

$$\begin{aligned}&a_n := \|u_n - p_n\|_c, \\ &b_n := (1 - \alpha_n + 2\delta\alpha_n)(1+\delta)\|p_n - q\|_c, \\ &h := 1 - \sup_n \alpha_n.\end{aligned}$$

Since $\lim_{n\to\infty}\|p_n - q\|_c = 0$, by using Lemma 1.1, we get

$$\lim_{n\to\infty}\|u_n - p_n\|_c = 0.$$

Thus

$$0 \le \|u_n - q\|_c \le \|u_n - p_n\|_c + \|p_n - q\|_c \to 0,$$

as $n \to \infty$. This completes the proof.

Now we prove (ii) $\Rightarrow$ (i). Suppose that $\lim_{n\to\infty}\|u_n - q\|_c = 0$. Applying (3) and (4), we have

$$\begin{aligned}\|u_{n+1} - p_{n+1}\|_c &\le (1-\alpha_n)\|u_n - Tp_n\|_c + \alpha_n\|Tu_n - Tp_n\|_c \\ &\le (1-\alpha_n)\|u_n - Tu_n\|_c + \|Tu_n - Tp_n\|_c.\end{aligned} \tag{10}$$

Using (2) with $x := u_n$, $y := p_n$, we obtain

$$\|Tu_n - Tp_n\|_c \le \delta\|u_n - p_n\|_c + 2\delta\|u_n - Tu_n\|_c. \tag{11}$$

Therefore, from (10) and (11), we get

$$\begin{aligned}\|u_{n+1} - p_{n+1}\|_c &\le \delta\|u_n - p_n\|_c + (1 - \alpha_n + 2\delta)\|u_n - Tu_n\|_c \\ &\le \delta\|u_n - p_n\|_c + (1 - \alpha_n + 2\delta)\big(\|u_n - q\|_c + \|Tu_n - Tq\|_c\big) \\ &\le \delta\|u_n - p_n\|_c + (1 - \alpha_n + 2\delta)(1+\delta)\|u_n - q\|_c.\end{aligned} \tag{12}$$

Put

$$a_n := \|u_n - p_n\|_c,$$
$$b_n := (1 - \alpha_n + 2\delta)(1 + \delta)\|u_n - q\|_c,$$
$$h := \delta.$$

Since $\lim_{n\to\infty} b_n = 0$, by Lemma 1.1 and relation (12), we get $\lim_{n\to\infty} \|u_n - p_n\|_c = 0$. Thus

$$\|p_n - q\| \le \|p_n - u_n\|_c + \|u_n - q\|_c \to 0,$$

as $n \to \infty$ and so the proof is complete. □

Theorem 2.2 *Let X be a cone normed space and P be a normal cone. Suppose that T is a Zamfirescu self-map of X and $q \in F(T)$. Then the following are equivalent*:

(i) *the Picard iteration converges to q,*
(ii) *the Krasnoselskij iteration converges to q.*

Proof For $\alpha_n = \lambda$, the Mann iteration reduces to the Krasnoselskij iteration. Now apply the proof of Theorem 2.1. □

Theorem 2.3 *Let X be a cone normed space and P be a normal cone. Suppose that T is a Zamfirescu operator of X and $q \in F(T)$. Then the following are equivalent*:

(i) *the Mann iteration converges to q,*
(ii) *the Ishikawa iteration converges to q.*

Proof Let $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq [0,1)$ be given. We prove the implication (i) ⇒ (ii). Suppose that $\lim_{n\to\infty} u_n = q$. Using

$$\lim_{n\to\infty} \|x_n - u_n\|_c = 0, \tag{13}$$

and

$$0 \le \|q - x_n\|_c \le \|u_n - q\|_c + \|x_n - u_n\|_c,$$

we get $\lim_{n\to\infty} x_n = q$. The proof is complete if we prove relation (13).

Using (2), (4) and (5) with $x := u_n$, $y := z_n$, we have

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\|_c &\le \big\|(1 - \alpha_n)(u_n - x_n) + \alpha_n(Tu_n - Tz_n)\big\|_c \\
&\le (1 - \alpha_n)\|u_n - x_n\|_c + \alpha_n\|Tu_n - Tz_n\|_c \\
&\le (1 - \alpha_n)\|u_n - x_n\|_c + \alpha_n\delta\|u_n - z_n\|_c + 2\alpha_n\delta\|u_n - Tu_n\|_c.
\end{aligned} \tag{14}$$

Using (2) with $x := u_n$, $y := x_n$, we have

$$\begin{aligned}
\|u_n - z_n\|_c &\le \big\|(1 - \beta_n)(u_n - x_n) + \beta_n(u_n - Tx_n)\big\|_c \\
&\le (1 - \beta_n)\|u_n - x_n\|_c + \beta_n\|u_n - Tx_n\|_c \\
&\le (1 - \beta_n)\|u_n - x_n\|_c + \beta_n\|u_n - Tu_n\|_c + \beta_n\|Tu_n - Tx_n\|_c
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)\|u_n - x_n\|_c + \beta_n\|u_n - Tu_n\|_c + \beta_n\delta\|u_n - x_n\|_c + 2\beta_n\delta\|u_n - Tu_n\|_c \\
&= \bigl(1-\beta_n(1-\delta)\bigr)\|u_n - x_n\|_c + \beta_n(1+2\delta)\|u_n - Tu_n\|_c. \qquad (15)
\end{aligned}
$$

Relations (14) and (15) lead to

$$
\begin{aligned}
\|u_{n+1} - x_{n+1}\|_c &\leq (1-\alpha_n)\|u_n - x_n\|_c + \alpha_n\delta\bigl(1-\beta_n(1-\delta)\bigr)\|u_n - x_n\|_c \\
&\quad + \alpha_n\beta_n\delta(1+2\delta)\|u_n - Tu_n\| + 2\alpha_n\delta\|u_n - Tu_n\|_c \\
&= \Bigl(1-\alpha_n\bigl(1-\delta\bigl(1-\beta_n(1-\delta)\bigr)\bigr)\Bigr)\|u_n - x_n\|_c \\
&\quad + \alpha_n\delta\bigl(\beta_n(1+2\delta)+2\bigr)\|u_n - Tu_n\|_c.
\end{aligned}
$$

Put

$$
\begin{aligned}
&a_n := \|u_n - x_n\|_c, \\
&b_n := \alpha_n\delta\bigl(\beta_n(1+2\delta)+2\bigr)\|u_n - Tu_n\|_c, \\
&h := 1 - \sup_n \alpha_n.
\end{aligned}
$$

Note that $\lim_{n\to\infty}\|u_n - q\|_c = 0$, T is Zamfirescu and $q \in F(T)$. By (2) we obtain

$$
0 \leq \|u_n - Tu_n\|_c \leq \|u_n - q\|_c + \|q - Tu_n\|_c \leq (\delta+1)\|u_n - q\|_c.
$$

Hence $\lim_{n\to\infty}\|u_n - Tu_n\|_c = 0$; that is, $\lim_{n\to\infty} b_n = 0$. Lemma 1.1 leads to

$$
\lim_{n\to\infty}\|u_n - x_n\|_c = 0.
$$

Now we will prove that (ii) $\Rightarrow$ (i). Using (2) with $x := z_n$, $y := u_n$, we obtain

$$
\begin{aligned}
\|x_{n+1} - u_{n+1}\|_c &\leq \bigl\|(1-\alpha_n)(x_n - u_n) + \alpha_n(Tz_n - Tu_n)\bigr\|_c \\
&\leq (1-\alpha_n)\|x_n - u_n\|_c + \alpha_n\|Tz_n - Tu_n\|_c \\
&\leq (1-\alpha_n)\|x_n - u_n\|_c + \alpha_n\delta\|z_n - u_n\|_c + 2\alpha_n\delta\|z_n - Tz_n\|_c. \qquad (16)
\end{aligned}
$$

Also, the following relation holds:

$$
\begin{aligned}
\|z_n - u_n\|_c &\leq \bigl\|(1-\beta_n)(x_n - u_n) + \beta_n(Tx_n - u_n)\bigr\|_c \\
&\leq (1-\beta_n)\|x_n - u_n\|_c + \beta_n\|Tx_n - u_n\|_c \\
&\leq (1-\beta_n)\|x_n - u_n\|_c + \beta_n\|Tx_n - x_n\|_c + \beta_n\|x_n - u_n\|_c \\
&\leq \|x_n - u_n\|_c + \beta_n\|Tx_n - x_n\|_c. \qquad (17)
\end{aligned}
$$

Substituting (17) in (16), we obtain

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|_c \\
&\quad \leq (1-\alpha_n)\|x_n - u_n\|_c + \alpha_n\delta\bigl(\|x_n - u_n\|_c + \beta_n\|Tx_n - x_n\|_c\bigr) + 2\alpha_n\delta\|z_n - Tz_n\| \\
&\quad \leq \bigl(1-(1-\delta)\alpha_n\bigr)\|x_n - u_n\|_c + \alpha_n\beta_n\delta\|Tx_n - x_n\|_c + 2\alpha_n\delta\|z_n - Tz_n\|_c. \qquad (18)
\end{aligned}
$$

Put

$$\begin{aligned}
&a_n := \|x_n - u_n\|_c, \\
&b_n := \alpha_n \beta_n \delta \|Tx_n - x_n\|_c + 2\alpha_n \delta \|z_n - Tz_n\|_c, \\
&h := 1 - \sup_n \alpha_n.
\end{aligned}$$

From $\lim_{n\to\infty} \|x_n - q\|_c = 0$, T is Zamfirescu, $q \in F(T)$ and by (2) we obtain

$$0 \leq \|x_n - Tx_n\|_c \leq \|x_n - q\|_c + \|q - Tx_n\|_c \leq (\delta + 1)\|x_n - q\|_c,$$

and

$$\begin{aligned}
0 &\leq \|z_n - Tz_n\|_c \\
&\leq \|z_n - q\|_c + \|q - Tz_n\|_c \\
&\leq (\delta + 1)\|z_n - q\|_c \\
&\leq (\delta + 1)\bigl[(1 - \beta_n)\|x_n - q\|_c + \beta_n \|q - Tx_n\|_c\bigr] \\
&\leq (\delta + 1)\bigl[(1 - \beta_n)\|x_n - q\|_c + \delta \beta_n \|q - x_n\|_c\bigr] \\
&\leq (\delta + 1)\bigl(1 - \beta_n(1 - \delta)\bigr)\|q - x_n\|_c.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - Tx_n\|_c = 0$ and $\lim_{n\to\infty} \|z_n - Tz_n\|_c = 0$; that is, $\lim_{n\to\infty} b_n = 0$.

Lemma 1.1 and (18) lead to $\lim_{n\to\infty} \|x_n - u_n\|_c = 0$. Thus, we get

$$\|q - u_n\| \leq \|x_n - u_n\|_c + \|x_n - q\|_c \to 0,$$

and the proof is complete. □

Corollary 2.4 *Let X be a cone Banach space, P be a normal cone and T be a Zamfirescu self-map of X. Then T has a unique fixed point in X and the Picard, Mann, Krasnoselskij and Ishikawa iterative sequences converge to the fixed point of T.*

Corollary 2.5 *Let X be a cone Banach space, P be a normal cone and T be a quasi-contraction mapping of X with $0 < \lambda < 1/2$. Then T has a unique fixed point in X and the Picard, Mann, Krasnoselskij and Ishikawa iterative sequences converge to the fixed point of T.*

Theorem 2.6 *Let X be a cone Banach space and P be a normal cone. Suppose that T is a self-map of X and that every Picard and Mann iteration converges to a fixed point of T. Then the following are equivalent*:

(i) *the Picard iteration is semistable with respect to T,*
(ii) *the Mann iteration is semistable with respect to T.*

Proof Suppose that q is a fixed point of T such that every Picard and Mann iteration converges to q. Let $\{y_n\}$ be an arbitrary sequence in X. For (i) ⇒ (ii), let

$$\lim_{n\to\infty} \bigl\|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ty_n\bigr\|_c = 0$$

and $\|y_n - Ty_n\|_c = o(t_n)$ for some $\{t_n\} \subseteq \mathbb{R}^+$. We have

$$\|y_{n+1} - Ty_n\|_c \leq \left\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\right\|_c + (1-\alpha_n)\|y_n - Ty_n\|_c \to 0$$

as $n \to \infty$. By assumption (i), we get $\lim_{n\to\infty} y_n = q$.

Conversely, we prove (ii) $\Rightarrow$ (i). Let $\lim_{n\to\infty} \|y_{n+1} - Ty_n\|_c = 0$ and $\|y_n - Ty_n\|_c = o(t_n)$ for some $\{t_n\} \subseteq \mathbb{R}^+$. We have

$$\begin{aligned}
&\left\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\right\|_c \\
&\quad \leq \|y_{n+1} - Ty_n\|_c + (1-\alpha_n)\|y_{n+1} - y_n\|_c + (1-\alpha_n)\|y_{n+1} - Ty_n\|_c \\
&\quad \leq (2-\alpha_n)\|y_{n+1} - Ty_n\|_c + (1-\alpha_n)\|y_{n+1} - y_n\|_c \\
&\quad \leq (2-\alpha_n)\|y_{n+1} - Ty_n\|_c + (1-\alpha_n)\big(\|y_{n+1} - Ty_n\|_c + \|y_n - Ty_n\|_c\big) \\
&\quad = (3-2\alpha_n)\|y_{n+1} - Ty_n\|_c + (1-\alpha_n)\|y_n - Ty_n\|_c \to 0
\end{aligned}$$

as $n \to \infty$. Thus $\lim_{n\to\infty} y_n = q$ and so the Picard iteration is semistable with respect to T. □

Theorem 2.7 *Let X be a cone Banach space and P be a normal cone. Suppose that T is a self-map of X and that every Picard and Krasnoselskij iteration converges to a fixed point of T. Then the following are equivalent*:

(i) *the Picard iteration is semistable with respect to T,*
(ii) *the Krasnoselskij iteration is semistable with respect to T.*

Proof In Theorem 2.7, put $\alpha_n = \lambda$. Then by the same method used in the proof of Theorem 2.7, we can complete the proof. □

Theorem 2.8 *Let X be a cone Banach space and P be a normal cone. Suppose that $\{\alpha_n\}$ in Ishikawa iteration procedure satisfies $\lim_{n\to\infty} \alpha_n = 0$, T is a self-map of X with bounded above range and also every Picard and Ishikawa iterative sequence converges to a fixed point of T. Then the following are equivalent*:

(i) *the Picard iteration is semistable with respect to T,*
(ii) *the Ishikawa iteration is semistable with respect to T.*

Proof Suppose that q is a fixed point of T such that every Picard and Ishikawa iterative sequence converges to q. Let $\{y_n\} \subseteq X$ and $\{\beta_n\} \subseteq [0,1)$ be given and set

$$\begin{aligned}
s_n &:= (1-\beta_n)y_n + \beta_n Ty_n, \\
\gamma_n &:= \|y_{n+1} - Ty_n\|_c, \\
\delta_n &:= \left\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\right\|_c, \\
M &:= \sup\big\{N(Tx) : x \in X\big\},
\end{aligned}$$

where N is the norm type with respect to $(\|\cdot\|_c)$. It is assumed that T has bounded above range and so, by Lemma 1.7, $M < \infty$.

Now we prove that (i) $\Rightarrow$ (ii). Let $\lim_{n\to\infty}\delta_n = 0$ and $\|y_n - Ty_n\|_c = o(t_n)$ for some $\{t_n\} \subseteq \mathbb{R}^+$. Observe that

$$\begin{aligned}
\gamma_n &= \|y_{n+1} - Ty_n\|_c \\
&\leq \big\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\big\|_c + \big\|(1-\alpha_n)y_n + \alpha_n Ts_n - Ty_n\big\|_c \\
&\leq \big\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\big\|_c + \alpha_n\big(\|y_n\|_c + \|Ts_n\|_c\big) + \|y_n - Ty_n\|_c \\
&\leq \delta_n + \alpha_n\big(\|y_n - Ty_n\|_c + \|Ty_n\|_c + \|Ts_n\|_c\big) + \|y_n - Ty_n\|_c \\
&\leq \delta_n + \alpha_n\big(\|Ty_n\|_c + \|Ts_n\|_c\big) + (1+\alpha_n)\|y_n - Ty_n\|_c.
\end{aligned}$$

By Lemma 1.7 we have

$$\begin{aligned}
N(\gamma_n) &\leq kN\big(\delta_n + \alpha_n\big(\|Ty_n\|_c + \|Ts_n\|_c\big) + (1+\alpha_n)\|y_n - Ty_n\|_c\big) \\
&\leq kN(\delta_n) + 2kM\alpha_n + k(1+\alpha_n)N(y_n - Ty_n) \to 0
\end{aligned}$$

as $n \to \infty$ (here k is the normal constant of P). So, by Lemma 1.9, $\lim_{n\to\infty}\gamma_n = 0$ and the condition (i) assures that $\lim_{n\to\infty} y_n = q$. Thus the Ishikawa iteration is semistable with respect to T.

Conversely, we prove (ii) $\Rightarrow$ (i). Let $\lim_{n\to\infty}\gamma_n = 0$ and $\|y_n - Ty_n\|_c = o(t_n)$ for some $\{t_n\} \subseteq \mathbb{R}^+$. We have

$$\begin{aligned}
\delta_n &= \big\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\big\|_c \\
&\leq \|y_{n+1} - Ty_n\|_c + \big\|Ty_n - (1-\alpha_n)y_n - \alpha_n Ts_n\big\|_c \\
&= \|y_{n+1} - Ty_n\|_c + \big\|Ty_n - (1-\alpha_n)y_n - \alpha_n Ts_n + \alpha_n Ty_n - \alpha_n Ty_n\big\|_c \\
&\leq \gamma_n + (1-\alpha_n)\|y_n - Ty_n\|_c + \alpha_n\|Ty_n - Ts_n\|_c.
\end{aligned}$$

By Lemmas 1.7 and 1.9, we get

$$\begin{aligned}
N(\delta_n) &\leq kN\big(\gamma_n + (1-\alpha_n)\|y_n - Ty_n\|_c + \alpha_n\|Ty_n - Ts_n\|_c\big) \\
&\leq kN(\gamma_n) + k(1-\alpha_n)N(y_n - Ty_n) + k\alpha_n N(Ty_n - Ts_n) \\
&\leq kN(\gamma_n) + k(1-\alpha_n)N(y_n - Ty_n) + 2kM\alpha_n \to 0
\end{aligned}$$

as $n \to \infty$, where k is the normal constant of P. So $\lim_{n\to\infty}\delta_n = 0$ and by assumption (ii), we have $\lim_{n\to\infty} y_n = q$. Thus the Picard iteration is semistable with respect to T. □

Theorem 2.9 *Let X be a cone Banach space and P be a normal cone. Suppose that $\{\alpha_n\}$ in Mann and Ishikawa procedures satisfies $\lim_{n\to\infty}\alpha_n = 0$, T is a self-map of X with bounded above range and also every Mann and Ishikawa iterative sequence converges to a fixed point of T. Then the following are equivalent*:

(i) *the Mann iteration is T-stable*,

(ii) *the Ishikawa iteration is T-stable*.

Proof Let q be a fixed point of T and every Mann and Ishikawa iterative sequence converge to q. Suppose that k is the normal constant of P and put

$$M := \sup\{N(Tx) : x \in X\},$$

where N is the norm type with respect to $\|\cdot\|_c$. Since T has bounded above range, then $M < \infty$. Now let $\{y_n\}$ be an arbitrary sequence in X. We prove (i) $\Rightarrow$ (ii). For this suppose that

$$\lim_{n\to\infty} \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\|_c = 0,$$

where $s_n = (1-\beta_n)y_n + \beta_n Ty_n$ and $\{\beta_n\} \subseteq [0,1)$. We show that $\lim_{n\to\infty} y_n = q$. Note that

$$\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_c \le \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\|_c + \|\alpha_n Ts_n - \alpha_n Ty_n\|_c.$$

By Lemma 1.7 and Lemma 1.9, we obtain

$$N\big(y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\big) \le kN\big(y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\big) + 2kM\alpha_n \to 0,$$

as $n \to \infty$ and so

$$\lim_{n\to\infty} \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_c = 0.$$

Condition (i) assures that $\lim_{n\to\infty} y_n = q$. Thus the Ishikawa iteration is T-stable.

Conversely, we prove (ii) $\Rightarrow$ (i). Suppose that

$$\lim_{n\to\infty} \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_c = 0.$$

We show that $\lim_{n\to\infty} y_n = q$. Put

$$s_n := (1-\beta_n)y_n + \beta_n Ty_n,$$

and observe that

$$\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\|_c \le \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_c + \|\alpha_n Ty_n - \alpha_n Ts_n\|_c.$$

By Lemma 1.7 and Lemma 1.9, we obtain

$$N\big(y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\big) \le kN\big(y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\big) + 2kM\alpha_n \to 0$$

as $n \to \infty$ and hence $\lim_{n\to\infty} \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ts_n\|_c = 0$. By assumption (ii), we get $\lim_{n\to\infty} y_n = q$ and the proof is complete. □

Corollary 2.10 *Let $(X, \|\cdot\|_c)$ be a cone normed space, P be a normal cone and T be a self-map of X and $q \in F(T)$. Suppose that there exist nonnegative real numbers a and b with $b < 1$ such that*

$$\|Tx - q\|_c \le a\|x - Tx\|_c + b\|x - q\|_c$$

for each $x \in X$. Assume that for $\{\alpha_n\} \subseteq (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, and let every Picard, Mann, Krasnoselskij and Ishikawa iterative sequence converge to q. Then the Picard, Mann and Krasnoselskij iterations are T-semistable. Moreover, if T has bounded above range, then the Ishikawa iteration is T-semistable.

Corollary 2.11 *Let $(X, \|\cdot\|_c)$ be a cone normed space, P be a normal cone and T be a f-Zamfirescu or quasi-contraction self-map of X and $q \in F(T)$. Assume that $\{\alpha_n\}$ in Mann and Ishikawa iteration procedures satisfies $\{\alpha_n\} \subseteq (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Also, let every Picard, Mann, Krasnoselskij and Ishikawa iterative sequence converge to q. Then the Picard, Mann and Krasnoselskij iterations are T-semistable. Moreover, if T has bounded above range, then the Ishikawa iteration is T-semistable.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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