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Existence of clustering high dimensional bump solutions of superlinear elliptic problems on expanding annuli

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Abstract

We consider the nonlinear elliptic problem

$$-\Delta u = u^p \quad \text{in } \Omega_R, \qquad u > 0 \quad \text{in } \Omega_R, \qquad u = 0 \quad \text{in } \Omega_R$$

where $p > 1$ and $\Omega_R = \{x \in \mathbb{R}^N\colon\ R < |x| < R + 1\}$ with $N \geqslant 3$. It is known that as $R \to \infty$, the number of nonequivalent solutions of the above problem goes to ∞ when $p \in (1, (N+2)/(N-2))$, $N \geqslant 3$. Here we prove the same phenomenon for any $p > 1$ by finding $O(N-1)$-symmetric clustering bump solutions which concentrate near the set $\{(x_1, \dots, x_N) \in \Omega_R\colon\ x_N = 0\}$ for large $R > 0$.

Keywords: Concentration phenomena; Expanding annuli; Supercritical problem

1. Introduction

This paper deals with the semilinear elliptic equation

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega_R, \\ u > 0 & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R \end{cases} \tag{1.1}$$

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where $1 < p < \infty$ and Ω_R is an expanding annulus in $\mathbb{R}^N$, $N \geqslant 3$, i.e.

$$\Omega_R := \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N\colon\ R^2 < |x|^2 := \sum_{i=1}^{N} x_i^2 < (R+1)^2 \right\} \tag{1.2}$$

with R large enough.

If the domain Ω_R is a ball and $p \in (1, (N+2)/(N-2))$, problem (1.1) has a unique solution which is radially symmetric in virtue of the classical result of Gidas, Ni and Nirenberg [12]. On the other hand, it is not difficult to show for $p \in (1, (N+2)/(N-2))$ that even though the annulus Ω_R has a rotational symmetry, a least energy solution of (1.1) is not radially symmetric for large $R > 0$. In past decades, Coffman [9], Li [15], Byeon [4–6], Catrina and Wang [8] found many nonequivalent (nonradial) solutions of (1.1) in the subcritical case, i.e. $p < (N+2)/(N-2)$. Here we say functions u and v on Ω_R are nonequivalent if $u(\cdot) \neq v(g\,\cdot)$ for any $g \in O(N)$. Even more, it is known in [15] and [6] that for some supercritical exponent $p > (N+2)/(N-2)$, there exist nonradial solutions of (1.1) for large $R > 0$. In fact, for a typical closed subgroup $O(k) \times O(N-k) \subset O(N)$ with any integer $2 \leqslant k \leqslant N/2$, it is known in [6] that for $p \in (1, (N-k+2)/(N-k-2))$, there are two $O(k) \times O(N-k)$-symmetric solutions u_R and v_R of (1.1) such that u_R concentrates near $\{(x_1, \dots, x_N) \in \Omega_R \mid x_{k+1} = \dots = x_N = 0\}$ for large $R > 0$ and v_R near $\{(x_1, \dots, x_N) \in \Omega_R \mid x_1 = \dots = x_k = 0\}$ for large $R > 0$. The concentration sets are special cases of locally minimal orbital sets defined in [6], where the solutions concentrating around locally minimal sets were found. All solutions in the works cited above are locally minimal energy solutions in the class of G-symmetric functions for some closed subgroup $G \subset O(N)$. Even though those solutions were found for some supercritical exponents $p > (N+2)/(N-2)$, only finite type of solutions of (1.1) have been known. On the other hand, it was shown by Kazdan and Warner [14] that problem (1.1) always has a radial solution even for any $p > 1$. So it is natural to wonder whether for any $p > 1$, there exist many nonequivalent nonradial solutions of (1.1) for large $R > 0$.

In this paper, we answer positively to the question by finding clustering bumps in the class of $O(N-1)$-symmetric functions. More precisely, we look for solutions to (1.1) which are radial with respect to $x_1, \dots, x_{N-1}$ variables. We define $s = \sqrt{\sum_{i=1}^{N-1} x_i^2}$ and $z = x_N$. Then, a function $u(x_1, \dots, x_N)$ solves problem (1.1) if and only if $v(s,z) := u(x_1, \dots, x_{N-1}, x_N)$ solves the following two dimensional problem

$$\begin{cases} \Delta_{s,z} v + \dfrac{N-2}{s}\dfrac{\partial v}{\partial s} + v^p = 0 & \text{in } A_R, \\ v > 0 & \text{in } A_R, \\ v = 0 & \text{on } \partial A_R \end{cases} \tag{1.3}$$

where

$$A_R := \left\{ (s,z) \in \mathbb{R}^2\colon\ R < \left|(s,z)\right| < R+1 \right\} \tag{1.4}$$

is an expanding annulus in the plane as $R \to \infty$. Therefore, we are led to look for solutions of (1.3) which is even with respect to $s \in \mathbb{R}$. We note that for any $t(R) > 0$ with $\lim_{R\to\infty} t(R) = 0$,

$$\lim_{R\to\infty} \inf\left\{ |s|\colon\ (s,z) \in A_R,\ |z| \leqslant t(R)R \right\} = \infty.$$

Thus, as $R \to +\infty$, we are brought to consider the following limit problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } S, \\ u > 0 & \text{in } S, \\ u = 0 & \text{on } \partial S \end{cases} \tag{1.5}$$

where S is an infinite strip $S := (0,1) \times \mathbb{R}$ (see Section 2.2). The basic cell in our construction is a solution of (1.5) which does exist for any $p > 1$ and is unique up to a translation. Then, for any integer k, provided R is large enough, we build solutions to (1.3) gluing together k basic cells, which are suitably rotated and translated. The solution we find concentrates at k different points $(s_1^R, z_1^R), \dots, (s_k^R, z_k^R)$ as $R \to \infty$, where $\lim_{R\to\infty}(s_i^R/R, z_i^R/R) = (1,0)$ for any $i = 1, \dots, k$. It is clear that if a solution v of (1.3) concentrates at a point (s_i^R, z_i^R) as $R \to +\infty$, then the corresponding solution u of (1.1) concentrates on the $(N-2)$-dimensional set

$$\Gamma_i^R = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N \colon \sum_{i=1}^{N-1} x_i^2 = s_i^R,\ x_N = z_i^R \right\} \quad \text{as } R \to +\infty.$$

Therefore, for any integer k provided R is large enough, we construct solutions of (1.1) which possess $O(N-1)$-symmetry and concentrate on k different $(N-2)$-dimensional spheres $\Gamma_1^R, \dots, \Gamma_k^R$ whose normalized sets $\Gamma_1^R/R, \dots, \Gamma_k^R/R$ collapse to the unit sphere $\mathbb{S}^{N-2} = \{(x_1, \dots, x_{N-1}, 0) \in \mathbb{R}^N \colon (x_1)^2 + \dots + (x_{N-1})^2 = 1\}$ as $R \to +\infty$. More precisely, our main result reads as follows.

Theorem 1.1. *For any $p \in (1,\infty)$ and $l \in \mathbb{N}$ there exists an $R_l = R_l(p) > 0$ such that for all $R > R_l$, Eq. (1.1) has a solution $U_{R,l}$ such that*

(i) *for any $\Theta \in O(N-1)$, $s = \sqrt{(x_1)^2 + \dots + (x_{N-1})^2}$ and $z = x_N$,*

$$U_{R,l}\big(\Theta(x_1, \dots, x_{N-1}), x_N\big) = U_{R,l}(x_1, \dots, x_{N-1}, x_N) := U_{R,l}(s,z);$$

(ii) *there exist l different points $(s_1^R, z_1^R), \dots, (s_l^R, z_l^R)$ with $|(s_i^R, z_i^R)| = R + 1/2$, $s_i > 0$ for $i = 1, \dots, l$ such that*

$$\lim_{R\to\infty} \min\big\{\big|\big(s_i^R, z_i^R\big) - \big(s_j^R, z_j^R\big)\big| \colon 1 \leqslant i \neq j \leqslant l\big\} = \infty, \qquad \lim_{R\to\infty} \big(s_i^R/R, z_i^R/R\big) = (1,0)$$

and

$$\lim_{R\to\infty} \left\| U_{R,l}(s,z) - \sum_{i=1}^{l} u_0\Big(\big(\Theta_i^R\big)^{-1}(s,z) - (R,0)\Big) \right\|_{L^\infty(\Omega_R \cap \{s>0\})} = 0,$$

where the matrix $\Theta_i^R \in O(2)$ is defined so that

$$\big(s_i^R, z_i^R\big) = \Theta_i^R \left(R + \frac{1}{2}, 0 \right)$$

and u_0 is the unique solution of (1.5) which is even with respect to the second variable.

In particular, for $l_1 \neq l_2 \in \mathbb{N}$, the solutions U_{R,l_1} and U_{R,l_2} of Eq. (1.1) are nonequivalent in the sense that $U_{R,l_1}(\cdot) \neq U_{R,l_2}(g\,\cdot)$ for any $g \in O(N)$.

From Theorem 1.1, we see that for any $p > 1$, the number of nonequivalent solutions of (1.1) goes to ∞ as $R \to \infty$. Let us recall some recent results concerning existence of multi-bump positive solutions in expanding tubular domains. Let M be a compact m-dimensional smooth submanifold of $\mathbb{R}^N$ and $M_R = \{Rx \in \mathbb{R}^N\colon x \in M\}$. We define

$$D_R := \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, M_R) \leqslant 1\}.$$

The set D_R is an annulus if $M = S^{N-1}$. For $p \in (1, (N+2)/(N-2))$ with $N \geqslant 3$ and $p > 1$ with $N = 2$, we consider a problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } D_R, \\ u > 0 & \text{in } D_R, \\ u = 0 & \text{on } \partial D_R. \end{cases} \tag{1.6}$$

For general M, there is no radial symmetry of the domain D_R. Thus, we cannot use the principle of symmetric criticality by Palais [16] to get multi-bump solutions. Even though, the existence of multi-bump solutions to the subcritical problem (1.6) was established by Dancer and Yan in [11] and by Ackermann, Clapp and Pacella in [1] under a nondegeneracy assumption for a solution of the limit problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } B(0,1) \times \mathbb{R}^m \subset \mathbb{R}^N, \\ u > 0 & \text{in } B(0,1) \times \mathbb{R}^m, \\ u = 0 & \text{on } \partial B(0,1) \times \mathbb{R}^m. \end{cases} \tag{1.7}$$

The same result in a more general context – possibly in a degenerate setting – was obtained by Byeon and Tanaka [7] using a variational method. More precisely, their results claim that for any integer k, provided R is large enough, there exists a k-bumps solution which is obtained by gluing k different bubbles which solve the limit problem (1.7). It is not certain whether we can find many nonequivalent solutions of (1.3) for any $p > 1$ without any symmetry of M when $R > 0$ is large. On the other hand, if M is rotationally invariant with respect to a fixed line, that is, $O(N-1)$-symmetric, we can obtain the same result with Theorem 1.1 by the same argument in this paper.

The proof of our result relies on the Lyapunov–Schmidt reduction argument. The paper is organized as follows. In Section 2 we construct a set of approximate solutions. In Section 3 we study a linear problem and in Section 5 we reduce the problem to a finite dimensional one. In Section 6 we study the reduced problem and in Section 7 we prove Theorem 1.1.

Notation.

- The letters c and C will be used throughout the paper to denote positive constants which may vary from line to line. On the other hand, constants with subscripts $C_0, C_1, \ldots$ are reserved for fixed quantities (particularly independent of R).
- We will use big O and small o notations to describe the limit behavior of a certain quantity as $R \to \infty$.

– The Laplacian Δ represents $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$ or $\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial z^2}$, depending on the dimension of the domain of functions for which the operator Δ acts.
– Given any domain D, $\lambda_1(D)$ is the first Dirichlet eigenvalue of the Laplacian in D.
– For a domain D, $C_c^\infty(D)$ is the space of compactly supported smooth functions in D. If D is a domain such that $\lambda_1(D) > 0$, e.g., $D = \Omega_R$ or S, then $H_0^1(D)$ is defined as the completion of $C_c^\infty(D)$ with respect to the norm $\|v\|_{H^1(D)} := (\int_D |\nabla v|^2)^{1/2}$.
– $H^1(\mathbb{R}^2)$ is the completion of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm $\|v\|_{H^1(\mathbb{R}^2)} := (\int_{\mathbb{R}^2} |\nabla v|^2 + |v|^2)^{1/2}$.
– $c_+ = \max\{c, 0\}$ for any $c \in \mathbb{R}$.
– $B((s,z); r)$ denotes the ball of radius $r > 0$ with the center $(s,z) \in \mathbb{R}^2$.

2. Preliminaries

2.1. *The symmetric Sobolev space $\mathcal{H}$ with a weighted Sobolev norm*

Let

$$\widetilde{\mathcal{H}} = \{U \in H_0^1(\Omega_R)\colon\ U(\Theta x) = U(x) \text{ for any } \Theta \in O(N-1) \times \{1\} \subset O(N)\}$$

be a Hilbert space whose inner product and norm are given by

$$\langle u, v\rangle_{H^1(\Omega_R)} = \int_{\Omega_R} \nabla u(x) \cdot \nabla v(x)\, dx, \qquad \|u\|_{H^1(\Omega_R)} = \left(\int_{\Omega_R} \left|\nabla u(x)\right|^2 dx\right)^{1/2}.$$

Also for the two dimensional annulus A_R defined in (1.4), let $\mathcal{H}$ be the completion of $\{u \in C_c^\infty(A_R)\colon\ u(s,z) = u(-s,z)\}$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \left(\int_{A_R} \left|\nabla u(s,z)\right|^2 |s|^{N-2}\, ds\, dz\right)^{1/2},$$

which becomes a Hilbert space endowed with the inner product

$$\langle u, v\rangle_{\mathcal{H}} = \int_{A_R} \nabla u(s,z) \cdot \nabla v(s,z) |s|^{N-2}\, ds\, dz.$$

It is easy to check that the map $\Phi : \widetilde{\mathcal{H}} \to \mathcal{H}$ defined by

$$\Phi(U)(s,z) = U(x_1, \ldots, x_{N-1}, x_N) \quad \text{where } s^2 = \sum_{i=1}^{N-1} x_i^2 \text{ and } z = x_N \tag{2.1}$$

gives an isomorphism.

Note that if $1 < p \leqslant 2^* - 1 := \frac{N+2}{N-2}$ then $U \in \widetilde{\mathcal{H}}$ is a solution of (1.1) if and only if $u = \Phi(U) \in \mathcal{H}$ is a critical point of the functional $I_R : \mathcal{H} \to \mathbb{R}$ given by

$$I_R(u)=\frac{1}{2}\int\limits_{A_R}\left|\nabla u(s,z)\right|^2|s|^{N-2}\,ds\,dz-\frac{1}{p+1}\int\limits_{A_R}u(s,z)_+^{p+1}|s|^{N-2}\,ds\,dz. \tag{2.2}$$

We will look for a critical point in $\mathcal{H}$ of the functional I_R with the required properties. In Section 7 we will use the same argument even when p is supercritical ($p>2^*-1$) provided the functional I_R is suitable modified.

2.2. *Properties of u_0 and an approximation for solutions*

In [3], [11, Proposition A.1] (see also [7, Lemma 6.1]), it was proved that (1.5) has a solution u_0 such that u_0 is symmetric with respect to the s-axis and the line $\{s=1/2\}$, and

$$u_0(s,z)=c\big(1+o(1)\big)e^{-\pi|z|}\sin\pi s \quad \text{in } S \text{ for some } c>0 \tag{2.3}$$

and

$$\left|u_0(s,z)\right|,\ \left|\nabla u_0(s,z)\right|,\ \left|D^2u_0(s,z)\right|\leqslant c_1e^{-c_2|z|} \quad \text{in } S \text{ for some } c_1,c_2>0.$$

Moreover, Dancer [10] proved that u_0 is the unique solution of (1.5) up to translation and has the following nondegeneracy property.

Lemma 2.1. *Given any $p\in(1,\infty)$, suppose that $\phi_0\in L^\infty(S)\cap H^1_{\mathrm{loc}}(S)$ solves the linear problem*

$$\begin{cases}\Delta\phi+pu_0^p\phi=0 & \text{in } S,\\ \phi=0 & \text{on } \partial S.\end{cases}$$

Then $\phi_0=c\frac{\partial u_0}{\partial z}$ for some $c\in\mathbb{R}$.

Here the nondegeneracy is known only for two dimensional strips.

Using u_0 as a building block, we are going to construct an approximation of solutions (1.1) in the following way.

First, we fix $\alpha\in(0,1)$ and we set

$$\widetilde{S}_R:=\left\{(s,z)\colon\ s\in\left(R^{-\alpha},1-R^{-\alpha}\right),\ |z|\leqslant R^{\frac{1-\alpha}{2}}\right\}$$

so that $S_R:=(R,0)+\widetilde{S}_R\subset A_R$. Then we define

$$u_R(s,z):=\frac{1}{(1-2R^{-\alpha})^{\frac{2}{p-1}}}\cdot u_0\left(\frac{s-R-R^{-\alpha}}{1-2R^{-\alpha}},\frac{z}{1-2R^{-\alpha}}\right)\quad \text{in } S_R. \tag{2.4}$$

It is straightforward to check that u_R satisfies $\Delta u_R+u_R^p=0$ in S_R and $u=0$ on $\partial S_R\cap\{s=R+R^{-\alpha}$ or $R+1-R^{-\alpha}\}$.

Moreover, in order to extend u_R to a function in $H_0^1(A_R)$ or $H^1(\mathbb{R}^2)$, we need to introduce a truncation function. Choose a function $\psi_R\in C^\infty(\mathbb{R}^2)$ satisfying

$$0 \leqslant \psi_R(s,z) = \psi_R(z) \leqslant 1, \quad \psi_R(z) = \psi_R(-z), \tag{2.5}$$

$$\psi_R(z) = \begin{cases} 1 & \text{if } |z| \leqslant R^{\frac{1-\alpha}{2}}/2, \\ 0 & \text{if } |z| \geqslant R^{\frac{1-\alpha}{2}}, \end{cases} \tag{2.6}$$

$$\left\| \frac{\partial \psi_R}{\partial z} \right\|_{L^\infty(\mathbb{R}^2)} = O\big(R^{\frac{\alpha-1}{2}}\big) \quad \text{and} \quad \left\| \frac{\partial^2 \psi_R}{\partial z^2} \right\|_{L^\infty(\mathbb{R}^2)} = O\big(R^{\alpha-1}\big). \tag{2.7}$$

Now let Λ_R be the configuration space

$$\Lambda_R := \bigg\{ P = (p_1, \ldots, p_l) = \big((s_1,z_1), \ldots, (s_l,z_l)\big) \colon |p_i| = R + \frac{1}{2}, \\ s_i \geqslant \left(R + \frac{1}{2}\right) - \frac{R}{R^{\pi(1+\epsilon_1)M_1}}, \ |p_i - p_j| \geqslant M_1 \log R \text{ for } i \neq j \bigg\} \tag{2.8}$$

where ϵ_1 and M_1 are small positive numbers which will be determined later. Define also $\Theta_i \in O(2)$ by

$$\Theta_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \tag{2.9}$$

where $\theta_i \in (-\pi/2, \pi/2)$ is determined by the relation $p_i = \Theta_i(R + 1/2, 0) = (R + 1/2)(\cos\theta_i, \sin\theta_i)$. Then, we set

$$v_{R,P}(s,z) = \begin{cases} \sum_{i=1}^{l} (\psi_R u_R)(\Theta_i^{-1}(s,z)) & \text{if } s \geqslant 0, \\ v_{R,P}(-s,z) & \text{if } s < 0 \end{cases} \tag{2.10}$$

for $P = (p_1, \ldots, p_l) \in \Lambda_R$. It is useful to point out that

$$\operatorname{supp} v_{R,P} \subset C_R := A_R \cap \left\{ (s,z) \colon \left(1 - \frac{2}{R^{\pi(1+\epsilon_1)M_1}}\right) R < s < R+1 \right\}. \tag{2.11}$$

We will find a solution of (1.1) having the form $V_{R,P} + W$ where

$$V_{R,P} = \Phi^{-1}(v_{R,P}) \tag{2.12}$$

and Φ is the isomorphism between $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ defined in (2.1). Note that (1.1) is equivalent to an equation of $W \in \widetilde{\mathcal{H}}$ given by

$$L(W) = -\big(E + N(W)\big) \quad \text{in } \Omega_R, \qquad W = 0 \quad \text{on } \partial\Omega_R \tag{2.13}$$

where Ω_R is the annulus in $\mathbb{R}^N$ defined in (1.2),

$$L(W) = \Delta W + p V_{R,P}^{p-1} W, \tag{2.14}$$

$$E = \Delta V_{R,P} + V_{R,P}^{p} \tag{2.15}$$

and

$$N(W) = (Vv_{R,P} + W)_+^p - V_{R,P}^p - pV_{R,P}^{p-1}W. \tag{2.16}$$

2.3. A technical lemma

Before starting the proof of Theorem 1.1, we introduce the following elementary but quite useful lemma.

Lemma 2.2. *For each $M > 0$, there is $C = C(M) > 0$ such that*

1. $$\left|\left(\sum_{i=1}^k t_{i+}\right)^p - \sum_{i=1}^k (t_i)_+^p\right| \leqslant C \sum_{i\neq j} |t_i t_j|^{\min\{\frac{p}{2},1\}}$$

and

2. $$\left|\left(\sum_{i=1}^k t_{i+}\right)^{p+1} - \sum_{i=1}^k (t_i)_+^{p+1} - (p+1)\sum_{i\neq j}(t_i)_+^{p-1} t_j\right| \leqslant C \sum_{i\neq j} |t_i t_j|^{\min\{\frac{p+1}{2},2\}}$$

for $|t_1|, \ldots, |t_k| \leqslant M$.

3. Invertibility of the operator L

Let $R > 0$ be fixed and let $\mathcal{V}_R : \Lambda_R \to \mathcal{H}$ be a map such that $\mathcal{V}_R(P) = v_{R,P}$. If each component p_i of $P = (p_1, \ldots, p_l) \in \Lambda_R$ is written as $p_i = (R+1/2)(\cos\theta_i, \sin\theta_i)$, we choose a vector $\tau_i := (0, \ldots, (-\sin\theta_i, \cos\theta_i), \ldots, 0)$ in the tangent space $T_P\Lambda_R$ of Λ_R at P and then we define

$$(Z_i)_{R,P} := \frac{\partial \mathcal{V}_R}{\partial \tau_i}(P) = d(\mathcal{V}_R)_P \tau_i \in \mathcal{H}, \tag{3.1}$$

i.e. the directional derivative of $\mathcal{V}_R$ along τ_i at P. It is easy to check that

$$\begin{aligned}(Z_i)_{R,P}(s,z) = \left(R+\frac{1}{2}\right)^{-1} \cdot \bigg[&(-\sin\theta_i s + \cos\theta_i z)\cdot \frac{\partial(\psi_R u_R)}{\partial s}\big(\Theta_i^{-1}(s,z)\big) \\ &- (\cos\theta_i s + \sin\theta_i z)\cdot \frac{\partial(\psi_R u_R)}{\partial z}\big(\Theta_i^{-1}(s,z)\big)\bigg] \quad \text{if } s \geqslant 0\end{aligned} \tag{3.2}$$

and $(Z_i)_{R,P}(s,z) = (Z_i)_{R,P}(-s,z)$ for all (s,z) in the domain A_R (see (1.4)) where Θ_i is the matrix given in (2.9). It is useful to point out that (see (2.11)) $\operatorname{supp}(Z_i)_{R,P} \subset C_R$.

Here the notation $\Theta_i^{-1}(s,z)$ is understood as $\Theta_i^{-1}\binom{s}{z} \in \mathbb{R}^2$.

In the following, we will often omit subscripts R and P for simplicity if no ambiguity arises. (For example, $v = v_{R,P}$, $Z_i = (Z_i)_{R,P}$ and so on.)

In this section, we study the invertibility of the linear operator L in the subspace of $\widetilde{\mathcal{H}}$ orthogonal to $\operatorname{span}\{\overline{Z}_1, \ldots, \overline{Z}_l\}$ where

$$\overline{Z}_i := \Phi^{-1}(Z_i). \tag{3.3}$$

Proposition 3.1. *For $R > 0$ sufficiently large and $P \in \Lambda_R$, if $H \in \widetilde{\mathcal{H}}$, then the problem*

$$\begin{cases} L(W) = \Delta H + \sum_{i=1}^{l} c_i \Delta \overline{Z}_i & \text{in } \Omega_R, \\ W = 0 & \text{on } \partial\Omega_R, \\ \langle \overline{Z}_i, W \rangle_{H^1(\Omega_R)} = 0 & \text{for } i = 1, \dots, l \end{cases} \tag{3.4}$$

admits a unique solution $W \in \widetilde{\mathcal{H}}$ and $\{c_1, \dots, c_l\} \subset \mathbb{R}$ that satisfy

$$\|W\|_{H^1(\Omega_R)} \leqslant C_1 \|H\|_{H^1(\Omega_R)} \tag{3.5}$$

for some $C_1 > 0$ independent of R.

The following a priori estimate will be used in an essential way in the proof of Proposition 3.1.

Lemma 3.2. *For $R > 0$ sufficiently large and $P \in \Lambda_R$, a solution of (3.4) satisfies (3.5) with $C_1 > 0$ independent of R.*

Proof. Suppose that (3.5) does not hold. Then there are sequences of numbers $R_n \to \infty$, $(c_{1n}, \dots, c_{ln}) \in \mathbb{R}^l$, points $P_n = (p_{1n}, \dots, p_{ln}) \in \Lambda_{R_n}$ and functions $V_n = V_{R_n, P_n}$, $\overline{Z}_{in} = (\overline{Z}_i)_{R_n, P_n} \in \widetilde{\mathcal{H}}$ (see (2.12) and (3.3)), $W_n, H_n \in \widetilde{\mathcal{H}}$ such that

$$\|W_n\|_{H^1(\Omega_R)} = R_n^{\frac{N-2}{2}}, \qquad \|H_n\|_{H^1(\Omega_R)} = o\big(R_n^{\frac{N-2}{2}}\big) \tag{3.6}$$

and

$$\Delta W_n + p V_n^{p-1} W_n = \Delta H_n + \sum_{i=1}^{l} c_{in} \Delta \overline{Z}_{in} \quad \text{in } \Omega_{R_n}. \tag{3.7}$$

We will show that a contradiction arises. The proof is divided into three steps.

Step 1. We claim that $c_{1n}, \dots, c_{ln} = o(1)$.

It is clear that $\bigcup_{i=1}^{l} \Theta_i(S_{R_n}) \subset C_n := C_{R_n}$ (see (2.11)) and also that the support of $v_n := \Phi(V_n)$ is contained in C_n. Multiplying (3.7) by $\overline{Z}_{jn}$, integrating the result over Ω_{R_n} and using the symmetry of elements in $\widetilde{\mathcal{H}}$, we get

$$\int_{C_n} \big(\nabla w_n \cdot \nabla Z_{jn} - p v_n^{p-1} w_n Z_{jn}\big) \cdot \left(\frac{|s|}{R_n}\right)^{N-2}$$
$$= \int_{C_n} (\nabla h_n \cdot \nabla Z_{jn}) \cdot \left(\frac{|s|}{R_n}\right)^{N-2} + \sum_{i=1}^{l} c_{in} \int_{C_n} (\nabla Z_{in} \cdot \nabla Z_{jn}) \cdot \left(\frac{|s|}{R_n}\right)^{N-2}$$

where $w_n := \Phi(W_n)$, $Z_{jn} := \Phi(\overline{Z}_{jn})$ and $h_n := \Phi(H_n) \in \mathcal{H}$. By (3.6), $\|w_n\|_{H^1(C_n)}$ is bounded, while $\|h_n\|_{H^1(C_n)} = o(1)$. In addition, $\|w_n\|_{L^2(C_n)}$ is bounded since $\lambda_1(\Omega_{R_n})$ is bounded away from 0 for n large. Therefore

$$\int_{\operatorname{supp} Z_{jn}} \left(\nabla w_n \cdot \nabla Z_{jn} - p v_n^{p-1} w_n Z_{jn}\right) = \sum_{i=1}^{l} c_{in} \int_{C_n} (\nabla Z_{in} \cdot \nabla Z_{jn}) + o(1). \tag{3.8}$$

Furthermore, if we denote the rotation matrix corresponding to p_{jn} by $\Theta_{jn} \in O(2)$ (see (2.9)) and set $\widetilde{u}_n(s,z) = u_{R_n}(s + R_n, z)$, $\psi_n(s,z) = \psi_{R_n}(z)$ (refer to (2.4) and (2.5)) and $\widetilde{w}_n(s,z) = w_n(\Theta_{jn}(s + R_n, z))$ for $(s,z) \in \widetilde{S}_{R_n}$, then we obtain

$$\begin{aligned}&\big(\text{LHS of (3.8)}\big)\\ &= -\int_{\widetilde{S}_{R_n}} \left\{\nabla(\psi_n \widetilde{w}_n) \cdot \nabla \frac{\partial(\psi_n \widetilde{u}_n)}{\partial z} - p(\psi_n \widetilde{u}_n)^{p-1} (\psi_n \widetilde{w}_n) \frac{\partial(\psi_n \widetilde{u}_n)}{\partial z}\right\} + o(1).\end{aligned}$$

However, since $\|\psi_n \widetilde{w}_n\|_{H^1(\mathbb{R}^2)}$ is bounded, it converges (up to a subsequence) to $w_0 \in H_0^1(S)$ weakly in $H^1(\mathbb{R}^2)$ and it follows that

$$\big(\text{LHS of (3.8)}\big) = -\int_S \left[\nabla w_0 \cdot \nabla \frac{\partial u_0}{\partial z} - p u_0^{p-1} w_0 \frac{\partial u_0}{\partial z}\right] + o(1) = o(1).$$

On the other hand, we have

$$\int_{C_n} (\nabla Z_{in} \cdot \nabla Z_{jn}) = \delta_{ij} \int_S \left|\nabla \frac{\partial u_0}{\partial z}\right|^2 + o(1) \tag{3.9}$$

where δ_{ij} is the Kronecker delta. Thus $c_{1n}, \dots, c_{ln} = o(1)$ and the first claim is proved.

Let $\varphi_n = w_n - h_n - \sum_{i=1}^{l} c_{in} Z_{in} \in \mathcal{H}$. By (3.6) and Step 1 we deduce that $\|\varphi_n\|_{\mathcal{H}} = R_n^{\frac{N-2}{2}}(1 + o(1))$. Moreover, $\Phi^{-1}(\varphi_n)$ solves

$$\Delta \Phi^{-1}(\varphi_n) + p V_n^{p-1} \Phi^{-1}(\varphi_n) = -p V_n^{p-1} \left(H_n + \sum_{i=1}^{l} c_{in} \overline{Z}_{in}\right) \quad \text{in } \Omega_{R_n}. \tag{3.10}$$

Step 2. We claim that

$$\lim_{n \to \infty} \int_{C_n} p v_n^{p-1} \varphi_n^2 = 1. \tag{3.11}$$

In fact, if we multiply both sides of (3.10) by $\Phi^{-1}(\varphi_n)/R_n^{N-2}$ and integrate over Ω_{R_n}, we get

$$o(1) = \int_{A_{R_n}} |\nabla \varphi_n|^2 \left(\frac{|s|}{R_n}\right)^{N-2} - \int_{C_n} p v_n^{p-1} \varphi_n^2 \left(\frac{|s|}{R_n}\right)^{N-2}$$
$$= \frac{1}{R_n^{N-2}} \cdot \|\varphi_n\|_{\mathcal{H}}^2 - \int_{C_n} p v_n^{p-1} \varphi_n^2 + o(1) = 1 - \int_{C_n} p v_n^{p-1} \varphi_n^2 + o(1)$$

and the claim follows.

Step 3. We claim that

$$\lim_{n\to\infty} \int_{C_n} p v_n^{p-1} \varphi_n^2 = 0. \tag{3.12}$$

This contradicts (3.11) and the proof of lemma is completed.

For any $j = 1, \ldots, l$, we set

$$\widetilde{\varphi}_{jn}(s,z) = \begin{cases} \psi_n(s+R_n,z)\varphi_n(\Theta_{jn}(s+R_n,z)) & \text{if } s > -R_n^{-\alpha}, \\ 0 & \text{otherwise} \end{cases}$$

where ψ_n is the function defined in the previous step. It is clear that $\widetilde{\varphi}_{jn} \in H_0^1(\widehat{S}_n) \subset H^1(\mathbb{R}^2)$ where $\widehat{S}_n = (-R_n^{-\alpha}, 1) \times (-R_n^{\frac{1-\alpha}{2}}, R_n^{\frac{1-\alpha}{2}})$.

First we show that (up to a subsequence)

$$\widetilde{\varphi}_{jn} \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbb{R}^2). \tag{3.13}$$

Indeed, (up to a subsequence) $\widetilde{\varphi}_{jn} \rightharpoonup \phi_j$ weakly in $H^1(\mathbb{R}^2)$ for some $\phi_j \in H_0^1(S)$ and $\widetilde{\varphi}_{jn}$ solves

$$\begin{aligned} &\Delta\widetilde{\varphi}_{jn} + \frac{N-2}{s+R_n}\frac{\partial\widetilde{\varphi}_{jn}}{\partial s} + p(\psi_n\widetilde{u}_n)^{p-1}\widetilde{\varphi}_{jn} \\ &= \psi_n\left[p v_n^{p-1}\left(h_n + \sum_{i=1}^{l} c_{in} Z_{in}\right)\big(\Theta_j(s+R_n,z)\big)\right] \\ &\quad + \psi_n \cdot \frac{N-2}{s+R_n} \cdot \left[\left\{(\cos\theta_j - 1)\frac{\partial\varphi_n}{\partial s} + \sin\theta_j \frac{\partial\varphi_n}{\partial z}\right\}\big(\Theta_j(s+R_n,z)\big)\right] \\ &\quad + \left[(\Delta\psi_n)\cdot\varphi_n\big(\Theta_j(s+R_n,z)\big) + 2\nabla\psi_n\cdot\nabla\big\{\varphi_n\big(\Theta_j(s+R_n,z)\big)\big\}\right] \\ &\quad + p\left[\left\{(\psi_n\widetilde{u}_n) + \sum_{i\neq j}(\psi_n u_{R_n})\big(\Theta_{in}^{-1}\big(\Theta_{jn}(s+R_n,z)\big)\big)\right\}^{p-1} - (\psi_n\widetilde{u}_n)^{p-1}\right]\widetilde{\varphi}_{jn} \end{aligned}$$

for $(s,z) \in \widehat{S}_n$. Therefore, by letting $n \to \infty$, we find

$$\Delta\phi_j + pu_0^{p-1}\phi_j = 0 \quad \text{in } S \quad \text{and} \quad \phi_j = 0 \quad \text{on } \partial S. \tag{3.14}$$

On the other hand, by the orthogonality assumption made in the statement of the proposition, it holds that

$$0 = \int_{C_n} \nabla Z_{jn} \cdot \nabla\left(\varphi_n + h_n + \sum_{i=1}^{l} c_{in} Z_{in}\right)\left(\frac{|s|}{R_n}\right)^{N-2} = \int_{C_n} \nabla Z_{jn} \cdot \nabla\varphi_n + o(1)$$
$$= \int_{\widehat{S}_n} \nabla \frac{\partial(\psi_n \widetilde{u}_n)}{\partial z} \cdot \nabla\widetilde{\varphi}_{jn} + o(1)$$

for $j = 1, \dots, l$. Therefore

$$\int_S \nabla \frac{\partial u_0}{\partial z} \cdot \nabla\phi_j = 0. \tag{3.15}$$

Now, from (3.14), (3.15) and Lemma 2.1, we deduce $\phi_j = 0$ and (3.13) is proved.

Finally, we deduce from (3.13) that

$$\int_{C_n} pv_n^{p-1}\varphi_n^2 \leqslant Cp \sum_{i=1}^{l} \int_{C_n} (\psi_n u_{R_n})^{p-1}\left(\Theta_{in}^{-1}(s,z)\right)\varphi_n^2$$
$$= Cp \sum_{i=1}^{l} \int_{\widetilde{S}_{R_n}} (\psi_n \widetilde{u}_n)^{p-1}\widetilde{\varphi}_{jn}^2 + o(1) = o(1)$$

for some $C > 0$ and so (3.12) follows. □

Proof of Proposition 3.1. Estimate (3.5) is proved in Lemma 3.2. Let us prove solvability of problem (3.4). Set

$$\mathcal{K}_R = \mathcal{K}_{R,P} = \left\{\sum_{i=1}^{l} c_i \overline{Z}_i\colon c_1, \dots, c_l \in \mathbb{R}\right\}$$

and

$$\mathcal{K}_R^\perp = \mathcal{K}_{R,P}^\perp = \left\{W \in \widetilde{\mathcal{H}}\colon \int_{\Omega_R} \nabla\overline{Z}_i \cdot \nabla W = 0 \text{ for any } i = 1, \dots, l\right\}$$

for $P \in \Lambda_R$. Also, let $\Pi_R = \Pi_{R,P} : \widetilde{\mathcal{H}} \to \mathcal{K}_R$ be the projection map given as $\Pi_R(W) = \sum_{i=1}^{l} c_i \overline{Z}_i$ for $W \in \widetilde{\mathcal{H}}$ where c_i $(i = 1, \dots, l)$ is determined by the system

$$\sum_{i=1}^{l} c_i \int_{C_R} \nabla Z_i \cdot \nabla Z_j \left(\frac{|s|}{R}\right)^{N-2} = \int_{C_R} \nabla\Phi(W) \cdot \nabla Z_j \left(\frac{|s|}{R}\right)^{N-2} \quad \text{for } j = 1, \dots, l.$$

(By (3.9), c_i's are well-defined.) If we denote $\Pi_R^\perp = Id_{\widetilde{\mathcal{H}}} - \Pi_R : \widetilde{\mathcal{H}} \to \mathcal{K}_R^\perp$, problem (3.4) can be rewritten as

$$(Id_{\mathcal{K}_R^\perp} - K)W = \Pi_R^\perp(H)$$

where $K(W) := -\Pi_R^\perp \Delta^{-1}(pV^{p-1}W)$ is a compact operator in $\mathcal{K}_R^\perp$. By Lemma 3.2, $Id_{\mathcal{K}_R^\perp} - K$ is an injective operator on $\mathcal{K}_R^\perp$. Consequently, from the Fredholm alternative, we can conclude that $Id_{\mathcal{K}_R^\perp} - K$ is also surjective in $\mathcal{K}_R^\perp$, which implies the unique solvability of problem (3.4). That concludes the proof. □

4. The nonlinear problem

In this section, we will solve the following auxiliary problem for the function W and the parameters $(c_1, \ldots, c_l)$ (see Proposition 4.3)

$$\begin{cases} L(W) = -\big(E + N(W)\big) + \sum_{i=1}^{l} c_i \Delta \overline{Z}_i & \text{in } \Omega_R, \\ w = 0 & \text{on } \partial\Omega_R, \\ \langle \overline{Z}_i, W \rangle_{H^1(\Omega_R)} = 0 & \text{for } i = 1, \ldots, l. \end{cases} \tag{4.1}$$

Here Ω_R is the annulus in $\mathbb{R}^N$ defined in (1.2), L is the linear operator defined in (2.14), E is the error term defined in (2.15), $N(W)$ is the nonlinear term defined in (2.16) and $\overline{Z}_i$'s are the functions defined in (3.3).

Let us rewrite problem (4.1) in an equivalent way. First, for any $U \in \widetilde{\mathcal{H}}$, let $H = Q(U) \in \widetilde{\mathcal{H}}$ be the unique solution of

$$\Delta H = U \quad \text{in } \Omega_R, \qquad H = 0 \quad \text{on } \partial\Omega_R.$$

Next, let $J_R : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ be the operator defined by $J_R(H) = W$ where W is the unique solution of problem (3.4). Notice that the existence of W is established in Proposition 3.1 and J_R is linear. Then problem (4.1) can be rewritten as $W = -J_R(Q(E + N(W)))$ and it reduces to finding a fixed point of the operator $T_R : \mathcal{H} \to \mathcal{H}$ defined by

$$T_R(W) := -J_R\big(Q\big(E + N(W)\big)\big).$$

In the following, we are going to prove that $T_R : \mathcal{F}_R \to \mathcal{F}_R$ is a contraction mapping where

$$\mathcal{F}_R = \Big\{w \in \widetilde{\mathcal{H}}:\ \|W\|_{H^1(\Omega_R)} \leqslant C_2 R^{\frac{N-2}{2} - \frac{\pi}{2}(1+\epsilon_2)M_1}, \\ \|W\|_{L^\infty(\Omega_R)} \leqslant C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\Big\}. \tag{4.2}$$

Here $\epsilon_2 \in (0, \min\{(p-1)/2, 1\})$, $M_1 > 0$ is the small constant used in the definition of (2.8) (actually not yet fixed) and $C_2, C_3 > 0$ will be determined later. We remark that the choice of the set $\mathcal{F}_R$ is motivated from the work of Ambrosetti, Malchiodi and Ni [2].

First of all, it is necessary to estimate the error term $Q(E)$ and the nonlinear term $Q(N(W))$. This is done in the next two lemmas.

Lemma 4.1. *Suppose that $M_1 > 0$ is small enough. Then there exists a constant $C_4 > 0$ such that*

$$\|Q(E)\|_{H^1(\Omega_R)} \leqslant C_4 R^{\frac{N-2}{2}-\frac{\pi}{2}(1+\epsilon_2)M_1}.$$

Proof. We remark that

$$\begin{aligned}\|Q(E)\|_{H^1(\Omega_R)} &= \sup\left\{\int_{\Omega_R}(\nabla Q(E)\cdot\nabla f)\colon f\in C_c^\infty(\Omega_R),\ \|f\|_{H^1(\Omega_R)}\leqslant 1\right\}\\ &\leqslant \sup\left\{\int_{\Omega_R} Ef\colon f\in C_c^\infty(\Omega_R),\ \|f\|_{L^2(\Omega_R)}\leqslant \lambda_1(\Omega_R)^{-\frac{1}{2}}\right\}\\ &\leqslant \lambda_1(\Omega_R)^{-\frac{1}{2}}\|E\|_{L^2(\Omega_R)}\leqslant C\left(\int_{A_R}\left|\Phi(E)\right|^2|s|^{N-2}\right)^{\frac{1}{2}}\end{aligned}$$

for some $C > 0$ independent of R, so it is enough to estimate the weighted L^2-norm of $\Phi(E)$. By (2.11), Lemma 2.2 and Lemma 6.2, we get

$$\begin{aligned}\int_{A_R}\left|\Phi(E)\right|^2|s|^{N-2} &\leqslant CR^{N-2}\left(\int_{C_R}\left|\Delta v+v^p\right|^2+\frac{1}{R^2}\int_{C_R}\left|\frac{\partial v}{\partial s}\right|^2\right)\\ &\leqslant CR^{N-2}\left(\int_{C_R}\left|\left(\sum_{i=1}^{l}v_i\right)^p-\sum_{i=1}^{l}v_i^p\right|^2+R^{-2}\right)\\ &\leqslant CR^{N-2}\left(\sum_{i\neq j}\int_{C_R}|v_iv_j|^{\min\{p,2\}}+R^{-2}\right)\\ &\leqslant CR^{N-2}\left(R^{-\pi(1+\epsilon_2)M_1}+R^{-2}\right)\end{aligned}\tag{4.3}$$

where $v_i(s,z)=(\psi_R u_R)(\Theta_i^{-1}(s,z))$ and $\epsilon_2\in(0,\min\{(p-1)/2,1\})$. (At this stage, we may pick $\epsilon_2\in(0,\min\{p-1,1\})$, but for Lemma 6.5 it is necessary to choose it smaller.) Therefore, if $M_1 > 0$ is small,

$$\|Q(E)\|_{H^1(\Omega_R)}\leqslant C\left(\int_{A_R}\left|\Phi(E)\right|^2|s|^{N-2}\right)^{\frac{1}{2}}\leqslant C_4R^{\frac{N-2}{2}-\frac{\pi}{2}(1+\epsilon_2)M_1}$$

for some $C_4 > 0$. This concludes the proof. □

Lemma 4.2. *There is a constant $C_5 > 0$ such that for any $w \in \mathcal{F}_R$*

$$\|Q(N(W))\|_{H^1(\Omega_R)}\leqslant C_5\left(C_3R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\right)^{\min\{1,p-1\}}\|W\|_{H^1(\Omega_R)}.$$

Proof. We remark that we can write $N(W) = \int_0^1 p[(V + tW)_+^{p-1} - V^{p-1}]W\,dt$. Then we have $|N(W)| \leqslant C(|W| + |W|^{p-1})W$, so

$$\begin{aligned}\int_{\Omega_R} |N(W)|^2 &\leqslant C\big(C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\big)^{2\min\{1,p-1\}} \|W\|^2_{L^2(\Omega_R)} \\ &\leqslant C\big(C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\big)^{2\min\{1,p-1\}} \|W\|^2_{H^1(\Omega_R)} \qquad (4.4)\end{aligned}$$

and arguing as in the previous lemma, we get

$$\big\|Q\big(N(W)\big)\big\|_{H^1(\Omega_R)} \leqslant C\bigg(\int_{\Omega_R} |N(W)|^2\bigg)^{\frac{1}{2}}.$$

This completes the proof. □

Now, we can deduce the unique solvability of Eq. (4.1).

Proposition 4.3. *For $R > 0$ sufficiently large and $P \in \Lambda_R$, there is a unique solution $W_{R,P} \in \widetilde{\mathcal{H}}$ and $\{(c_1)_{R,P}, \dots, (c_l)_{R,P}\}$ for problem (4.1) such that*

$$\|W_{R,P}\|_{H^1(\Omega_R)} \leqslant C_2 R^{\frac{N-2}{2} - \frac{\pi}{2}(1+\epsilon_2)M_1}, \qquad \|W_{R,P}\|_{L^\infty(\Omega_R)} \leqslant C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}. \qquad (4.5)$$

Proof. The proof proceeds in two steps.

Step 1. We first show that T_R maps $\mathcal{F}_R$ to itself. From (3.5), Lemmas 4.1 and 4.2, we obtain

$$\big\|T_R(W)\big\|_{H^1(\Omega_R)} \leqslant C_1\big(C_4 + C_2 C_5\big(C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\big)^{\min\{1,p-1\}}\big) R^{\frac{N-2}{2} - \frac{\pi}{2}(1+\epsilon_2)M_1}.$$

Therefore, if we take $C_2 = 2C_1C_4$, then

$$\big\|T_R(W)\big\|_{H^1(\Omega_R)} \leqslant C_2 R^{\frac{N-2}{2} - \frac{\pi}{2}(1+\epsilon_2)M_1} \qquad (4.6)$$

once C_3 is appropriately chosen and R is taken sufficiently large according to the magnitude of C_3.

Let us estimate the L^∞-norm of $T_R(W) \in \widetilde{\mathcal{H}}$. For simplicity, we write $w_T = (w_T)_R := \Phi(T_R(W))$. Then

$$\begin{aligned}&-\Delta w_T - \frac{N-2}{s}\frac{\partial w_T}{\partial s} \\ &\quad = p v^{p-1} w_T + \big\{\Phi(E) + \Phi\big(N(W)\big)\big\} - \sum_{i=1}^{l} c_i \bigg(\Delta Z_i + \frac{N-2}{s}\frac{\partial Z_i}{\partial s}\bigg) \qquad (4.7)\end{aligned}$$

in $A_R \setminus \{s = 0\}$. By elliptic regularity [13, Theorems 9.20, 9.26], we have for any $(s,z) \in C_R$ (see (2.11)),

$$\|w_T\|_{L^\infty(A_{R,1})} \leqslant C_{61}\big(\|w_T\|_{L^2(A_{R,2})} + \|g\|_{L^2(A_{R,2})}\big) \tag{4.8}$$

where $C_{61} > 0$, g denotes the right-hand side of (4.7) and

$$A_{R,1} = A_R \cap B\big((s,z); r\big) \quad \text{and} \quad A_{R,2} = A_R \cap B\big((s,z); 2r\big)$$

for some $r > 0$ sufficiently small.

Because of (4.6), it holds that

$$\|w_T\|_{L^2(A_{R,2})} \leqslant C_{62} R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \tag{4.9}$$

for some $C_{62} > 0$. Besides, applying (4.3) and (4.4), we can get

$$\begin{aligned}
\|g\|_{L^2(A_{R,2})} &\leqslant C_{63}\|w_T\|_{L^2(A_{R,2})} + \big\|\Phi(E)\big\|_{L^2(A_{R,2})} + \big\|\Phi\big(N(w)\big)\big\|_{L^2(A_{R,2})} + C_{63}\sum_{i=1}^{l}|c_i| \\
&\leqslant C_{62}C_{63}R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} + C_{64}R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} + o\big(R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}\big) + C_{63}\sum_{i=1}^{l}|c_i|
\end{aligned}$$

for some $C_{63}, C_{64} > 0$. However, arguing as in Step 1 of the proof of Lemma 3.2, we estimate c_i's as $|c_i| \leqslant C_{65}R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}$ for some constant $C_{65} > 0$ and $i = 1, \ldots, l$. As a result, we have

$$\|g\|_{L^2(A_{R,2})} \leqslant 2(C_{62}C_{63} + C_{64} + C_{63}C_{65}l)R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we get

$$\begin{aligned}
\|w_T\|_{L^\infty(C_R)} &\leqslant 2C_{61}(C_{62} + C_{62}C_{63} + C_{64} + C_{63}C_{65}l)R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \\
&=: C_6 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}.
\end{aligned} \tag{4.11}$$

Now, we define

$$\Xi_R = \left\{(x_1, \ldots, x_{n-1}, z) \in \Omega_R \colon \sqrt{\sum_{i=1}^{n-1} x_i^2} < \left(1 - \frac{2}{R^{\pi(1+\epsilon_1)M_1}}\right)R\right\}.$$

Then by (4.7), using the fact that v and Z_i's vanish in the corresponding two dimensional set $\{(s,z) \in A_R \colon s < (1 - 2R^{-\pi(1+\epsilon_1)M_1})R\}$, we get

$$-\Delta W_{T,R} = W_+^p \quad \text{in } \Xi_R$$

where $W_{T,R} := T_R(W)$.

If we let

$$\Psi_R(x_1, \ldots, x_n) = C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \cos\frac{\pi}{2}\left(r - R - \frac{1}{2}\right) \quad \text{where } r = \sqrt{\sum_{i=1}^{n} x_i^2},$$

with $C_3 = \sqrt{2}C_6$, then we have

$$-\Delta\big((\pm W_{T,R}) - \Psi_R\big) \leqslant C_3^p R^{-\frac{\pi}{2}(1+\epsilon_2)M_1 p} - \left(\frac{\pi}{2}\right)^2 C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \cos\frac{\pi}{2}\left(r - R - \frac{1}{2}\right)$$
$$- \frac{(N-1)\pi}{2r} C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \sin\frac{\pi}{2}\left(r - R - \frac{1}{2}\right)$$
$$\leqslant -\frac{\pi^2}{8} C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \leqslant 0 \quad \text{in } \Xi_R$$

and by (4.11)

$$\pm W_{T,R} - \Psi_R \leqslant \left(C_6 - \frac{C_3}{\sqrt{2}}\right) R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} = 0 \quad \text{on } \partial\Xi_R.$$

Therefore the maximum principle gives

$$|W_{T,R}| \leqslant \Psi_R \leqslant C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1} \quad \text{in } \Xi_R. \tag{4.12}$$

Finally, (4.11) and (4.12) imply $\|w_T\|_{L^\infty(\Omega_R)} \leqslant C_3 R^{-\frac{\pi}{2}(1+\epsilon_2)M_1}$.

Step 2. We prove that T_R is a contraction mapping.
Indeed, arguing as in the proof of Lemma 4.2, we get

$$\big\|T_R(w_1) - T_R(w_2)\big\|_{H^1(\Omega_R)} \leqslant O\big(R^{-\frac{\pi}{2}(1+\epsilon_2)M_1\cdot\min\{1,p-1\}}\big)\|w_1 - w_2\|_{H^1(\Omega_R)}$$

for any $w_1, w_2 \in \mathcal{F}_R$.

The claim follows by the contraction mapping principle. □

5. The finite dimensional reduction

For each fixed $R > 0$ large, we introduce the reduced energy

$$F_R(P) := I_R\big(v_{R,P} + \Phi(W_{R,P})\big), \quad P \in \Lambda_R \tag{5.1}$$

where $v_{R,P}$ is defined in (2.10) and $W_{R,P}$ is given in Proposition 4.3. The following result shows that critical points of the reduced energy F_R give rise to critical points of the original functional I_R or equivalently to solutions of problem (1.1).

Proposition 5.1. *$F_R : \Lambda_R \to \mathbb{R}$ is of class C^1. Furthermore, if $F'_R(P) = 0$, i.e. $d(F_R)_P\tau = 0$ for all $\tau \in T_P\Lambda_R$, then $(c_1)_{R,P} = \cdots = (c_l)_{R,P} = 0$ and in particular $V_{R,P} + W_{R,P}$ is a solution of* (1.1) *($V_{R,P}$ is defined in* (2.12)*).*

Proof. Denote $V_P = V_{R,P}$, $W_P = W_{R,P}$, $c_{i,P} = (c_i)_{R,P}$ and $\overline{Z}_{i,P} = (\overline{Z}_i)_{R,P}$ for $i = 1, \dots, l$. To prove $F_R \in C^1(\Lambda_R)$, it is enough to check that the map $P \in \Lambda_R \mapsto W_P \in \widetilde{\mathcal{H}}$ is in C^1. Define a map $G_R : \Lambda_R \times \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ by

$$G_R(P, U) = U + \Pi^{\perp}_{R,P}\big(V_P + \Delta^{-1}(V_P + U)^p_+\big)$$

where $\Pi^{\perp}_{R,P} : \widetilde{\mathcal{H}} \to \mathcal{K}^{\perp}_{R,P}$ is the projection map (see the proof of Proposition 3.1). Clearly $G_R(P, W_P) = 0$ and by the Fredholm alternative the operator

$$\frac{\partial G_R}{\partial U}(P, W_P) = Id_{\widetilde{\mathcal{H}}} + p\Pi^{\perp}_{R,P}\Delta^{-1}(V_P + W_P)^{p-1}_+$$

is a Fredholm operator of index 0. Thus, to deduce that it is invertible, it is enough to check the injectivity of $\frac{\partial G_R}{\partial U}(P, W_P)$. Suppose that $f \in \widetilde{\mathcal{H}}$ is contained in its kernel. Then there is $(c_1, \dots, c_l) \in \mathbb{R}^l$ such that

$$\Delta f + pV_P^{p-1} f = p\big[V_P^{p-1} - (V_P + W_P)^{p-1}\big] f + \sum_{i=1}^{l} c_i \Delta \overline{Z}_{i,P} \quad \text{in } \Omega_R.$$

By Proposition 3.1, it follows that

$$\|f\|_{H^1(\Omega_R)} \leqslant C\|W_P\|^{\min\{1,p-1\}}_{L^\infty(\Omega_R)}\|f\|_{H^1(\Omega_R)} \leqslant CR^{-\frac{\pi}{2}(1+\epsilon_2)M_1\cdot\min\{1,p-1\}}\|f\|.$$

Hence $f = 0$ given R sufficiently large. From the implicit function theorem, we conclude that the map $P \in \Lambda_R \mapsto W_P \in \widetilde{\mathcal{H}}$ is indeed in C^1.

Now suppose that $F'_R(P) = 0$. Let us write the ith component of P as $p_i = (R + 1/2) \times (\cos\theta_i, \sin\theta_i)$ and let $\tau_i := (0, \dots, (-\sin\theta_i, \cos\theta_i), \dots, 0) \in T_P\Lambda_R$ and $\partial_i = \frac{\partial}{\partial \tau_i}$ (see the first paragraph of Section 3). Denote also $w_P := \Phi(W_P)$. Then we have, for any $j = 1, \dots, l$,

$$\begin{aligned}
0 &= d(F_R)_P\tau_j = I'_R(v_P + w_P)(\partial_j v_P + \partial_j w_P) \\
&= \sum_{i=1}^{l} c_{i,P}\bigg[\int_{C_R} \nabla\partial_j v_P \cdot \nabla Z_{i,P}\bigg(\frac{|s|}{R}\bigg)^{N-2} - \int_{C_R} \nabla w_P \cdot \nabla\partial_j Z_{i,P}\bigg(\frac{|s|}{R}\bigg)^{N-2}\bigg] \\
&= c_{j,P}\bigg(\int_S \bigg|\nabla\frac{\partial u_0}{\partial z}\bigg|^2 + o(1)\bigg) + \sum_{i\neq j} c_{i,P}\cdot o(1)
\end{aligned}$$

where the third equality comes from $W_P \in \mathcal{K}^{\perp}_{R,P}$, while the fourth one is due to (3.1), (3.9) and (4.5). This implies $c_{1,P} = \dots = c_{l,P} = 0$. $\square$

6. The reduced energy

This section is devoted to study the reduced energy F_R defined in (5.1). More precisely, our aim is to deduce that F_R has a critical point in the interior of Λ_R.

We first prove that the contribution of the term $w_{R,P} = \Phi(W_{R,P})$ to the energy functional I_R is quite small.

Lemma 6.1. *The following expansion holds*:

$$F_R(P) = I_R(v_{R,P}) + O\big(R^{N-2-\pi(1+\epsilon_2)M_1}\big) \quad \textit{for any } P \in \Lambda_R.$$

Proof. Using Taylor's theorem and the fact that $I'_R(v_{R,P} + w_{R,P})w_{R,P} = 0$, we get

$$F_R(P) - I_R(v_{R,P}) = -\int_0^1 t I''_R(v_{R,P} + t w_{R,P}) w_{R,P}^2 \, dt.$$

On the other hand, we have from (4.5) that

$$I''_R(v_{R,P} + t w_{R,P}) w_{R,P}^2 = \int_{A_R} |\nabla w_{R,P}|^2 |s|^{N-2} - \int_{A_R} p(v_{R,P} + t w_{R,P})_+^{p-1} w_{R,P}^2 |s|^{N-2}$$
$$= O\big(R^{N-2-\pi(1+\epsilon_2)M_1}\big).$$

Therefore the proof is finished. □

Given $P = (p_1, \ldots, p_l) \in \Lambda_R$, let Θ_i be the corresponding orthogonal matrix to $p_i = (R + 1/2)(\cos\theta_i, \sin\theta_i)$. Also, denote

$$v_{i,P}(s,z) = v_i(s,z) = (\psi_R u_R)\big(\Theta_i^{-1}(s,z)\big) \quad \text{for } i = 1, \ldots, l \tag{6.1}$$

as before. We give upper and lower estimates of interaction terms $\int_{A_R} (v_i v_j)^q$ or $\int_{A_R} v_i^p v_j$ for $p > 1$, $q > 0$ and $i \neq j$.

Lemma 6.2. *For any $q > 0$ and small $\epsilon > 0$, there exists a constant $C = C(q, \epsilon) > 0$ such that*

$$\int_{A_R} (v_i v_j)^q \leqslant C R^{-q(1-\epsilon)\pi M_1}$$

if R is sufficiently large.

Proof. We may assume $p_i = (R + 1/2, 0)$ and $p_j = (R + 1/2)(\cos\theta_0, \sin\theta_0)$. Let L_0 be the s-axis and L_1 the line through the origin and p_j. Since $\theta_0(1 + o(1)) \geqslant M_1 \log R/(R + 1/2)$, by (2.11), (2.3) and (2.4), we have

$$\int_{C_R} (v_i v_j)^q \leqslant \int_R^{R+1} \int_0^{\theta_0} e^{-q(1-\epsilon)\pi \cdot \mathrm{dist}((r,\theta), L_0)} e^{-q(1-\epsilon)\pi \cdot \mathrm{dist}((r,\theta), L_1)} r \, d\theta \, dr + (\mathrm{S})$$
$$= \int_R^{R+1} \int_0^{\theta_0} e^{-q(1-\epsilon)\pi r[\sin\theta + \sin(\theta_0 - \theta)]} r \, d\theta \, dr + (\mathrm{S})$$

$$\leqslant \int_R^{R+1}\int_0^{\theta_0} e^{-q(1-\epsilon)\pi r[(1-\epsilon)\theta+(1-2\epsilon)(\theta_0-\theta)]} r\,d\theta\,dr + (\mathrm{S})$$

$$\leqslant \int_R^{R+1} e^{-q(1-3\epsilon)\pi r\theta_0}\left[\int_0^{\theta_0} e^{-q(1-\epsilon)\epsilon\pi r\theta}\,d\theta\right] r\,dr + (\mathrm{S})$$

$$\leqslant \frac{2}{q\epsilon\pi}\int_R^{R+1} e^{-q\pi(1-3\epsilon)\theta_0 r}\,dr + (\mathrm{S})$$

$$\leqslant \frac{2}{q\epsilon\pi} e^{-q\pi(1-3\epsilon)\theta_0 R} + (\mathrm{S}) \leqslant \frac{2}{q\epsilon\pi} R^{-q(1-4\epsilon)\pi M_1} + (\mathrm{S})$$

where (S) = (a smaller term). Hence the result follows. □

Lemma 6.3. *For any $\epsilon > 0$ and $p > 1$, if there is a pair of indices (i_0, j_0) such that $i_0 \neq j_0$ and $|p_{i_0} - p_{j_0}| = M_1 \log R$, then we have a constant $C = C(p,\epsilon) > 0$ independent of (i_0, j_0) and $P \in \Lambda_R$ such that*

$$\int_{A_R} v_{i_0}^p v_{j_0} \geqslant C R^{-\pi(1+\epsilon)M_1}$$

provided R is large.

Proof. Assume $p_i = (R + 1/2, 0)$ and $p_j = (R + 1/2)(\cos\theta_0, \sin\theta_0)$ as in the previous proof. By our assumption, $\theta_0(1 + o(1)) = M_1 \log R/(R + 1/2)$. Therefore, we obtain from (2.3) and (2.4) that

$$\int_{A_R} v_{i_0}^p v_{j_0} = \int_{S_R} v_{i_0}^p v_{j_0}$$

$$\geqslant \int_0^1 \int_{R+\frac{1}{4}}^{R+\frac{3}{4}} u_R^p(s,z)\cdot u_R(\cos\theta_0 s + \sin\theta_0 z, -\sin\theta_0 s + \cos\theta_0 z)\,ds\,dz$$

$$\geqslant \frac{C}{(1-2R^{-\alpha})^{\frac{2(p+1)}{p-1}}} \int_0^1 \int_{R+\frac{1}{4}}^{R+\frac{3}{4}} \exp\left(\frac{-p\pi z}{1-2R^{-\alpha}}\right)\exp\left(\frac{\pi(-\sin\theta_0 s + \cos\theta_0 z)}{1-2R^{-\alpha}}\right) ds\,dz$$

$$\geqslant C\left[\int_0^1 \exp\left(\frac{-\pi z(p-\cos\theta_0)}{1-2R^{-\alpha}}\right) dz\right]\cdot\left[\int_{R+\frac{1}{4}}^{R+\frac{3}{4}} \exp\left(\frac{-\pi\sin\theta_0 s}{1-2R^{-\alpha}}\right) ds\right]$$

$$\geqslant C\cdot R^{-\pi(1+\epsilon)M_1}$$

for some constant $C > 0$. □

By applying the previous interaction estimates, we can now prove that F_R has a maximum in the interior of the admissible set Λ_R. To achieve this, we will estimate $F_R(P)$ for any fixed point $P=(p_1,\dots,p_l)$ lying on the boundary of Λ_R and deduce that it is strictly smaller than the value of F_R at a point in the interior of Λ_R.

Note that two possibilities exist if $P\in\partial\Lambda_R$: either

$$|p_{i_0}-p_{j_0}|=M_1\log R \quad \text{for some } i_0\neq j_0, \tag{6.2}$$

or

$$s_{i_0}=\left(R+\frac{1}{2}\right)-\frac{R}{R^{\pi(1+\epsilon_1)M_1}} \quad \text{for some } i_0. \tag{6.3}$$

Assume first that the case (6.2) happens. If we choose $\epsilon_1\in(0,\epsilon_2)$ arbitrarily, we get then

Lemma 6.4. *Fix $P=(p_1,\dots,p_l)\in\partial\Lambda_R$. If $|p_{i_0}-p_{j_0}|=M_1\log R$ for some $i_0\neq j_0$, then*

$$F_R(P)\leqslant 2R^{N-2}\left[l\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_S|\nabla u_0|^2-CR^{-\pi(1+\epsilon_3)M_1}+O\left(R^{-\pi(1+\epsilon_1)M_1}\right)\right]$$

for some $C>0$ and $0<\epsilon_3<\epsilon_1$.

For the proof of Lemma 6.4, the following expansion for F_R is necessary.

Lemma 6.5. *For sufficiently small $M_1>0$, it holds that*

$$F_R(P)=2R^{N-2}\left[l\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_S|\nabla u_0|^2-\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}\int_{C_R}v_{i,P}^p v_{j,P}\right.$$
$$\left.-\sum_{i=2}^{l}\sum_{j=1}^{i-1}C_{ij}(P)+O\left(R^{-\pi(1+\epsilon_1)M_1}\right)\right]$$

for any $P\in\Lambda_R$. Here, the functions $v_{i,P}$ and $v_{j,P}$ are defined in (6.1) and $P\mapsto C_{ij}(P)$ are positive maps in Λ_R whose definition is given in the proof.

Proof. By Lemma 2.2 and (2.11),

$$\frac{1}{2R^{N-2}}\cdot I_R(v_{R,P})$$
$$=\left(\frac{1}{2}\sum_{i=1}^{l}\int_{C_R}|\nabla v_i|^2-\frac{1}{p+1}\sum_{i=1}^{l}\int_{C_R}v_i^{p+1}\right)+\left(\sum_{i>j}\int_{C_R}\nabla v_i\cdot\nabla v_j-\sum_{i\neq j}\int_{C_R}v_i^p v_j\right)$$
$$+\sum_{i\neq j}O\left(\int_{C_R}(v_iv_j)^{\min\{\frac{p+1}{2},2\}}\right)+O\left(R^{-\pi(1+\epsilon_1)M_1}\right).$$

Since $\epsilon_2\in(\epsilon_1,\min\{(p-1)/2,1\})$, we can pick a small $\epsilon_4>0$ such that

$$\int_{C_R} (v_i v_j)^{\min\{\frac{p+1}{2},2\}} = O\big(R^{-\pi \min\{\frac{p+1}{2},2\}(1-\epsilon_4)M_1}\big)$$
$$= o\big(R^{-\pi(1+\epsilon_2)M_1}\big) = o\big(R^{-\pi(1+\epsilon_1)M_1}\big) \tag{6.4}$$

for $i \neq j$ (see Lemma 6.2). Also, we have

$$\frac{1}{2}\int_{C_R} |\nabla v_i|^2 - \frac{1}{p+1}\int_{C_R} v_i^{p+1} = \frac{1}{2}\int_{S_R} \big|\nabla(\psi_R u_R)\big|^2 - \frac{1}{p+1}\int_{S_R} (\psi_R u_R)^{p+1}$$
$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_S |\nabla u_0|^2 + O\big(R^{-\alpha}\big) \tag{6.5}$$

for any $i = 1, \ldots, l$, and

$$\sum_{i>j}\int_{C_R} \nabla v_i \cdot \nabla v_j - \sum_{i\neq j}\int_{C_R} v_i^p v_j = \sum_{i>j}\int_{C_R} \big(\nabla v_i \cdot \nabla v_j - v_i^p v_j\big) - \sum_{i<j}\int_{C_R} v_i^p v_j. \tag{6.6}$$

However, for $i > j$,

$$\int_{C_R} \big(\nabla v_i \cdot \nabla v_j - v_i^p v_j\big) = -\int_{\check{S}_R} \big(\Delta u_R + u_R^p\big)(s,z) \cdot u_R\big(\Theta_j^{-1}\Theta_i(s,z)\big)$$
$$+ \int_{\Gamma_1\cup\Gamma_2} \frac{\partial u_R}{\partial \nu}(s,z) \cdot u_R\big(\Theta_j^{-1}\Theta_i(s,z)\big) + O\big(e^{-\frac{\pi}{2}R^{\frac{1-\alpha}{2}}}\big)$$
$$= \left(-\int_{\Gamma_1} + \int_{\Gamma_2}\right)\frac{\partial u_R}{\partial s}(s,z) \cdot u_R\big(\Theta_j^{-1}\Theta_i(s,z)\big) + O\big(e^{-\frac{\pi}{2}R^{\frac{1-\alpha}{2}}}\big)$$
$$=: -C_{ij}(P) + O\big(e^{-\frac{\pi}{2}R^{\frac{1-\alpha}{2}}}\big) \tag{6.7}$$

where $\check{S}_R = (R + R^{-\alpha}, R + 1 - R^{-\alpha}) \times \mathbb{R}$, $\Gamma_1 := \{s = R + R^{-\alpha}\} \cap \partial(\check{S}_R \cap \Theta_i^{-1}\Theta_j(\check{S}_R))$, $\Gamma_2 := \{s = R + 1 - R^{-\alpha}\} \cap \partial(\check{S}_R \cap \Theta_i^{-1}\Theta_j(\check{S}_R))$ and ν is the unit outward vector normal to the boundary $\partial\check{S}_R$, i.e., $\nu = ((-1)^i, 0)$ on Γ_i for $i = 1, 2$. Note that by Hopf's lemma, $\frac{\partial u_R}{\partial s} > 0$ (< 0) on Γ_1 (Γ_2, respectively) and so $C_{ij}(P) > 0$. Consequently, from Lemma 6.1 and the above estimates, the lemma follows. □

Proof of Lemma 6.4. By Lemmas 6.5 and 6.3,

$$F_R(P) \leqslant 2R^{N-2}\left[l\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_S |\nabla u_0|^2 - \int_{C_R} v_{i_0}^p v_{j_0} + O\big(R^{-\pi(1+\epsilon_1)M_1}\big)\right]$$
$$\leqslant 2R^{N-2}\left[l\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_S |\nabla u_0|^2 - CR^{-\pi(1+\epsilon_3)M_1} + O\big(R^{-\pi(1+\epsilon_1)M_1}\big)\right]$$

for some $C > 0$ and $\epsilon_3 > 0$. We can select $\epsilon_3 > 0$ so small that $\epsilon_3 < \epsilon_1$. □

If (6.3) occurs, we have the following lemma.

Lemma 6.6. *Let $P = ((s_1, z_1), \dots, (s_l, z_l)) \in \Lambda_R$. If there is an index i_0 such that $s_{i_0} = (R + 1/2) - R^{1-\pi(1+\epsilon_1)M_1}$, then*

$$F_R(P) \leqslant 2R^{N-2}\left[l\left(\frac{1}{2} - \frac{1}{p+1}\right)\int\limits_S |\nabla u_0|^2 - CR^{-\pi(1+\epsilon_1)M_1} + O\left(R^{-\pi(1+\epsilon_2)M_1}\right)\right] \tag{6.8}$$

for some $C > 0$.

Proof. It is sufficient to prove that

$$I_R(v_{R,P}) \leqslant 2R^{N-2}\left[l\left(\frac{1}{2} - \frac{1}{p+1}\right)\int\limits_S |\nabla u_0|^2 - CR^{-\pi(1+\epsilon_1)M_1} + o\left(R^{-\pi(1+\epsilon_2)M_1}\right)\right] \tag{6.9}$$

since with Lemma 6.1 it implies (6.8).

Decompose

$$\begin{aligned}\frac{1}{2R^{N-2}} I_R(v_{R,P}) &= \frac{1}{2}\int\limits_{C_R} |\nabla v_{R,P}|^2 \left(\frac{s}{R}\right)^{N-2} - \frac{1}{p+1}\int\limits_{C_R} v_{R,P}^{p+1}\left(\frac{s}{R}\right)^{N-2} \\ &=: I_1 - I_2.\end{aligned} \tag{6.10}$$

We will estimate I_1 and I_2 respectively.

We keep using the notation v_i and θ_i. By Lemma 6.2, it holds that

$$\begin{aligned} I_1 &= \frac{1}{2}\sum_{i=1}^{l}\int\limits_{C_R} |\nabla v_i|^2 \left(\frac{s}{R}\right)^{N-2} + \sum_{i>j}\int\limits_{C_R} \nabla v_i \cdot \nabla v_j \left(\frac{s}{R}\right)^{N-2} \\ &= \frac{1}{2}\sum_{i=1}^{l}\int\limits_{S_R} \left|\nabla(\psi_R u_R)\right|^2 \left(\frac{\cos\theta_i s - \sin\theta_i z}{R}\right)^{N-2} \\ &\quad + \sum_{i>j}\int\limits_{C_R} \nabla v_i \cdot \nabla v_j + O\left(R^{-\pi((1+\epsilon_1)+(1-\epsilon_4))M_1}\right) \end{aligned}$$

where ϵ_4 is the small number chosen in the proof of Lemma 6.5 so that $(1+\epsilon_1) + (1-\epsilon_4) > 2(1-\epsilon_4) > 1+\epsilon_2$. Therefore

$$I_1 = \frac{1}{2}\left(\sum_{i=1}^{l} \cos^{N-2}\theta_i\right)\int\limits_S |\nabla u_0|^2 + \sum_{i>j}\int\limits_{C_R} \nabla v_i \cdot \nabla v_j + o\left(R^{-\pi(1+\epsilon_2)M_1}\right). \tag{6.11}$$

Furthermore, applying Lemma 2.2 and (6.5), we see that

$$I_2 = \frac{1}{p+1}\sum_{i=1}^{l}\int_{C_R} v_i^{p+1}\left(\frac{s}{R}\right)^{N-2} + \sum_{i\neq j}\int_{C_R} v_i^p v_j\left(\frac{s}{R}\right)^{N-2} + o\left(R^{-\pi(1+\epsilon_2)M_1}\right)$$
$$= \frac{1}{p+1}\left(\sum_{i=1}^{l}\cos^{N-2}\theta_i\right)\int_S |\nabla u_0|^2 + \sum_{i\neq j}\int_{C_R} v_i^p v_j + o\left(R^{-\pi(1+\epsilon_2)M_1}\right). \tag{6.12}$$

On the other hand, we have $\cos\theta_{i_0} = 1 - R^{1-\pi(1+\epsilon_1)M_1}(R+1/2)^{-1}$. Hence from (6.10), (6.11) and (6.12), we get

$$\frac{1}{2R^{N-2}} I_R(v_{R,P})$$
$$\leqslant l\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_S |\nabla u_0|^2 - \frac{R}{R+1/2}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_S |\nabla u_0|^2 \cdot R^{-\pi(1+\epsilon_1)M_1}$$
$$+ \left(\sum_{i>j}\int_{C_R}\nabla v_i\cdot\nabla v_j - \sum_{i\neq j}\int_{C_R} v_i^p v_j\right) + o\left(R^{-\pi(1+\epsilon_2)M_1}\right).$$

Finally, by (6.6) and (6.7),

$$\sum_{i>j}\int_{C_R}\nabla v_i\cdot\nabla v_j - \sum_{i\neq j}\int_{C_R} v_i^p v_j \leqslant O\left(e^{-\frac{\pi}{2}R^{\frac{1-\alpha}{2}}}\right).$$

Thus (6.9) holds. □

Finally, we deduce a lower energy estimate of F_R in Λ_R.

Lemma 6.7. *We have the following estimate*:

$$\max_{P\in\Lambda_R} F_R(P) \geqslant 2R^{N-2}\left[l\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_S |\nabla u_0|^2 + O\left(R^{-\pi(1+\epsilon_2)M_1}\right)\right]. \tag{6.13}$$

Proof. Assume that l is odd, that is, l is written as $l = 2l'+1$ for some $l' \in \mathbb{N}$. Let $p_0 = (R+1/2, 0)$, $p_{2i-1} = (R+1/2)(\cos 2i\theta_0, \sin 2i\theta_0)$ and $p_{2i} = (R+1/2)(\cos 2i\theta_0, -\sin 2i\theta_0)$ for $i = 1,\ldots,l'$ where $\theta_0 \in (0,\pi/2)$ is determined by the relation $\tan\theta_0 = \frac{R^{\frac{1-\alpha}{2}}}{R+R^{-\alpha}}$. Then one can check that given the orthogonal matrix Θ_i which corresponds to p_i for each $i = 1,\ldots,l$ (see (2.9)), it is satisfied that $\Theta_i(S_R)\cap\Theta_j(S_R) = \emptyset$ unless $i = j$.

Now by applying the fact that

$$\theta_0^2\left(1+o(1)\right) = \sin^2\theta_0 = \left[\left(R^{\frac{1+\alpha}{2}} + R^{-\frac{1+\alpha}{2}}\right)^2 + 1\right]^{-1},$$

we derive

$$\cos 2l'\theta_0 = 1 - 2(l')^2\theta_0^2(1+o(1)) \geqslant 1 - 4(l')^2 R^{-(1+\alpha)} \geqslant 1 - \frac{R^{-\pi(1+\epsilon_1)M_1}}{2},$$

provided M_1 small, which implies

$$\left(R+\frac{1}{2}\right)\cos 2l'\theta_0 \geqslant \left(R+\frac{1}{2}\right) - R^{1-\pi(1+\epsilon_1)M_1}.$$

Furthermore, we have

$$\min_{i\neq j}|p_i - p_j|^2 = 4\left(R+\frac{1}{2}\right)^2 \sin^2\theta_0 = 4\left(R+\frac{1}{2}\right)^2\left[\left(R^{\frac{1+\alpha}{2}} + R^{-\frac{1+\alpha}{2}}\right)^2 + 1\right]^{-1}$$
$$\geqslant 2R^{1-\alpha} \geqslant M_1^2 \log^2 R.$$

Therefore $P_0 := (p_0, p_1, p_2, \ldots, p_{2l'-1}, p_{2l'}) \in \Lambda_R$, and we obtain (6.13) from (6.5). Note that v_i and v_j have disjoint compact support and $(|s|/R)^{N-2} = O(R^{-\alpha(N-2)})$ in the support of v_{R,P_0}, whose effect on the value of $F_R(P_0)$ is negligible.

If l is even, we can use the point $P_1 := (p_1, p_2, \ldots, p_{2l'-1}, p_{2l'}) \in \Lambda_R$ to check that (6.13) holds again. □

Collecting Lemmas 6.7, 6.4 and 6.6 and reminding that we chose $0 < \epsilon_3 < \epsilon_1 < \epsilon_2$, we obtain

Proposition 6.8. *For sufficiently small $M_1 > 0$, there exists $R_0 > 0$ such that the maximum of F_R in Λ_R is attained by a point P_R in the interior of Λ_R for all $R > R_0$.*

7. Completion of the proof of Theorem 1.1

Putting together the results obtained in previous sections, we can now conclude the proof of the main theorem.

Completion of the proof of Theorem 1.1. If $R > 0$ is large enough and fixed, from Proposition 5.1 and Proposition 6.8, we get a critical point $P_R \in \Lambda_R$ of I_R which gives rise to a solution $V_{R,P_R} + W_{R,P_R} = \Phi^{-1}(v_{R,P_R} + w_{R,P_R})$ of (1.1). It is not hard to show that this solution has the properties described in the statement of the main theorem, by employing (2.10) and (4.2). Consequently, Theorem 1.1 is valid for $1 < p \leqslant 2^* - 1$.

Now, we consider the supercritical case, i.e. $p > 2^* - 1$. Since there is a constant $\rho > 0$ such that $\|v_{R,P} + \Phi(W)\|_{L^\infty(A_R)} \leqslant \rho$ for every $W \in \mathcal{F}_R$ (see (4.2)) given R sufficiently large, if we pick a function $F \in C^2(\mathbb{R})$ satisfying

$$F(u) = \begin{cases} u_+^{p+1} & \text{for } u \in (-\infty, \rho), \\ (\rho+1)^{p+1} & \text{for } u \in (\rho+1, \infty), \end{cases}$$

then a critical point of the functional

$$\widetilde{I}_R(u) = \frac{1}{2}\int_{A_R} |\nabla u(s,z)|^2 |s|^{n-2}\, ds\, dz - \frac{1}{p+1}\int_{A_R} F(u(s,z))|s|^{n-2}\, ds\, dz, \quad u \in \mathcal{H}$$

gives a solution of (1.1) substituting u^p with $F(u)$. The analogue of previous results replacing u^p with $F(u)$ remains to hold. Thus, Eq. (1.1) with $F(u)$ instead of u^p has the desired solution $U_{R,l}$. However, it is a solution of the original problem since $F(U_{R,l}) = (U_{R,l})^p$ by the property $\|U_{R,l}\|_{L^\infty(A_R)} \leqslant \rho$. This proves the remained case $p > 2^* - 1$. $\square$

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