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Functional equations and inequalities in matrix paranormed spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of the Cauchy additive functional inequality, the Cauchy additive functional equation and the quadratic functional equation in matrix paranormed spaces.

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1 Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently, and since then several generalizations and applications of this notion have been investigated by various authors (see [3–7]). This notion was defined in normed spaces by Kolk [8].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1 [9] Let X be a vector space. A paranorm $P(\cdot): X \to [0,\infty)$ is a function on X such that

(1) $P(0) = 0$;

(2) $P(-x) = P(x)$;

(3) $P(x+y) \le P(x) + P(y)$ (triangle inequality);

(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair $(X, P(\cdot))$ is called a *paranormed space* if $P(\cdot)$ is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from the question of Ulam [10] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [11] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [12] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, during the 27th International Symposium on Functional Equations, Rassias [15] asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [16], following the same approach as in Rassias [13], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Rassias and Šemrl [17], that one cannot prove a Rassias-type theorem when $p = 1$ (*cf.* the books of Czerwik [18] and Hyers *et al.* [19]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [20] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [21] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [22] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [23–27]).

In [28], Gilányi showed that if f satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f\left(xy^{-1}\right)\right\| \leq \left\|f(xy)\right\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right).$$

See also [29]. Gilányi [30] and Fechner [31] proved the Hyers-Ulam stability of functional inequality (1.1).

Park *et al.* [32] proved the Hyers-Ulam stability of the following functional inequality:

$$\left\|f(x) + f(y) + f(z)\right\| \leq \left\|f(x+y+z)\right\|. \tag{1.2}$$

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [33] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [34]).

The proof given in [33] appealed to the theory of ordered operator spaces [35]. Effros and Ruan [36] showed that one can give a purely metric proof of this important theorem by using the technique of Pisier [37] and Haagerup [38] (as modified in [39]).

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero.

For $x \in M_n(X)$, $y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Definition 1.2 Let $(X, P(\cdot))$ be a paranormed space.

(1) $(X, \{P_n(\cdot)\})$ is a *matrix paranormed space* if $(M_n(X), P_n(\cdot))$ is a paranormed space for each positive integer n, $P_n(E_{kl} \otimes x) = P(x)$ for $x \in X$, and $P(x_{kl}) \le P_n([x_{ij}])$ for $[x_{ij}] \in M_n(X)$.

(2) $(X, \{P_n(\cdot)\})$ is a *matrix Fréchet space* if X is a Fréchet space and $(X, \{P_n(\cdot)\})$ is a *matrix paranormed space*.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

In Section 2, we prove the Hyers-Ulam stability of Cauchy additive functional inequality (1.2) in matrix paranormed spaces. In Section 3, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix paranormed spaces. In Section 4, we prove the Hyers-Ulam stability of the quadratic functional equation in matrix paranormed spaces.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix Banach space and $(Y, \{P_n(\cdot)\})$ be a matrix Fréchet space.

2 Hyers-Ulam stability of additive functional inequality (1.2) in matrix paranormed spaces

In this section, we prove the Hyers-Ulam stability of additive functional inequality (1.2) in matrix paranormed spaces.

Lemma 2.1 *Let* $(X, \{P_n(\cdot)\})$ *be a matrix paranormed space. Then*

(1) $P(x_{kl}) \le P_n([x_{ij}]) \le \sum_{i,j=1}^n P(x_{ij})$ *for* $[x_{ij}] \in M_n(X)$;

(2) $\lim_{s\to\infty} x_s = x$ *if and only if* $\lim_{s\to\infty} x_{sij} = x_{ij}$ *for* $x_s = [x_{sij}], x = [x_{ij}] \in M_k(X)$.

Proof (1) By Definition 1.2, $P(x_{kl}) \le P_n([x_{ij}])$.

Since $[x_{ij}] = \sum_{i,j=1}^n E_{ij} \otimes x_{ij}$,

$$P_n([x_{ij}]) = P_n\left(\sum_{i,j=1}^n E_{ij} \otimes x_{ij}\right) \le \sum_{i,j=1}^n P_n(E_{ij} \otimes x_{ij}) = \sum_{i,j=1}^n P(x_{ij}).$$

(2) By (1), we have

$$P(x_{skl}-x_{kl}) \le P_n\big([x_{sij}-x_{ij}]\big) = P_n\big([x_{sij}]-[x_{ij}]\big) \le \sum_{i,j=1}^{n} P(x_{sij}-x_{ij}).$$

So, we get the result. □

Lemma 2.2 *Let $(X,\{\|\cdot\|_n\})$ be a matrix normed space. Then*

(1) $\|E_{kl}\otimes x\|_n = \|x\|$ *for* $x \in X$;

(2) $\|x_{kl}\| \le \|[x_{ij}]\|_n \le \sum_{i,j=1}^{n}\|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$;

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Proof (1) Since $E_{kl}\otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\|E_{kl}\otimes x\|_n \le \|x\|$. Since $e_k(E_{kl}\otimes x)e_l^* = x$, $\|x\| \le \|E_{kl}\otimes x\|_n$. So, $\|E_{kl}\otimes x\|_n = \|x\|$.

(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k\| = \|e_l^*\| = 1$, $\|x_{kl}\| \le \|[x_{ij}]\|_n$. Since $[x_{ij}] = \sum_{i,j=1}^{n} E_{ij}\otimes x_{ij}$,

$$\big\|[x_{ij}]\big\|_n = \left\|\sum_{i,j=1}^{n} E_{ij}\otimes x_{ij}\right\|_n \le \sum_{i,j=1}^{n}\|E_{ij}\otimes x_{ij}\|_n = \sum_{i,j=1}^{n}\|x_{ij}\|.$$

(3) By

$$\|x_{kln}-x_{kl}\| \le \big\|[x_{ijn}-x_{ij}]\big\|_n = \big\|[x_{ijn}]-[x_{ij}]\big\|_n \le \sum_{i,j=1}^{n}\|x_{ijn}-x_{ij}\|,$$

we get the result. □

We need the following result.

Lemma 2.3 *Let $f: X \to Y$ be an odd mapping such that*

$$P\big(f(a)+f(b)+f(c)\big) \le P\big(f(a+b+c)\big) \tag{2.1}$$

for all $a,b,c \in X$. Then $f: X\to Y$ is additive.

Proof Letting $c = -a-b$ in (2.1), we get $P(f(a)+f(b)+f(-a-b)) \le P(f(0)) = 0$ for all $a,b \in X$. So,

$$f(a)+f(b)-f(a+b) = f(a)+f(b)+f(-a-b) = 0$$

for all $a,b\in X$. Thus $f: X\to Y$ is additive. □

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 2.4 *Let r,θ be positive real numbers with $r > 1$. Let $f: X\to Y$ be an odd mapping such that*

$$\begin{aligned} P_n\big(f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])\big) &\le P_n\big(f_n([x_{ij}]+[y_{ij}]+[z_{ij}])\big) \\ &\quad + \sum_{i,j=1}^{n}\theta\big(\|x_{ij}\|^r+\|y_{ij}\|^r+\|z_{ij}\|^r\big) \end{aligned} \tag{2.2}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$P_n\big(f_n([x_{ij}]) - A_n([x_{ij}])\big) \le \sum_{i,j=1}^{n} \frac{2^r + 2}{2^r - 2}\theta \|x_{ij}\|^r \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof When $n = 1$, (2.2) is equivalent to

$$P\big(f(a) + f(b) + f(c)\big) \le P\big(f(a + b + c)\big) + \theta\big(\|a\|^r + \|b\|^r + \|c\|^r\big) \tag{2.4}$$

for all $a, b, c \in X$.

Letting $b = a$ and $c = -2a$ in (2.4), we get

$$P\big(f(2a) - 2f(a)\big) \le \big(2 + 2^r\big)\theta \|a\|^r,$$

and so

$$P\left(f(a) - 2f\left(\frac{a}{2}\right)\right) \le \frac{2 + 2^r}{2^r}\theta \|a\|^r$$

for all $a, b \in X$.

One can easily show that

$$P\left(2^p f\left(\frac{a}{2^p}\right) - 2^q f\left(\frac{a}{2^q}\right)\right) \le \sum_{l=p}^{q-1} \frac{(2 + 2^r)2^l}{2^{(l+1)r}}\theta \|a\|^r \tag{2.5}$$

for all $a, b \in X$ and nonnegative integers p, q with $p < q$. It follows from (2.5) that the sequence $\{2^l f(\frac{a}{2^l})\}$ is Cauchy for all $a \in X$. Since Y is complete, the sequence $\{2^l f(\frac{a}{2^l})\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(a) = \lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right)$$

for all $a \in X$.

Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (2.5), we get

$$P\big(f(a) - A(a)\big) \le \frac{2^r + 2}{2^r - 2}\theta \|a\|^r \tag{2.6}$$

for all $a \in X$.

It follows from (2.4) that

$$\begin{aligned} &P\left(2^l\left(f\left(\frac{a}{2^l}\right) + f\left(\frac{b}{2^l}\right) + f\left(\frac{c}{2^l}\right)\right)\right) \\ &\quad \le 2^l P\left(f\left(\frac{a + b + c}{2^l}\right)\right) + \frac{2^l}{2^{lr}}\theta\big(\|a\|^r + \|b\|^r + \|c\|^r\big) \end{aligned}$$

for all $a, b, c \in X$. Passing the limit $l \to \infty$ in the above inequality, we get

$$P\big(A(a) + A(b) + A(c)\big) \le P\big(A(a + b + c)\big)$$

for all $a, b, c \in X$. Since $f : X \to Y$ is an odd mapping, the mapping $A : X \to Y$ is odd. By Lemma 2.3, $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.6). Then we have

$$\begin{aligned} P\big(A(a) - T(a)\big) &= P\left(2^q A\left(\frac{a}{2^q}\right) - 2^q T\left(\frac{a}{2^q}\right)\right) \\ &\le P\left(2^q\left(A\left(\frac{a}{2^q}\right) - g\left(\frac{a}{2^q}\right)\right)\right) + P\left(2^q\left(T\left(\frac{a}{2^q}\right) - g\left(\frac{a}{2^q}\right)\right)\right) \\ &\le 2\frac{2^r + 2}{2^r - 2}\frac{2^q}{2^{qr}}\theta\|a\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $a \in X$. So, we can conclude that $A(a) = T(a)$ for all $a \in X$. This proves the uniqueness of A.

By Lemma 2.1 and (2.6),

$$P_n\big(f_n\big([x_{ij}]\big) - A_n\big([x_{ij}]\big)\big) \le \sum_{i,j=1}^{n} P\big(f(x_{ij}) - A(x_{ij})\big) \le \sum_{i,j=1}^{n} \frac{(2 + 2^r)}{2^r - 2}\theta\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (2.3), as desired. □

Theorem 2.5 *Let r, θ be positive real numbers with $r < 1$. Let $f : Y \to X$ be an odd mapping such that*

$$\begin{aligned} \big\|f_n\big([x_{ij}]\big) + f_n\big([y_{ij}]\big) + f_n\big([z_{ij}]\big)\big\|_n &\le \big\|f_n\big([x_{ij}] + [y_{ij}] + [z_{ij}]\big)\big\|_n \\ &\quad + \sum_{i,j=1}^{n} \theta\big(P(x_{ij})^r + P(y_{ij})^r + P(z_{ij})^r\big) \end{aligned} \tag{2.7}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(Y)$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$\big\|f_n\big([x_{ij}]\big) - A_n\big([x_{ij}]\big)\big\|_n \le \sum_{i,j=1}^{n} \frac{2 + 2^r}{2 - 2^r}\theta P(x_{ij})^r \tag{2.8}$$

for all $x = [x_{ij}] \in M_n(Y)$.

Proof Let $n = 1$ in (2.7). Then (2.7) is equivalent to

$$\big\|f(a) + f(b) + f(c)\big\| \le \big\|f(a + b + c)\big\| + \theta\big(P(a)^r + P(b)^r + P(c)^r\big) \tag{2.9}$$

for all $a, b, c \in Y$.

Letting $b = a$ and $c = -2a$ in (2.9), we get

$$\big\|f(2a) - 2f(a)\big\| \le \big(2 + 2^r\big)\theta P(a)^r,$$

and so

$$\left\| f(a) - \frac{1}{2}f(2a) \right\| \le \frac{2+2^r}{2}\theta P(a)^r$$

for all $a \in Y$.

One can easily show that

$$\left\| \frac{1}{2^p}f\left(2^p a\right) - \frac{1}{2^q}f\left(2^q a\right) \right\| \le \sum_{l=p}^{q-1} \frac{2^{lr}}{2^l}\frac{2+2^r}{2}\theta P(a)^r \tag{2.10}$$

for all $a \in Y$ and nonnegative integers p, q with $p < q$. It follows from (2.10) that the sequence $\{\frac{1}{2^l}f(2^l a)\}$ is Cauchy for all $a \in Y$. Since X is complete, the sequence $\{\frac{1}{2^l}f(2^l a)\}$ converges. So, one can define the mapping $A : Y \to X$ by

$$A(a) = \lim_{l\to\infty} \frac{1}{2^l}f\left(2^l a\right)$$

for all $a \in Y$.

Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (2.10), we get

$$\left\| f(a) - A(a) \right\| \le \frac{2+2^r}{2-2^r}\theta P(a)^r \tag{2.11}$$

for all $a \in Y$.

It follows from (2.9) that

$$\left\| \frac{1}{2^l}\left(f\left(2^l a\right) + f\left(2^l b\right) + f\left(2^l c\right)\right) \right\| \le \left\| \frac{1}{2^l}f\left(2^l(a+b+c)\right) \right\| + \frac{2^{lr}}{2^l}\theta\left(\|a\|^r + \|b\|^r + \|c\|^r\right)$$

for all $a, b, c \in Y$. Passing the limit $l \to \infty$ in the above inequality, we get

$$\left\| A(a) + A(b) + A(c) \right\| \le \left\| A(a+b+c) \right\|$$

for all $a, b, c \in Y$. By [32, Lemma 3.1], the mapping $A : Y \to X$ is additive.

Now, let $T : Y \to X$ be another additive mapping satisfying (2.11). Let $n = 1$. Then we have

$$\begin{aligned} \left\| A(a) - T(a) \right\| &= \left\| \frac{1}{2^q}A\left(2^q a\right) - \frac{1}{2^q}T\left(2^q a\right) \right\| \\ &\le \left\| \frac{1}{2^q}\left(A\left(2^q a\right) - g\left(2^q a\right)\right) \right\| + \left\| \frac{1}{2^q}\left(T\left(2^q a\right) - g\left(2^q a\right)\right) \right\| \\ &\le 2\frac{2+2^r}{2-2^r}\frac{2^{qr}}{2^q}\theta P(a)^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $a \in Y$. So, we can conclude that $A(a) = T(a)$ for all $a \in Y$. This proves the uniqueness of A.

By Lemma 2.2 and (2.11),

$$\left\| f_n\left([x_{ij}]\right) - A_n\left([x_{ij}]\right) \right\|_n \le \sum_{i,j=1}^{n} \left\| f(x_{ij}) - A(x_{ij}) \right\| \le \sum_{i,j=1}^{n} \frac{2+2^r}{2^r-2}\theta P(x_{ij})^r$$

for all $x = [x_{ij}] \in M_n(Y)$. Thus $A : Y \to X$ is a unique additive mapping satisfying (2.8), as desired. □

3 Hyers-Ulam stability of the Cauchy additive functional equation in matrix paranormed spaces

In this section, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix paranormed spaces.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned} &Df(a,b) = f(a+b) - f(a) - f(b), \\ &Df_n\big([x_{ij}],[y_{ij}]\big) := f_n\big([x_{ij}+y_{ij}]\big) - f_n\big([x_{ij}]\big) - f_n\big([y_{ij}]\big) \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.1 *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping such that*

$$P_n\big(Df_n\big([x_{ij}],[y_{ij}]\big)\big) \le \sum_{i,j=1}^{n} \theta\big(\|x_{ij}\|^r + \|y_{ij}\|^r\big) \tag{3.1}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$P_n\big(f_n\big([x_{ij}]\big) - A_n\big([x_{ij}]\big)\big) \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2}\|x_{ij}\|^r \tag{3.2}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$ in (3.1). Then (3.1) is equivalent to

$$P\big(f(a+b) - f(a) - f(b)\big) \le \theta\big(\|a\|^r + \|b\|^r\big) \tag{3.3}$$

for all $a, b \in X$.

Letting $b = a$ in (3.3), we get

$$P\big(f(2a) - 2f(a)\big) \le 2\theta\|a\|^r,$$

and so

$$P\left(f(a) - 2f\left(\frac{a}{2}\right)\right) \le \frac{2}{2^r}\theta\|a\|^r$$

for all $a, b \in X$.

One can easily show that

$$P\left(2^p f\left(\frac{a}{2^p}\right) - 2^q f\left(\frac{a}{2^q}\right)\right) \le \sum_{l=p}^{q-1} \frac{2 \cdot 2^l}{2^{(l+1)r}}\theta\|a\|^r \tag{3.4}$$

for all $a, b \in X$ and nonnegative integers p, q with $p < q$. It follows from (3.4) that the sequence $\{2^l f(\frac{a}{2^l})\}$ is Cauchy for all $a \in X$. Since Y is complete, the sequence $\{2^l f(\frac{a}{2^l})\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(a) = \lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right)$$

for all $a \in X$.

Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (3.4), we get

$$P\bigl(f(a) - A(a)\bigr) \le \frac{2\theta}{2^r - 2}\|a\|^r \tag{3.5}$$

for all $a \in X$.

It follows from (3.3) that

$$\begin{aligned} P\left(2^l\left(f\left(\frac{a+b}{2^l}\right) - f\left(\frac{a}{2^l}\right) - f\left(\frac{b}{2^l}\right)\right)\right) &\le 2^l P\left(f\left(\frac{a+b}{2^l}\right) - f\left(\frac{a}{2^l}\right) - f\left(\frac{b}{2^l}\right)\right) \\ &\le \frac{2^l}{2^{lr}}\theta\left(\|a\|^r + \|b\|^r\right), \end{aligned}$$

which tends to zero as $l \to \infty$. So, $P(A(a + b) - A(a) - A(b)) = 0$, *i.e.*, $A(a + b) = A(a) + A(b)$ for all $a, b \in X$. Hence $A : X \to Y$ is additive.

The proof of the uniqueness of A is similar to the proof of Theorem 2.4.

By Lemma 2.1 and (3.5),

$$P_n\bigl(f_n([x_{ij}]) - A_n([x_{ij}])\bigr) \le \sum_{i,j=1}^{n} P\bigl(f(x_{ij}) - A(x_{ij})\bigr) \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (3.2), as desired. □

Theorem 3.2 *Let r, θ be positive real numbers with $r < 1$. Let $f : Y \to X$ be a mapping such that*

$$\bigl\|Df_n([x_{ij}], [y_{ij}])\bigr\|_n \le \sum_{i,j=1}^{n} \theta\bigl(P(x_{ij})^r + P(y_{ij})^r\bigr) \tag{3.6}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(Y)$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$\bigl\|f_n([x_{ij}]) - A_n([x_{ij}])\bigr\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2 - 2^r} P(x_{ij})^r \tag{3.7}$$

for all $x = [x_{ij}] \in M_n(Y)$.

Proof Let $n = 1$ in (3.6). Then (3.6) is equivalent to

$$\bigl\|f(a + b) - f(a) - f(b)\bigr\| \le \theta\bigl(P(a)^r + P(b)^r\bigr) \tag{3.8}$$

for all $a, b \in Y$.

Letting $b = a$ in (3.8), we get

$$\|f(2a) - 2f(a)\| \le 2\theta P(a)^r,$$

and so

$$\left\|f(a) - \frac{1}{2}f(2a)\right\| \le \theta P(a)^r$$

for all $a \in Y$.

One can easily show that

$$\left\|\frac{1}{2^p}f(2^p a) - \frac{1}{2^q}f(2^q a)\right\| \le \sum_{l=p}^{q-1}\frac{2^{lr}}{2^l}\theta P(a)^r \tag{3.9}$$

for all $a \in Y$ and nonnegative integers p, q with $p < q$. It follows from (3.9) that the sequence $\{\frac{1}{2^l}f(2^l a)\}$ is Cauchy for all $a \in Y$. Since X is complete, the sequence $\{\frac{1}{2^l}f(2^l a)\}$ converges. So, one can define the mapping $A : Y \to X$ by

$$A(a) = \lim_{l\to\infty}\frac{1}{2^l}f(2^l a)$$

for all $a \in Y$.

Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (3.9), we get

$$\|f(a) - A(a)\| \le \frac{2\theta}{2 - 2^r}P(a)^r \tag{3.10}$$

for all $a \in Y$.

It follows from (3.8) that

$$\left\|\frac{1}{2^l}\left(f\left(2^l(a+b)\right) - f\left(2^l a\right) - f\left(2^l b\right)\right)\right\| \le \frac{2^{lr}}{2^l}\theta\left(\|a\|^r + \|b\|^r\right),$$

which tends to zero as $l \to \infty$. So, $\|A(a+b) - A(a) - A(b)\| = 0$, *i.e.*, $A(a+b) = A(a) + A(b)$ for all $a, b \in Y$. Hence $A : Y \to X$ is additive.

The proof of the uniqueness of A is similar to the proof of Theorem 2.5.

By Lemma 2.2 and (3.10),

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \le \sum_{i,j=1}^{n}\left\|f(x_{ij}) - A(x_{ij})\right\| \le \sum_{i,j=1}^{n}\frac{2\theta}{2^r - 2}P(x_{ij})^r$$

for all $x = [x_{ij}] \in M_n(Y)$. Thus $A : Y \to X$ is a unique additive mapping satisfying (3.7), as desired. □

4 Hyers-Ulam stability of the quadratic functional equation in matrix paranormed spaces

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation in matrix paranormed spaces.

For a mapping $f: X \to Y$, define $Df: X^2 \to Y$ and $Df_n: M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = f(a+b) + f(a-b) - 2f(a) - 2f(b),$$
$$Df_n([x_{ij}],[y_{ij}]) := f_n([x_{ij}+y_{ij}]) + f_n([x_{ij}-y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 4.1 *Let r, θ be positive real numbers with $r > 2$. Let $f: X \to Y$ be a mapping such that*

$$P_n(Df_n([x_{ij}],[y_{ij}])) \le \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \tag{4.1}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$P_n(f_n([x_{ij}]) - Q_n([x_{ij}])) \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 4}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$ in (4.1). Then (4.1) is equivalent to

$$P(f(a+b) + f(a-b) - 2f(a) - 2f(b)) \le \theta(\|a\|^r + \|b\|^r) \tag{4.2}$$

for all $a, b \in X$.

Letting $a = b = 0$ in (4.2), we get $P(2f(0)) \le 0$ and so $f(0) = 0$.

Letting $b = a$ in (4.2), we get

$$P(f(2a) - 4f(a)) \le 2\theta\|a\|^r,$$

and so

$$P\left(f(a) - 4f\left(\frac{a}{2}\right)\right) \le \frac{2}{2^r}\theta\|a\|^r$$

for all $a, b \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 4.2 *Let r, θ be positive real numbers with $r < 2$. Let $f: Y \to X$ be a mapping such that*

$$\|Df_n([x_{ij}],[y_{ij}])\|_n \le \sum_{i,j=1}^{n} \theta(P(x_{ij})^r + P(y_{ij})^r) \tag{4.3}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(Y)$. Then there exists a unique quadratic mapping $Q: Y \to X$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{4 - 2^r} P(x_{ij})^r$$

for all $x = [x_{ij}] \in M_n(Y)$.

Proof Let $n = 1$ in (4.3). Then (4.3) is equivalent to

$$\left\| f(a+b) + f(a-b) - 2f(a) - 2f(b) \right\| \leq \theta \left(P(a)^r + P(b)^r \right) \tag{4.4}$$

for all $a, b \in Y$.

Letting $a = b = 0$ in (4.4), we get $\|2f(0)\| \leq 0$ and so $f(0) = 0$.

Letting $b = a$ in (4.4), we get

$$\left\| f(2a) - 4f(a) \right\| \leq 2\theta P(a)^r,$$

and so

$$\left\| f(a) - \frac{1}{4} f(2a) \right\| \leq \frac{\theta}{2} P(a)^r$$

for all $a, b \in Y$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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References

1. Fast, H: Sur la convergence statistique. Colloq. Math. **2**, 241-244 (1951)
2. Steinhaus, H: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**, 73-74 (1951)
3. Fridy, JA: On statistical convergence. Analysis **5**, 301-313 (1985)
4. Karakus, S: Statistical convergence on probabilistic normed spaces. Math. Commun. **12**, 11-23 (2007)
5. Mursaleen, M: λ-Statistical convergence. Math. Slovaca **50**, 111-115 (2000)
6. Mursaleen, M, Mohiuddine, SA: On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. J. Comput. Appl. Math. **233**, 142-149 (2009)
7. Šalát, T: On the statistically convergent sequences of real numbers. Math. Slovaca **30**, 139-150 (1980)
8. Kolk, E: The statistical convergence in Banach spaces. Tartu Ülik. Toim. **928**, 41-52 (1991)
9. Wilansky, A: Modern Methods in Topological Vector Space. McGraw-Hill, New York (1978)
10. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
11. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
12. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
13. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
14. Găvruta, G: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
15. Rassias, TM: Problem 16; 2. Report of the 27th international symp. on functional equations. Aequ. Math. **39**, 292-293; 309 (1990)
16. Gajda, Z: On stability of additive mappings. Int. J. Math. Math. Sci. **14**, 431-434 (1991)
17. Rassias, TM, Šemrl, P: On the behaviour of mappings which do not satisfy Hyers-Ulam stability. Proc. Am. Math. Soc. **114**, 989-993 (1992)
18. Czerwik, P: Functional Equations and Inequalities in Several Variables. World Scientific, Singapore (2002)

19. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Basel (1998)
20. Skof, F: Proprieta' locali e approssimazione di operatori. Rend. Semin. Mat. Fis. Milano **53**, 113-129 (1983)
21. Cholewa, PW: Remarks on the stability of functional equations. Aequ. Math. **27**, 76-86 (1984)
22. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh. Math. Semin. Univ. Hamb. **62**, 59-64 (1992)
23. Aczel, J, Dhombres, J: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
24. Eshaghi Gordji, M, Savadkouhi, MB: Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces. Appl. Math. Lett. **23**, 1198-1202 (2010)
25. Isac, G, Rassias, TM: On the Hyers-Ulam stability of ψ-additive mappings. J. Approx. Theory **72**, 131-137 (1993)
26. Jun, K, Lee, Y: A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations. J. Math. Anal. Appl. **297**, 70-86 (2004)
27. Park, C: Homomorphisms between Poisson JC^*-algebras. Bull. Braz. Math. Soc. **36**, 79-97 (2005)
28. Gilányi, A: Eine zur Parallelogrammgleichung äquivalente Ungleichung. Aequ. Math. **62**, 303-309 (2001)
29. Rätz, J: On inequalities associated with the Jordan-von Neumann functional equation. Aequ. Math. **66**, 191-200 (2003)
30. Gilányi, A: On a problem by K. Nikodem. Math. Inequal. Appl. **5**, 707-710 (2002)
31. Fechner, W: Stability of a functional inequalities associated with the Jordan-von Neumann functional equation. Aequ. Math. **71**, 149-161 (2006)
32. Park, C, Cho, Y, Han, M: Functional inequalities associated with Jordan-von Neumann-type additive functional equations. J. Inequal. Appl. **2007**, Article ID 41820 (2007)
33. Ruan, ZJ: Subspaces of C^*-algebras. J. Funct. Anal. **76**, 217-230 (1988)
34. Effros, E, Ruan, ZJ: On approximation properties for operator spaces. Int. J. Math. **1**, 163-187 (1990)
35. Choi, MD, Effros, E: Injectivity and operator spaces. J. Funct. Anal. **24**, 156-209 (1977)
36. Effros, E, Ruan, ZJ: On the abstract characterization of operator spaces. Proc. Am. Math. Soc. **119**, 579-584 (1993)
37. Pisier, G: Grothendieck's theorem for non-commutative C^*-algebras with an appendix on Grothendieck's constants. J. Funct. Anal. **29**, 397-415 (1978)
38. Haagerup, U: Decomp. of completely bounded maps. Unpublished manuscript
39. Effros, E: On multilinear completely bounded module maps. In: Operator Algebras and Mathematical Physics. Contemp. Math., vol. 62, pp. 479-501. Am. Math. Soc., Providence (1987)

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