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Boundary towers of layers for some supercritical problems

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ABSTRACT

We consider the supercritical problem

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathcal{D}, \qquad u = 0 \quad \text{on } \partial\mathcal{D},$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ and p is smaller than the κ-th critical Sobolev exponent $2^*_{N,\kappa} := \frac{N-\kappa+2}{N-\kappa-2}$ with $1 \leqslant \kappa \leqslant N-3$. We show that in some suitable torus-like domains $\mathcal{D}$ there exists an arbitrary large number of sign-changing solutions with alternate positive and negative layers which concentrate at different rates along a κ-dimensional submanifold of $\partial\mathcal{D}$ as p approaches $2^*_{N,\kappa}$ from below.

1. Introduction

This paper deals with the classical Lane–Emden–Fowler problem

$$\Delta v + |v|^{p-1}v = 0 \quad \text{in } \mathcal{D}, \qquad v = 0 \quad \text{on } \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, and $p > 1$. In particular, we are interested in exploring the role of the lower dimensional Sobolev exponents $2^*_{N,\kappa}$ on the existence and multiplicity of solutions to problem (1.1). For any integer κ between 0 and $N-2$ let us set

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$$2^*_{N,\kappa} := \frac{N-\kappa+2}{N-\kappa-2} \quad \text{if } 0 \leqslant \kappa \leqslant N-3 \quad \text{and} \quad 2^*_{N,N-2} := +\infty. \tag{1.2}$$

If $0 \leqslant \kappa \leqslant N-3$, then $2^*_{N,\kappa}+1$ is nothing but the κ-th critical Sobolev exponent in dimension $N-\kappa$.

It is well known that in the subcritical regime, i.e. $p < 2^*_{N,0}$, the compactness of the Sobolev embedding ensures the existence of at least one positive solution and infinitely many sign-changing solutions to (1.1).

In the critical case (i.e. $p = 2^*_{N,0}$) or in the supercritical case (i.e. $p > 2^*_{N,0}$) existence of solutions to problem (1.1) turns out to be a delicate issue. Indeed, if the domain $\mathcal{D}$ is star shaped Pohožaev's identity [25] implies that problem (1.1) has only the trivial solution.

In the critical case, if $\mathcal{D}$ has nontrivial reduced homology with $\mathbb{Z}_2$-coefficients, Bahri–Coron [4] proved that problem (1.1) has a positive solution in the critical case. Moreover, it was proved by Ge–Musso–Pistoia [15] and Musso–Pistoia [18] that if $\mathcal{D}$ has a small hole, problem (1.1) has many sign changing solutions, whose number increases as the diameter of the hole decreases.

In the supercritical regime the existence of a nontrivial homology class in $\mathcal{D}$ does not guarantee the existence of a nontrivial solution to (1.1). Passaseo in [21,22] exhibited a domain in $\mathbb{R}^N$ homotopically equivalent to the κ-dimensional sphere in which problem (1.1) with $p \geqslant 2^*_{N,\kappa}$ has only the trivial solution. Recently Clapp–Faya–Pistoia [7] built domains in $\mathbb{R}^N$ with a richer topology, namely the cup-length is $\kappa+1$, in which problem (1.1) with $p > 2^*_{N,\kappa}$ has only the trivial solution. When $p = 2^*_{N,\kappa}$ the existence of infinitely many positive solutions to (1.1) was proved by Wei–Yan [26] for suitable torus-like domains $\mathcal{D}$.

It is interesting to study problem (1.1) in the almost critical case, i.e. $p = 2^*_{N,\kappa} \pm \epsilon$, where ϵ is a small positive parameter.

The peculiarity of the almost critical case when $\kappa = 0$ is that problem (1.1) has solutions which blow up at one or more simple or multiple points in $\mathcal{D}$ as ϵ goes to zero. Indeed, if $p = 2^*_{N,0} - \epsilon$, positive and sign-changing solutions to (1.1) with different simple blow-up points were built by Bahri–Li–Rey [5] and Bartsch–Micheletti–Pistoia [6], respectively. Moreover, Pistoia–Weth [24] and Musso–Pistoia [19] proved that the number of sign-changing solutions to (1.1) with a multiple blow-up point increases as ϵ goes to zero. On the other hand, if $p = 2^*_{N,0} + \epsilon$, Ben Ayed–El Mehdi–Grossi–Rey [3] proved that problem (1.1) does not have any positive solutions with one positive blow-up point, while Del Pino–Felmer–Musso [9] and Pistoia–Rey [23] found solutions with two or more positive blow-up points provided the domain $\mathcal{D}$ has a hole. Up to our knowledge there are no results about existence of sign-changing solutions in this case. In particular, we quote Ben Ayed–Bouh [2] who proved that problem (1.1) does not have any sign-changing solutions with one positive and one or two negative blow-up points.

Having in mind what happens in the almost critical case when $\kappa = 0$, we wonder if the same phenomenon occurs for any $1 \leqslant \kappa \leqslant N-2$. More precisely, we ask if for some suitable domains $\mathcal{D}$ the problem (1.1) has solutions which blow up at one or more simple or multiple κ-dimensional manifolds in $\mathcal{D}$ as p approaches the κ-th Sobolev exponent $2^*_{N,\kappa}$ from below. A first result in this direction was obtained by Del Pino–Musso–Pacard [10]. If $\kappa = 1$ and $p = 2^*_{N,1} - \epsilon$, they proved that for some domains $\mathcal{D}$ if ϵ is different from an explicit set of values, problem (1.1) has a positive solution which concentrates along a 1-dimensional submanifold of the boundary of $\mathcal{D}$ when ϵ goes to zero. Recently, it has been showed that if $\kappa \geqslant 2$ and p approaches from below $2^*_{N,\kappa}$ it is possible to build torus-like domains $\mathcal{D}$ in which problem (1.1) has positive solutions which concentrate at a κ-dimensional submanifold of $\partial\mathcal{D}$. The construction was performed in the case $1 \leqslant \kappa \leqslant N-3$, $p = 2^*_{N,\kappa} - \epsilon$ and ϵ goes to zero and in the case $\kappa = N-2$ and p goes to $+\infty$ by Ackermann–Clapp–Pistoia [1] and Kim–Pistoia [16], respectively.

As far as it concerns existence of sign-changing solutions, when $1 \leqslant \kappa \leqslant N-3$, $p = 2^*_{N,\kappa} - \epsilon$ and ϵ is small enough or when $\kappa = N-2$ and p is large enough, Ackermann–Clapp–Pistoia [1] and Kim–Pistoia [16], respectively, constructed a sign-changing solution with a positive and a negative layer which concentrate with the same rate along the same κ-dimensional submanifold of the boundary of suitable torus-like domains $\mathcal{D}$, as ϵ goes to zero. In particular, Kim–Pistoia [16] proved that when $\kappa = N-2$ the number of sign changing solutions to (1.1) increases as p goes to $+\infty$, provided $\mathcal{D}$ satisfies some symmetric assumptions. Their solutions have an arbitrary number of alternate positive and negative layers which concentrate with the same rate along the same $(N-2)$-dimensional submanifold of $\partial\mathcal{D}$ as p goes to $+\infty$.

In this paper, we build domains $\mathcal{D}$ such that the number of sign-changing solutions of problem (1.1) when $1 \leqslant \kappa \leqslant N-3$ and $p = 2^*_{N,\kappa} - \epsilon$ increases as ϵ goes to zero. In particular, for each set of positive integers $\kappa_1, \dots, \kappa_m$ with $\kappa := \kappa_1 + \dots + \kappa_m \leqslant N-3$ we exhibit torus-like domains $\mathcal{D}$ for which the number of sign-changing solutions to problem (1.1) with $p = 2^*_{N,\kappa} - \epsilon$ increases as ϵ goes to zero. These solutions have an arbitrary large number of alternate positive and negative layer which concentrate with different rates along a κ-dimensional submanifold Γ_0 of $\partial\mathcal{D}$ which is diffeomorphic to the product of spheres $\mathbb{S}^{\kappa_1} \times \dots \times \mathbb{S}^{\kappa_m}$. This follows from our main results, which we next state.

Fix $\kappa_1, \dots, \kappa_m \in \mathbb{N}$ with $\kappa := \kappa_1 + \dots + \kappa_m \leqslant N-3$ and a bounded smooth domain Ω in $\mathbb{R}^{N-\kappa}$ such that

$$\overline{\Omega} \subset \{(x_1, \dots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{N-\kappa-m}\colon x_i > 0,\ i = 1, \dots, m\}. \tag{1.3}$$

Set

$$\mathcal{D} := \{(y^1, \dots, y^m, z) \in \mathbb{R}^{\kappa_1+1} \times \dots \times \mathbb{R}^{\kappa_m+1} \times \mathbb{R}^{N-\kappa-m}\colon (|y^1|, \dots, |y^m|, z) \in \Omega\}. \tag{1.4}$$

$\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ which is invariant under the action of the group $\Theta := O(\kappa_1+1) \times \dots \times O(\kappa_m+1)$ on $\mathbb{R}^N$ given by

$$(g_1, \dots, g_m)(y^1, \dots, y^m, z) := (g_1 y^1, \dots, g_m y^m, z),$$

for every $g_i \in O(\kappa_i+1)$, $y^i \in \mathbb{R}^{\kappa_i+1}$, $z \in \mathbb{R}^{N-\kappa-m}$. Here, as usual, $O(d)$ denotes the group of linear isometries of $\mathbb{R}^d$. For $p = 2^*_{N,\kappa} - \epsilon$ we shall look for Θ-invariant solutions to problem (1.1), i.e. solutions v of the form

$$v(y^1, \dots, y^m, z) = u(|y^1|, \dots, |y^m|, z). \tag{1.5}$$

A simple calculation shows that v solves problem (1.1) if and only if u solves

$$-\Delta u - \sum_{i=1}^m \frac{\kappa_i}{x_i}\frac{\partial u}{\partial x_i} = |u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

This problem can be rewritten as

$$-\operatorname{div}(a(x)\nabla u) = a(x)|u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $a(x_1, \dots, x_{N-\kappa}) := x_1^{\kappa_1} \cdots x_m^{\kappa_m}$. Note that $2^*_{N,\kappa}$ is the critical exponent in dimension $n := N-\kappa$ which is the dimension of Ω.

Thus, we are led to study the more general almost critical problem

$$-\operatorname{div}(a(x)\nabla u) = a(x)|u|^{\frac{4}{n-2}-\epsilon}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, $\epsilon > 0$ is a small parameter and $a \in C^2(\overline{\Omega})$ is strictly positive in $\overline{\Omega}$.

This is a subcritical problem, so standard variational methods yield one positive and infinitely many sign changing solutions to problem (1.6) for every $\epsilon \in (0, \frac{4}{n-2})$. Our goal is to construct solutions u_ϵ with an arbitrary large number of alternate positive and negative bubbles which accumulate with different rates at the same point ξ_0 of $\partial\Omega$ as $\epsilon \to 0$. They correspond, via (1.5), to Θ-invariant solutions v_ϵ of problem (1.1) with positive and negative layers which accumulate with different rates along the κ-dimensional submanifold

$$\Gamma_0 := \left\{\left(y^1, \dots, y^m, z\right) \in \mathbb{R}^{\kappa_1+1} \times \cdots \times \mathbb{R}^{\kappa_m+1} \times \mathbb{R}^{N-\kappa-m} \colon \left(\left|y^1\right|, \dots, \left|y^m\right|, z\right) = \xi_0\right\}$$

of the boundary of $\mathcal{D}$ as $\epsilon \to 0$. Note that M_0 is diffeomorphic to $\mathbb{S}^{\kappa_1} \times \cdots \times \mathbb{S}^{\kappa_m}$ where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$.

We will assume the following conditions.

(a1) There are constants a_1 and a_2 such that

$$0 < a_1 \leqslant a(x) \leqslant a_2 < +\infty \quad \text{for all } x \in \overline{\Omega}.$$

(a2) The restriction of a to $\partial\Omega$ has a critical point $\xi_0 \in \partial\Omega$ and

$$\partial_\nu a(\xi_0) := \left(\nabla a(\xi_0), \nu(\xi_0)\right) > 0$$

where $\nu := \nu(\xi_0)$ is the inward unit normal vector to $\partial\Omega$ at ξ_0.

(a3) The domain Ω and the function a are symmetric with respect to the direction given by $\nu(\xi_0)$, i.e.,

$$\begin{aligned} &(x,\nu)\nu + (x,\tau_1)\tau_1 + \cdots + (x,\tau_i)\tau_i + \cdots + (x,\tau_{n-1})\tau_{n-1} \in \Omega \\ \Leftrightarrow \quad &(x,\nu)\nu + (x,\tau_1)\tau_1 + \cdots - (x,\tau_i)\tau_i + \cdots + (x,\tau_{n-1})\tau_{n-1} \in \Omega \end{aligned}$$

and

$$\begin{aligned} &a\left((x,\nu)\nu + (x,\tau_1)\tau_1 + \cdots + (x,\tau_i)\tau_i + \cdots + (x,\tau_{n-1})\tau_{n-1}\right) \\ &\quad = a\left((x,\nu)\nu + (x,\tau_1)\tau_1 + \cdots - (x,\tau_i)\tau_i + \cdots + (x,\tau_{n-1})\tau_{n-1}\right) \end{aligned}$$

for $i = 1, \dots, n-1$. Here $(\cdot,\cdot)$ is the standard inner product in $\mathbb{R}^n$ and $\{\tau_1, \dots, \tau_{n-1}\}$ is an orthonormal basis of the tangent space $T_{\xi_0}\partial\Omega$.

For each $\delta > 0$, $\xi \in \mathbb{R}^n$, we consider the standard bubble

$$U_{\delta,\xi}(x) := \left[n(n-2)\right]^{\frac{n-2}{4}} \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}}.$$

We will prove the following result.

Theorem 1.1. *Suppose that* (a1)–(a3) *hold true for a and Ω. Also, assume that $n \geqslant 4$. For any integer k, there exists $\epsilon_k > 0$ such that for each $0 < \epsilon < \epsilon_k$ problem* (1.6) *has a sign changing solution u_ϵ which satisfies*

$$u_\epsilon = \sum_{i=1}^{k} (-1)^{i+1} U_{\delta_i(\epsilon),\xi_i(\epsilon)} + o(1) \quad \text{in } H_0^1(\Omega)$$

where

$$\epsilon^{-\frac{n-1+2(i-1)}{n-2}} \delta_i(\epsilon) \to d_i > 0, \qquad \xi_i(\epsilon) \to \xi_0 \in \partial\Omega \quad \text{as } \epsilon \to 0,$$

for $i = 1, \dots, k$.

The solutions we found resemble the towers of bubbles with alternating sign which concentrates at a point on the boundary of Ω. This kind of solutions is typical of almost critical problems (see [8,11,14,24,19]).

The symmetry of the domain Ω as stated in (a2) allows to simplify considerably the computations. We believe that the result is true if we only require that ξ_0 be a non-degenerate critical point of the restriction of a to the $\partial\Omega$. Moreover, the restriction on the dimension $n \geqslant 4$ is due to technical reasons as it is explained in Remark A.11. We also believe that it can be removed but it seems to be necessary to overcome some technical difficulties.

Now, we come back to problem (1.1). In the following theorem we assume that we are given $\kappa_1, \ldots, \kappa_m \in \mathbb{N}$ with $\kappa := \kappa_1 + \cdots + \kappa_m \leqslant N - 3$ and a bounded smooth domain Ω in $\mathbb{R}^{N-\kappa}$ which satisfies (1.3). We set $a(x_1, \ldots, x_{N-\kappa}) := x_1^{\kappa_1} \cdots x_m^{\kappa_m}$, $\mathcal{D}$ as in (1.4), $p = 2^*_{N,\kappa} - \epsilon$, $\Theta := O(\kappa_1 + 1) \times \cdots \times O(\kappa_m + 1)$ and

$$\widetilde{U}_{\delta,\xi}\left(y^1, \ldots, y^m, z\right) := U_{\delta,\xi}\left(\left|y^1\right|, \ldots, \left|y^m\right|, z\right)$$

for $\delta > 0$, $\xi \in \mathbb{R}^{N-\kappa}$.

Theorem 1.2. *Assume $n = N - \kappa \geqslant 4$. Then for any integer k there exists $\epsilon_k > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, problem (1.1) has a Θ-invariant solution v_ϵ which satisfies*

$$v_\epsilon(x) = \sum_{i=1}^{k} (-1)^{i+1} \widetilde{U}_{\delta_i(\epsilon),\xi_i(\epsilon)}(x) + o(1) \quad \text{in } H_0^1(\mathcal{D}),$$

with

$$\epsilon^{-\frac{n-1+2(i-1)}{n-2}} \delta_i(\epsilon) \to d_i > 0 \quad \text{and} \quad \xi_i(\epsilon) \to \xi_0 \in \partial\Omega,$$

for each $i = 1, \ldots, k$ as $\epsilon \to 0$.

The solutions we found resemble the towers of layers with alternating sign which concentrate at a κ-dimensional submanifold of the boundary of $\mathcal{D}$. This result extends the one obtained by Pistoia–Weth [24] and Musso–Pistoia [19] when $\kappa = 0$ to higher κ's. Moreover, we stress the fact that the profile of our solutions is different from the one found by Ackermann–Clapp–Pistoia [1] and Kim–Pistoia [16]. Indeed, their solutions look like a cluster of layers (i.e. all the layers concentrate at the same speed), while our solutions look like a tower of layers (i.e. one layer concentrates faster than the previous one).

It is interesting to prove that this kind of solutions also exists in the setting of [10]. Indeed, we conjecture that if Γ is a non-degenerate geodesic of the boundary of $\mathcal{D}$ with inner normal curvature it is possible to build towers of sign-changing solutions whose 1-dimensional layers concentrate at Γ as p approaches the first Sobolev critical exponent $2^*_{N,1}$ from below (up to a subsequence of values).

By the previous discussion Theorem 1.2 follows immediately from Theorem 1.1. The proof of Theorem 1.6 relies on a very well known Lyapunov–Schmidt reduction. We omit many details on the finite dimensional reduction because they can be found, up to some minor modifications, in the literature. We only compute what cannot be deduced from known results. In Section 2 we write the approximate solution, we sketch the proof of the Lyapunov–Schmidt procedure and we prove Theorem 1.2. In Section 3 we compute the rate of the error term, while in Section 4 and in Section 5 we give the C^0-estimate and the C^1-estimate of the reduced energy, respectively. In Appendix A we give some important estimates which are not available in the literature.

Notations.

- For the sake of convenience, we assume that $\xi_0 = 0 \in \mathbb{R}^n$, $\tau_i = e_i$ for $i = 1, \dots, n-1$ and $\nu = e_n$ where $\{e_1, \dots, e_n\}$ denotes the standard basis in $\mathbb{R}^n$. Thus assumption (a3) reads as Ω is symmetric with respect to the x_n-axis and $a(x_1, \dots, x_i, \dots, x_n) = a(x_1, \dots, -x_i, \dots, x_n)$ for $i = 1, \dots, n-1$.
- $D^{1,2}(\mathbb{R}^n)$ is the space of measurable and weakly differentiable functions such that the L^2-norms of their gradients are finite.
- $\mathcal{D}(\Omega)$ is the space of smooth functions whose supports are compactly contained in Ω and $H_0^1(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}} = (\int_\Omega a|\nabla u|^2)^{\frac{1}{2}}$. By virtue of (a1), this norm is equivalent to the usual one.
- $\mathcal{H}(\Omega)$ is a subspace of $H_0^1(\Omega)$ defined by

$$\mathcal{H}(\Omega) = \left\{u \in H_0^1(\Omega)\colon u(x_1, \dots, x_i, \dots, x_n) = u(x_1, \dots, -x_i, \dots, x_n) \text{ for each } i = 1, \dots, n-1\right\}.$$

  Also, $\mathcal{H}(\mathbb{R}^n)$ is a subspace of $D^{1,2}(\mathbb{R}^n)$ defined similarly.
- For any $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ is the open ball in $\mathbb{R}^n$ of radius r centered at x.
- $|B^n| = \pi^{n/2}/\Gamma(n/2+1)$ and $|S^{n-1}| = (2\pi^{n/2})/\Gamma(n/2)$ denotes the Lebesgue measure of the n-dimensional unit ball and $(n-1)$-dimensional unit sphere, respectively.
- We will use big O and small o notations to describe the limit behavior of a certain quantity as $\epsilon \to 0$.
- $C > 0$ is a generic constant that may vary from line to line.

2. Preliminaries and scheme of the proof of Theorem 1.1

2.1. An approximation for the solution

Set $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$ and let

$$U_{\delta,\xi}(x) := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}} \quad \text{for } \delta > 0,\ \xi = (\xi_1, \dots, \xi_{n-1}, 0) \in \mathbb{R}^n, \tag{2.1}$$

which are positive solutions to the problem

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n,\ u \in \mathcal{H}\big(\mathbb{R}^n\big). \tag{2.2}$$

Define also

$$\psi^0_{\delta,\xi}(x) := \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha_n \left(\frac{n-2}{2}\right) \delta^{\frac{n-4}{2}} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{\frac{n}{2}}} \tag{2.3}$$

and

$$\psi^i_{\delta,\xi}(x) := \frac{\partial U_{\delta,\xi}}{\partial \xi_i} = \alpha_n (n-2) \delta^{\frac{n-2}{2}} \frac{(x-\xi)_i}{(\delta^2 + |x-\xi|^2)^{\frac{n}{2}}}, \quad i = 1, \dots, n, \tag{2.4}$$

where $(x-\xi)_i$ is the i-th coordinate of $x - \xi \in \mathbb{R}^n$. Recall that the space spanned by $\psi^0_{\delta,\xi}, \psi^1_{\delta,\xi}, \dots, \psi^n_{\delta,\xi}$ is the set of bounded solutions to the linearized problem of (2.2) at $U_{\delta,\xi}$

$$-\Delta \psi = \left(\frac{n+2}{n-2}\right) \cdot U_{\delta,\xi}^{\frac{4}{n-2}} \psi \quad \text{in } \mathbb{R}^n,\ \psi \in D^{1,2}\big(\mathbb{R}^n\big). \tag{2.5}$$

In particular, the set of bounded solutions to the linear equation (2.5) in the space $\mathcal{H}(\mathbb{R}^n)$ is generated by the only two functions $\psi^0_{\delta,\xi}$ and $\psi^n_{\delta,\xi}$.

Let PW be the projection of the function $W \in D^{1,2}(\mathbb{R}^n)$ onto $H^1_0(\Omega)$, that is,

$$\Delta PW = \Delta W \quad \text{in } \Omega, \qquad PW = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

and k a fixed integer. (See Appendix A.1 for estimation of $PU_{\delta,\xi}$ in terms of $U_{\delta,\xi}$.) We look for a solution to problem (1.6) of the form

$$u = \sum_{i=1}^{k} (-1)^{i+1} PU_{\delta_i,\xi_i} + \phi \in \mathcal{H}(\Omega)$$

where the concentration parameters satisfy

$$\delta_i = \epsilon^{\frac{n-1+2(i-1)}{n-2}} d_i \quad \text{with } d_i > 0, \tag{2.7}$$

the concentration points satisfy

$$\xi_i = \big(\xi_0 + \epsilon t \nu(\xi_0)\big) + \delta_i s_i \nu(\xi_0) \quad \text{with } t > 0 \text{ and } s_i \in \mathbb{R},\ s_k = 0, \tag{2.8}$$

and $\|\phi\|$ is sufficiently small.

For simplicity we write $\mathbf{d} := (d_1, \ldots, d_k) \in (0, +\infty)^k$, $\mathbf{t} := (t, s_1, \ldots, s_{k-1}) \in (0, +\infty) \times \mathbb{R}^{k-1}$, $U_i = U_{\delta_i,\xi_i}$ and

$$V^{\epsilon}_{\mathbf{d},\mathbf{t}} = V_{\mathbf{d},\mathbf{t}} = \sum_{i=1}^{k} (-1)^{i+1} PU_i \in \mathcal{H}(\Omega). \tag{2.9}$$

Also, we define the admissible set Λ by

$$\Lambda = \big\{(\mathbf{d}, \mathbf{t}) \colon \mathbf{d} \in (0, +\infty)^k,\ \mathbf{t} \in (0, +\infty) \times \mathbb{R}^{k-1}\big\}.$$

2.2. Scheme of the proof of Theorem 1.1

First, we rewrite problem (1.6). Let $i^* : L^{\frac{2n}{n+2}}(\Omega) \to H^1_0(\Omega)$ be the adjoint operator to the embedding $i : H^1_0(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, i.e., $i^*(v) = u$ if and only if $\langle u, \phi\rangle = \int_\Omega a v \phi$ for all $\phi \in \mathcal{D}(\Omega)$, or $-\operatorname{div}(a(x)\nabla u) = av$ in Ω and $u = 0$ on $\partial\Omega$. Therefore (1.6) is equivalent to

$$u = i^*\big(|u|^{p-1-\epsilon} u\big), \quad u \in H^1_0(\Omega) \text{ where } p := \frac{n+2}{n-2}. \tag{2.10}$$

For the sake of simplicity, we write $\psi^j_i = \psi^j_{\delta_i,\xi_i}$ with δ_i and ξ_i defined in (2.7) and (2.8). We introduce the spaces

$$\begin{aligned} &K_{\mathbf{d},\mathbf{t}} = \operatorname{span}\big\{P\psi^j_i \colon i = 1, \ldots, k,\ j = 0, n\big\}, \\ &K^\perp_{\mathbf{d},\mathbf{t}} = \big\{\phi \in \mathcal{H}(\Omega) \colon \big\langle \phi, P\psi^j_i\big\rangle = 0 \text{ for } i = 1, \ldots, k,\ j = 0, n\big\}, \end{aligned} \tag{2.11}$$

and the projection operators $\Pi_{\mathbf{d},\mathbf{t}} : \mathcal{H}(\Omega) \to K_{\mathbf{d},\mathbf{t}}$ and $\Pi^\perp_{\mathbf{d},\mathbf{t}} = Id_{\mathcal{H}(\Omega)} - \Pi_{\mathbf{d},\mathbf{t}} : \mathcal{H}(\Omega) \to K^\perp_{\mathbf{d},\mathbf{t}}$.

As usual, we will solve problem (2.10) by finding parameters $(\mathbf{d},\mathbf{t})\in\Lambda$ and a function $\phi\in K^{\perp}_{\mathbf{d},\mathbf{t}}$ such that

$$\Pi^{\perp}_{\mathbf{d},\mathbf{t}}\big(V_{\mathbf{d},\mathbf{t}}+\phi-i^*\big(|V_{\mathbf{d},\mathbf{t}}+\phi|^{p-1-\epsilon}(V_{\mathbf{d},\mathbf{t}}+\phi)\big)\big)=0 \tag{2.12}$$

and

$$\Pi_{\mathbf{d},\mathbf{t}}\big(V_{\mathbf{d},\mathbf{t}}+\phi-i^*\big(|V_{\mathbf{d},\mathbf{t}}+\phi|^{p-1-\epsilon}(V_{\mathbf{d},\mathbf{t}}+\phi)\big)\big)=0. \tag{2.13}$$

The first step is to solve Eq. (2.12). More precisely, if ϵ is small enough for any fixed $(\mathbf{d},\mathbf{t})\in\Lambda$, we will find a function $\phi\in K^{\perp}_{\mathbf{d},\mathbf{t}}$ such that (2.12) holds.

First of all we define the linear operator $L_{\mathbf{d},\mathbf{t}}:K^{\perp}_{\mathbf{d},\mathbf{t}}\to K^{\perp}_{\mathbf{d},\mathbf{t}}$ by

$$L_{\mathbf{d},\mathbf{t}}\phi=\phi-(p-\epsilon)\cdot\Pi^{\perp}_{\mathbf{d},\mathbf{t}}i^*\big(|V_{\mathbf{d},\mathbf{t}}|^{p-1-\epsilon}\phi\big). \tag{2.14}$$

Arguing as in [19, Lemma 3.1] and using Lemma A.5 and Lemma A.7, we prove that it is invertible.

Proposition 2.1. *For any compact subset Λ_0 of Λ, there exist $\epsilon_0>0$ and $c>0$ such that for each $\epsilon\in(0,\epsilon_0)$ and $(\mathbf{d},\mathbf{t})\in\Lambda_0$ the operator $L_{\mathbf{d},\mathbf{t}}$ satisfies*

$$\|L_{\mathbf{d},\mathbf{t}}\phi\|\geqslant c\|\phi\|\quad\textit{for all }\phi\in K^{\perp}_{\mathbf{d},\mathbf{t}}.$$

Secondly, in Section 3 we estimate the error term

$$R_{\mathbf{d},\mathbf{t}}:=\Pi^{\perp}_{\mathbf{d},\mathbf{t}}\big(i^*\big(|V_{\mathbf{d},\mathbf{t}}|^{p-1-\epsilon}V_{\mathbf{d},\mathbf{t}}\big)-V_{\mathbf{d},\mathbf{t}}\big).$$

Lemma 2.2. *It holds true that*

$$\|R_{\mathbf{d},\mathbf{t}}\|=O\big(\epsilon^{\frac{1}{2}\cdot\frac{n+6}{n+2}}\big)=o(\sqrt{\epsilon}).$$

Finally, we use a standard contraction mapping argument (see [19, Section 5]) to solve Eq. (2.12).

Proposition 2.3. *For any compact set Λ_0 of Λ, there is $\epsilon_0>0$ such that for each $\epsilon\in(0,\epsilon_0)$ and $(\mathbf{d},\mathbf{t})\in\Lambda_0$, a unique $\phi^{\epsilon}_{\mathbf{d},\mathbf{t}}\in K^{\perp}_{\mathbf{d},\mathbf{t}}$ exists such that*

$$\Pi^{\perp}_{\mathbf{d},\mathbf{t}}\big(V_{\mathbf{d},\mathbf{t}}+\phi^{\epsilon}_{\mathbf{d},\mathbf{t}}-i^*\big(\big|V_{\mathbf{d},\mathbf{t}}+\phi^{\epsilon}_{\mathbf{d},\mathbf{t}}\big|^{p-1-\epsilon}\big(V_{\mathbf{d},\mathbf{t}}+\phi^{\epsilon}_{\mathbf{d},\mathbf{t}}\big)\big)\big)=0$$

and

$$\big\|\phi^{\epsilon}_{\mathbf{d},\mathbf{t}}\big\|=o(\sqrt{\epsilon}). \tag{2.15}$$

The second step is to solve Eq. (2.13). More precisely, for ϵ small enough we will find $(\mathbf{d},\mathbf{t})$ such that Eq. (2.13) is satisfied.

Let us introduce the energy functional $J_\epsilon:\mathcal{H}(\Omega)\to\mathbb{R}$ defined as

$$J_\epsilon(u)=\frac{1}{2}\int\limits_{\Omega}a(x)|\nabla u|^2\,dx-\frac{1}{p+1-\epsilon}\int\limits_{\Omega}a(x)|u|^{p+1-\epsilon}\,dx,$$

whose critical points are solutions to problem (1.6) and let us define the reduced energy functional $\widetilde{J}_\epsilon : \Lambda \to \mathbb{R}$ by

$$\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t}) = J_\epsilon\big(V_{\mathbf{d},\mathbf{t}} + \phi^\epsilon_{\mathbf{d},\mathbf{t}}\big). \tag{2.16}$$

First of all, arguing as in [19, Proposition 2.2] and using Lemma A.5 and Lemma A.8, we get

Proposition 2.4. *The function $V_{\mathbf{d},\mathbf{t}} + \phi^\epsilon_{\mathbf{d},\mathbf{t}}$ is a critical point of the functional J_ϵ if the point $(\mathbf{d},\mathbf{t})$ is a critical point of the function $\widetilde{J}_\epsilon$.*

Thus, the problem is reduced to search for critical points of $\widetilde{J}_\epsilon$, whose asymptotic expansion is needed. The C^0 and C^1 estimates are carried out in Section 4 and Section 5, respectively, and they read as follows.

Proposition 2.5. *It holds true that*

$$\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t}) = a(\xi_0)[c_1 + c_2\epsilon - c_3\epsilon\log\epsilon] + \epsilon\Phi(\mathbf{d},\mathbf{t}) + o(\epsilon), \tag{2.17}$$

C^1-uniformly on compact sets of Λ. Here, the function $\Phi : \Lambda \to \mathbb{R}$ is defined by

$$\begin{aligned}\Phi(\mathbf{d},\mathbf{t}) := \partial_\nu a(\xi_0)c_4 t + a(\xi_0)\left[c_5\left(\frac{d_1}{2t}\right)^{n-2} + c_6\sum_{i=1}^{k-1}\left(\frac{d_{i+1}}{d_i}\right)^{\frac{n-2}{2}}\frac{1}{(1+s_i^2)^{\frac{n-2}{2}}}\right]\\ - a(\xi_0)c_7\sum_{i=1}^{k}\log d_i\end{aligned} \tag{2.18}$$

where c_i's are all positive constants.

Finally, we can prove Theorem 1.1 by showing that $\widetilde{J}_\epsilon$ has a critical point in Λ.

Proof of Theorem 1.1. The fact that $\partial_\nu a(\xi_0)$ is positive (see assumption (a2)) ensures that the function Φ defined in (2.18) has a non-degenerate critical point of min–max type (a minimum in t and d_i's and a maximum in s_i's) which is stable under C^1-perturbations (see page 7 in [19]). Therefore, by Proposition 2.5, we deduce that if ϵ is small enough the function $\widetilde{J}_\epsilon$ has a critical point. The claim follows by Proposition 2.4. □

3. Estimate of the error term $R_{\mathbf{d},\mathbf{t}}$

This section is devoted to prove Lemma 2.2. For sake of brevity, we drop the subscript $\mathbf{d},\mathbf{t}$.
Using the definition of V in (2.9), we decompose first

$$\begin{aligned}R &:= \Pi^\perp\big(i^*\big(|V|^{p-1-\epsilon}V\big) - V\big)\\ &= \Pi^\perp\left(i^*\left(|V|^{p-1-\epsilon}V - \sum_{i=1}^{k}(-1)^{i+1}U_i^p + \sum_{i=1}^{k}(-1)^{i+1}\nabla\log a\cdot\nabla PU_i\right)\right)\\ &= \Pi^\perp\big(i^*\big(|V|^{p-1-\epsilon}V - |V|^{p-1}V\big)\big) + \Pi^\perp\left(i^*\left(|V|^{p-1}V - \sum_{i=1}^{k}(-1)^{i+1}PU_i^p\right)\right)\end{aligned}$$

$$+\sum_{i=1}^{k}(-1)^{i+1}\Pi^{\perp}\big(i^*\big(PU_i^p-U_i^p\big)\big)+\sum_{i=1}^{k}(-1)^{i+1}\Pi^{\perp}\big(i^*(\nabla\log a\cdot\nabla PU_i)\big)$$

$$=: R_1+R_2+\sum_{i=1}^{k}R_3^i+\sum_{i=1}^{k}R_4^i. \tag{3.1}$$

Estimate of R_1. Set $\tilde{p}:=\frac{2n}{n+2}$. By the boundedness of $i^*: L^{\frac{2n}{n+2}}(\Omega)\to H_0^1(\Omega)$, the mean value theorem and

$$|u|^q\big|\log|u|\big|=O\big(|u|^{q+\sigma}+|u|^{q-\sigma}\big)\quad\text{for any } q>1 \text{ and small } \sigma>0, \tag{3.2}$$

it holds

$$\begin{aligned}\|R_1\|^{\tilde{p}}&\leqslant\big\|i^*\big(\big(|V|^{p-1-\epsilon}-|V|^{p-1}\big)V\big)\big\|^{\tilde{p}}\leqslant C\big\|\big(|V|^{p-1-\epsilon}-|V|^{p}-1\big)V\big\|_{L^{\tilde{p}}(\Omega)}^{\tilde{p}}\\&=C\epsilon^{\tilde{p}}\int_\Omega\big|\log|V|\big|^{\tilde{p}}\cdot\sup_{\theta\in[0,1]}|V|^{(p-\theta\epsilon)\tilde{p}}\leqslant C\epsilon^{\tilde{p}}\int_\Omega\big(|V|^{p\tilde{p}-\sigma'}+|V|^{p\tilde{p}+\sigma'}\big)\\&\leqslant C\epsilon^{\tilde{p}}\sum_{i=1}^{k}\int_\Omega\big(U_i^{\frac{2n}{n-2}-\sigma'}+U_i^{\frac{2n}{n-2}+\sigma'}\big)=O\big(\epsilon^{\tilde{p}-\sigma''}\big)\end{aligned}$$

where σ' and $\sigma''>0$ are constants small enough. Hence

$$\|R_1\|=O\big(\epsilon^{1-\sigma}\big)\quad\text{for any small } \sigma>0. \tag{3.3}$$

Estimate of R_2. Let $f(s):=|s|^{p-1}s$ for $s\in\mathbb{R}$ and choose $\rho>0$ sufficiently small so that $\overline{B(\xi_k,\rho\epsilon)}\subset\Omega$. Following the approach introduced in [19], we divide the domain Ω into $k+1$ mutually disjoint subsets, namely,

$$\Omega=\left(\bigcup_{l=1}^{k}A_l\right)\cup\big(\Omega\setminus B(\xi_k,\rho\epsilon)\big)$$

where A_l's are annuli defined as

$$A_l=B(\xi_k,\sqrt{\delta_{l-1}\delta_l})\setminus B(\xi_k,\sqrt{\delta_l\delta_{l+1}})\quad\text{with } \delta_0=\frac{(\epsilon\rho)^2}{\delta_1},\ \delta_{k+1}=0. \tag{3.4}$$

Then by the mean value theorem,

$$\begin{aligned}\|R_2\|^{\tilde{p}}&\leqslant C\left\||V|^{p-1}V-\sum_{i=1}^{k}(-1)^{i+1}PU_i^p\right\|_{L^{\tilde{p}}(\Omega)}^{\tilde{p}}\\&=C\sum_{l=1}^{k}\int_{A_l}\left||V|^{p-1}V-\sum_{i=1}^{k}(-1)^{i+1}PU_i^p\right|^{\tilde{p}}+O\big(\epsilon^{\frac{n}{n-2}}\big)\end{aligned}$$

$$= C\sum_{l=1}^{k}\int_{A_l}\left|f\left((-1)^{l+1}PU_l+\sum_{i\neq l}(-1)^{i+1}PU_i\right)-f\left((-1)^{l+1}PU_l\right)\right|^{\tilde p}+O\left(\epsilon^{\frac{n}{n-2}}\right)$$

$$= O\left(\sum_{l=1}^{k-1}\int_{A_l}U_l^{(p-1)\tilde p}U_{l+1}^{\tilde p}\right)+O\left(\sum_{l=2}^{k}\int_{A_l}U_l^{(p-1)\tilde p}U_{l-1}^{\tilde p}\right)+O\left(\epsilon^{\frac{n}{n-2}}\right).$$

By (4.19) and (4.13) we deduce

$$\begin{aligned}\int_{A_l}U_l^{(p-1)\tilde p}U_{l+1}^{\tilde p} &\leqslant \left\|U_l^{\frac{8n}{n^2-4}}U_{l+1}^{\frac{8n}{(n+2)^2}}\right\|_{L^{\frac{(n+2)^2}{8n}}(\Omega)}\left\|U_{l+1}^{\frac{2n(n-2)}{(n+2)^2}}\right\|_{L^{\frac{(n+2)^2}{(n-2)^2}}(\Omega)}\\ &= \left(\int_{A_l}U_l^pU_{l+1}\right)^{\frac{8n}{(n+2)^2}}\cdot\left(\int_{A_l}U_{l+1}^{p+1}\right)^{\frac{(n-2)^2}{(n+2)^2}}\\ &= O\left(\epsilon^{\frac{8n}{(n+2)^2}}\right)\cdot O\left(\epsilon^{\frac{n(n-2)}{(n+2)^2}}\right)=O\left(\epsilon^{\frac{n(n+6)}{(n+2)^2}}\right)\end{aligned}$$

for $l=1,\dots,k-1$, and similarly

$$\int_{A_l}U_l^{(p-1)\tilde p}U_{l-1}=O\left(\epsilon^{\frac{n(n+6)}{(n+2)^2}}\right)$$

for $l=2,\dots,l$. Therefore we obtain

$$\|R_2\|=O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+6}{n+2}}\right)+O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+2}{n-2}}\right). \tag{3.5}$$

Estimate of R_3. By the mean value theorem again,

$$\left\|R_3^i\right\|^{\tilde p}\leqslant C\left\|PU_i^p-U_i^p\right\|_{L^{\tilde p}(\Omega)}^{\tilde p}\leqslant C\int_{\Omega}\left(U_i^{(p-1)\tilde p}|PU_i-U_i|^{\tilde p}+|PU_i-U_i|^{p+1}\right).$$

Arguing as in the proof of Lemma A.3, we get

$$\int_{\Omega}|PU_i-U_i|^{p+1}=O\left(\epsilon^{\frac{n}{n-2}}\right), \tag{3.6}$$

and

$$\begin{aligned}&\int_{\Omega}U_i^{(p-1)\tilde p}|PU_i-U_i|^{\tilde p}\\ &\leqslant \int_{B(\xi_i,\rho\epsilon)}U_i^{(p-1)\tilde p}|PU_i-U_i|^{\tilde p}+\left(\int_{\Omega\setminus B(\xi_k,\rho\epsilon)}U_i^{p+1}\right)^{\frac{4}{n+2}}\left(\int_{\Omega}|PU_i-U_i|^{p+1}\right)^{\frac{n-2}{n+2}}\end{aligned}$$

$$\leqslant \left(\int\limits_{B(\xi_i,\rho\epsilon)} U_i^p |PU_i - U_i| \right)^{\frac{8n}{(n+2)^2}} \cdot \left(\int\limits_{\Omega} |PU_i - U_i|^{p+1} \right)^{\frac{(n-2)^2}{(n+2)^2}} + O\left(\epsilon^{\frac{4n}{n^2-4}}\right) \cdot O\left(\epsilon^{\frac{n}{n+2}}\right)$$

$$= O\left(\epsilon^{\frac{n(n+6)}{(n+2)^2}}\right) + O\left(\epsilon^{\frac{n(n+2)}{n^2-4}}\right)$$

(see [1, Lemma C.2(64)] for the estimate of the term $\int_{B(\xi_i,\rho\epsilon)} U_i^p |PU_i - U_i|$). Thus

$$\|R_3\| = O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+6}{n+2}}\right) + O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+2}{n-2}}\right). \tag{3.7}$$

Estimate of R_4. Lemma A.10 yields

$$\|R_4\| \leqslant C\|\nabla PU_i\|_{L^{\tilde{p}}(\Omega)} = O(\epsilon) \tag{3.8}$$

In conclusion, from (3.1), (3.3), (3.5), (3.7) and (3.8), we obtain

$$\|R\| = O\left(\epsilon^{1-\sigma}\right) + O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+6}{n+2}}\right) + O\left(\epsilon^{\frac{1}{2}\cdot\frac{n-2}{n-2}}\right) + O(\epsilon) = O\left(\epsilon^{\frac{1}{2}\cdot\frac{n+6}{n+2}}\right).$$

This completes the proof of Proposition 2.3.

4. Energy expansion: The C^0-estimates

The main task of this section is to prove that estimate (2.17) holds in the C^0-sense. We recall that the function $V_{\mathbf{d},\mathbf{t}}$ is defined in (2.9) and the function $\phi_{\mathbf{d},\mathbf{t}}^\epsilon$ is given in Proposition 2.3. For the sake of brevity, we denote $V = V_{\mathbf{d},\mathbf{t}}$ and $\phi = \phi_{\mathbf{d},\mathbf{t}}^\epsilon$. We decompose the reduced functional into three parts

$$\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t}) = \left(J_\epsilon\left(V_{\mathbf{d},\mathbf{t}} + \phi_{\mathbf{d},\mathbf{t}}^\epsilon\right) - J_\epsilon(V_{\mathbf{d},\mathbf{t}})\right) + \left(\frac{1}{2}\int\limits_\Omega a(x)|\nabla V_{\mathbf{d},\mathbf{t}}|^2\,dx - \frac{1}{p+1}\int\limits_\Omega a(x)|V_{\mathbf{d},\mathbf{t}}|^{p+1}\,dx\right)$$
$$+ \left(\frac{1}{p+1}\int\limits_\Omega a(x)|V_{\mathbf{d},\mathbf{t}}|^{p+1}\,dx - \frac{1}{p+1-\epsilon}\int\limits_\Omega a(x)|V_{\mathbf{d},\mathbf{t}}|^{p+1-\epsilon}\,dx\right)$$

and we estimate each of them. The C^0-estimate will follow by the three lemmata: Lemma 4.1, Lemma 4.2 and Lemma 4.3.

Lemma 4.1. *It holds true that*

$$J_\epsilon(V+\phi) - J_\epsilon(V) = o(\epsilon). \tag{4.1}$$

Proof. Using Taylor's theorem and the fact that $J_\epsilon'(V+\phi)[\phi] = 0$, we get

$$J_\epsilon(V+\phi) - J_\epsilon(V) = -\int\limits_0^1 t J_\epsilon''(V+t\phi)[\phi,\phi]\,dt.$$

On the other hand, since $\|\phi\| = o(\sqrt{\epsilon})$,

$$\left|J_\epsilon''(V+t\phi)[\phi,\phi]\right| \leqslant C\left(\int_\Omega a|\nabla\phi|^2 + \sum_{i=1}^k \int_\Omega aU_i^{p-1-\epsilon}\phi^2 + \int_\Omega a|\phi|^{p+1-\epsilon}\right) = o(\epsilon)$$

for some $C > 0$. Therefore (4.1) follows. □

It is useful to introduce the following constants:

$$a_1 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n}\,dy, \tag{4.2}$$

$$a_2 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}}\,dy, \tag{4.3}$$

$$a_3 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} \log \frac{\alpha_n}{(1+|y|^2)^{\frac{n-2}{2}}}\,dy. \tag{4.4}$$

Here, $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$.

Lemma 4.2. *It holds true that*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega a(x)|\nabla V_{\mathbf{d},\mathbf{t}}|^2\,dx - \frac{1}{p+1}\int_\Omega a(x)|V_{\mathbf{d},\mathbf{t}}|^{p+1}\,dx \\
&\quad = \left(\frac{1}{2} - \frac{1}{p+1}\right) k a_1 \big(a(\xi_0) + \partial_\nu a(\xi_0) t\epsilon\big) \\
&\qquad + a(\xi_0)\left[\frac{a_2}{2}\left(\frac{d_1}{2t}\right)^{n-2} + \sum_{i=1}^{k-1}\left(\frac{d_{i+1}}{d_i}\right)^{\frac{n-2}{2}} \frac{\alpha_n^{p+1}|B^n|}{(1+s_i^2)^{\frac{n-2}{2}}}\right]\epsilon + o(\epsilon).
\end{aligned} \tag{4.5}$$

Proof. Using the definition of the annuli A_i $(i = 1, \ldots, k)$ in (3.4), we write

$$\begin{aligned}
&\frac{1}{2}\int_\Omega a|\nabla V|^2 \\
&\quad = \frac{1}{2}\sum_{l=1}^k \int_\Omega a|\nabla PU_l|^2 + \sum_{l<i}(-1)^{l+i}\int_\Omega a\nabla PU_l \cdot \nabla PU_i \\
&\quad = \frac{1}{2}\sum_{l=1}^k \left[\int_{A_l} aU_l^{p+1} + \int_\Omega aU_l^p(PU_l - U_l) + \int_{\Omega\setminus A_l} aU_l^{p+1} - \int_\Omega (\nabla a\cdot\nabla PU_l)PU_l\right] \\
&\qquad + \sum_{l<i}(-1)^{l+i}\left[\int_{A_l} aU_l^p U_i + \int_\Omega aU_l^p(PU_i - U_i) + \int_{\Omega\setminus A_l} aU_l^p U_i - \int_\Omega (\nabla a\cdot\nabla PU_l)PU_i\right]
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\frac{1}{p+1}\int_\Omega a|V|^{p+1}\,dx \\
&\quad = \frac{1}{p+1}\sum_{l=1}^k\int_{A_l} a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1} + \frac{1}{p+1}\int_{\Omega\setminus B(\xi_k,\rho\epsilon)} a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1} \\
&\quad = \frac{1}{p+1}\sum_{l=1}^k\int_{A_l} a\left(\left|(-1)^{l+1}PU_l+\sum_{i\neq l}(-1)^{i+1}PU_i\right|^{p+1} - U_l^{p+1}\right) + \frac{1}{p+1}\sum_{l=1}^k\int_{A_l} aU_l^{p+1} + o(\epsilon) \\
&\quad = \sum_{l=1}^k\left[\frac{1}{p+1}\int_{A_l} aU_l^{p+1} + \int_{A_l} aU_l^p(PU_l-U_l)\right] + \sum_{i\neq l}(-1)^{i+l}\left[\int_{A_l} aU_l^pU_i + \int_{A_l} aU_l^p(PU_i-U_i)\right] \\
&\qquad + p\int_0^1(1-\theta)\int_{A_l} a\left|(-1)^{l+1}U_l + \theta\left[(-1)^{l+1}(PU_l-U_l)+\sum_{i\neq l}(-1)^{i+1}PU_i\right]\right|^{p-1} \\
&\qquad \times\left((-1)^{l+1}(PU_l-U_l)+\sum_{i\neq l}(-1)^{i+1}PU_i\right)^2 dx\,d\theta + o(\epsilon).
\end{aligned} \tag{4.7}$$

First of all, we claim that

$$\sum_{i\neq l}(-1)^{i+l}\int_{A_l} aU_l^pU_i = 2\sum_{l<i}(-1)^{l+i}\int_{A_l} aU_l^pU_i + o(\epsilon). \tag{4.8}$$

Indeed, suppose $l>i$. By the fact that $-\Delta PU_i = U_i^p$ in Ω and $PU_i=0$ on $\partial\Omega$, it follows that

$$\begin{aligned}
\int_{A_l} U_l^pU_i &= \int_\Omega \nabla PU_l\cdot\nabla PU_i - \int_\Omega U_l^p(PU_i-U_i) - \int_{\Omega\setminus A_l} U_l^pU_i \\
&= \int_{A_i} U_i^pU_l + \int_\Omega U_i^p(PU_l-U_l) + \int_{\Omega\setminus A_i} U_i^pU_l - \int_\Omega U_l^p(PU_i-U_i) - \int_{\Omega\setminus A_l} U_l^pU_i.
\end{aligned} \tag{4.9}$$

By Lemmas A.1 and A.2 (see also (4.18)) we deduce

$$\int_\Omega U_i^p(PU_l-U_l),\ \int_\Omega U_l^p(PU_i-U_i) = o(\epsilon),$$

$$\begin{aligned}
\int_{\Omega\setminus A_l} U_i^pU_l &\leqslant \left(\frac{\delta_l}{\delta_i}\right)^{\frac{n-2}{2}}\left[\int_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})^c} + \int_{B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})}\right]\frac{\alpha_n^{p+1}}{(1+|y-s_i\nu(\xi_0)|^2)^{\frac{n+2}{2}}}\frac{1}{|y-(\delta_l/\delta_i)s_l\nu(\xi_0)|^{n-2}}\,dy \\
&= o(\epsilon)
\end{aligned} \tag{4.10}$$

and

$$\int\limits_{\Omega\setminus A_l} U_l^p U_i$$

$$\leqslant \Bigg[\int\limits_{B(\xi_k,\sqrt{\delta_{l-1}\delta_l})^c} + \int\limits_{B(\xi_k,\sqrt{\delta_l\delta_{l+1}})}\Bigg] \frac{\alpha_n^{p+1}\delta_l^{\frac{n+2}{2}}}{(\delta_l^2+|x-\xi_k-s_l\delta_l\nu(\xi_0)|^2)^{\frac{n+2}{2}}} \frac{\delta_i^{\frac{n-2}{2}}}{(\delta_i^2+|x-\xi_k-s_i\delta_i\nu(\xi_0)|^2)^{\frac{n-2}{2}}}\,dx$$

$$\leqslant \alpha_n^{p+1}\left(\frac{\delta_l}{\delta_i}\right)^{\frac{n-2}{2}} \Bigg[\int\limits_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})^c} + \int\limits_{B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})}\Bigg] \frac{1}{(1+|y-s_l\nu(\xi_0)|^2)^{\frac{n+2}{2}}}\,dy = o(\epsilon).$$

Therefore, Eq. (4.9) can be rewritten as

$$\int\limits_{A_l} U_l^p U_i = \int\limits_{A_i} U_i^p U_l + o(\epsilon). \tag{4.11}$$

Moreover, we have the estimates

$$\int\limits_{A_l} \big(a(x)-a(\xi_0)\big)U_l^p U_i\,dx,\ \int\limits_{A_i} \big(a(x)-a(\xi_0)\big)U_i^p U_l\,dx = o(\epsilon). \tag{4.12}$$

By (4.11) and (4.12), we deduce that

$$\int\limits_{A_l} aU_l^p U_i = \Bigg[a(\xi_0)\int\limits_{A_l} U_l^p U_i + \int\limits_{A_l}\big(a(x)-a(\xi_0)\big)U_l^p U_i\,dx\Bigg]$$
$$= a(\xi_0)\int\limits_{A_l} U_l^p U_i + o(\epsilon) = a(\xi_0)\int\limits_{A_l} U_i^p U_l + o(\epsilon) = \int\limits_{A_i} aU_i^p U_l + o(\epsilon),$$

which in particular implies (4.8).

Next, we claim that the term $I := p\int_0^1(1-\theta)\int_{A_l}\cdots$ in (4.7) is of order $o(\epsilon)$. Indeed, we first remark that

$$\int\limits_{A_l} |PU_l - U_l|^{p+1} = O\big(\epsilon^{\frac{n}{n-2}}\big) \quad \text{and} \quad \int\limits_{A_l} U_i^{p+1} = O\big(\epsilon^{\frac{n}{n-2}}\big) \quad \text{for } i\neq l, \tag{4.13}$$

where the first equality is obtained in the proof of Lemma A.3 and the second one is deduced in (6.19) of [19]. Moreover, by (4.18) and (4.19), we deduce

$$\int\limits_{A_l} U_l^p|PU_l-U_l|,\ \int\limits_{A_l} U_l^p U_i = O(\epsilon) \quad \text{if } i\neq l.$$

By these estimates, we get

$$
\begin{aligned}
I &\leqslant C\Bigg[\int\limits_{A_l} U_l^{p-1}|PU_l-U_l|^2+\int\limits_{A_l}|PU_l-U_l|^{p+1}+\sum_{i\neq l}\int\limits_{A_l}U_i^{p-1}|PU_l-U_l|^2\\
&\quad+\sum_{i\neq l}\int\limits_{A_l}|PU_l-U_l|^{p-1}U_i^2\Bigg]+C\Bigg[\sum_{i\neq l}\int\limits_{A_l}U_l^{p-1}U_i^2+\sum_{i\neq l}\sum_{j\neq l}\int\limits_{A_l}U_i^{p-1}U_j^2\Bigg]\\
&\leqslant C\Bigg[\Bigg(\int\limits_{A_l}U_l^p|PU_l-U_l|\Bigg)^{\frac{4}{n+2}}\Bigg(\int\limits_{A_l}|PU_l-U_l|^{p+1}\Bigg)^{\frac{n-2}{n+2}}+\int\limits_{A_l}|PU_l-U_l|^{p+1}\\
&\quad+\sum_{i\neq l}\Bigg(\int\limits_{A_l}U_i^{p+1}\Bigg)^{\frac{2}{n}}\Bigg(\int\limits_{A_l}|PU_l-U_l|^{p+1}\Bigg)^{\frac{n-2}{n}}+\sum_{i\neq l}\Bigg(\int\limits_{A_l}|PU_l-U_l|^{p+1}\Bigg)^{\frac{2}{n}}\Bigg(\int\limits_{A_l}U_i^{p+1}\Bigg)^{\frac{n-2}{n}}\Bigg]\\
&\quad+C\Bigg[\Bigg(\int\limits_{A_l}U_l^pU_i\Bigg)^{\frac{4}{n+2}}\Bigg(\int\limits_{A_l}U_i^{p+1}\Bigg)^{\frac{n-2}{n+2}}+\sum_{i\neq l}\sum_{j\neq l}\Bigg(\int\limits_{A_l}U_i^{p+1}\Bigg)^{\frac{2}{n}}\Bigg(\int\limits_{A_l}U_j^{p+1}\Bigg)^{\frac{n-2}{n}}\Bigg]\\
&\leqslant C\big[\epsilon^{\frac{n+4}{n+2}}+\epsilon^{\frac{n}{n-2}}\big]+C\big[\epsilon^{\frac{n+4}{n+2}}+\epsilon^{\frac{n}{n-2}}\big]=O\big(\epsilon^{\frac{n+4}{n+2}}\big)=o(\epsilon)
\end{aligned}
\tag{4.14}
$$

for some constant $C>0$.

Finally, by (4.12), (4.8) and (4.14), we get

$$
\begin{aligned}
\frac{1}{p+1}\int\limits_{\Omega}a|V|^{p+1}\,dx &= \sum_{l=1}^{k}\Bigg[\frac{1}{p+1}\int\limits_{A_l}aU_l^{p+1}+\int\limits_{A_l}aU_l^p(PU_l-U_l)\Bigg]\\
&\quad+2\sum_{i<l}(-1)^{i+l}\int\limits_{A_l}aU_l^pU_i+o(\epsilon).
\end{aligned}
\tag{4.15}
$$

Moreover, by (4.6), (4.10), (4.12), (4.13), (4.15) and the estimate

$$
\int\limits_{\Omega}(\nabla a\cdot\nabla PU_l)PU_i=o(\epsilon)\quad\text{for } i,l=1,\dots,k,
$$

which is easily deduced by Lemma A.9, we find that

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega}a|\nabla V|^2-\frac{1}{p+1}\int\limits_{\Omega}a|V|^{p+1}\,dx\\
&\quad=\bigg(\frac{1}{2}-\frac{1}{p+1}\bigg)\sum_{l=1}^{k}\int\limits_{A_l}aU_l^{p+1}-\frac{1}{2}\sum_{l=1}^{k}\int\limits_{\Omega}aU_l^p(PU_l-U_l)\\
&\qquad+\sum_{l<i}(-1)^{l+i+1}\int\limits_{A_l}aU_l^pU_i+o(\epsilon).
\end{aligned}
\tag{4.16}
$$

Now, we estimate each term in the right-hand side of the above equality. Firstly, we write the first term as

$$\int_{A_l} aU_l^{p+1} = a(\xi_0)\int_{A_l} U_l^{p+1} + \int_{A_l} \big(a(x)-a(\xi_0)\big)U_l^{p+1}\,dx$$

and then we estimate

$$\begin{aligned}
a(\xi_0)\int_{A_l} U_l^{p+1} &= a(\xi_0)a_1 - a(\xi_0)\alpha_n^{p+1}\Bigg[\int_{B(\xi_k,\sqrt{\frac{\delta_{l-1}}{\delta_l}})^c} + \int_{B(\xi_k,\sqrt{\frac{\delta_l}{\delta_{l+1}}})}\Bigg]\frac{\delta_l^n}{(\delta_l^2+|x-\xi_k-\delta_l s_l\nu(\xi_0)|^2)^n}\,dx \\
&= a(\xi_0)a_1 - a(\xi_0)\alpha_n^{p+1}\Bigg[\int_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})^c} + \int_{B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})}\Bigg]\frac{dy}{(1+|y-s_l\nu(\xi_0)|^2)^n} \\
&= a(\xi_0)a_1 + o(\epsilon)
\end{aligned}$$

and

$$\int_{A_l} \big(a(x)-a(\xi_0)\big)U_l^{p+1}\,dx = \partial_\nu a(\xi_0)a_1 t\epsilon + o(\epsilon)$$

(cf. [1, Lemma C.1]). This shows that

$$\int_{A_l} aU_l^{p+1} = a(\xi_0)a_1 + \partial_\nu a(\xi_0)a_1 t\epsilon + o(\epsilon). \tag{4.17}$$

Secondly, by Lemma A.1 and Lemma A.2 (using the mean value theorem) we deduce

$$\begin{aligned}
&\int_{\Omega} aU_l^p(PU_l - U_l) \\
&\quad= -\alpha_n\delta_l^{\frac{n-2}{2}}\int_{\Omega} aU_l^p H(\cdot,\xi_l) + o(\epsilon) \\
&\quad= -\alpha_n^{p+1}\delta_l^{n-2}a(\xi_0)\int_{B(0,\rho\epsilon\delta_l^{-1})}\frac{1}{(1+|y|^2)^{\frac{n+2}{2}}}\left[\frac{1}{(2\epsilon t)^{n-2}} + O\left(\frac{\delta_l(1+|y|)}{\epsilon^{n-1}}\right)\right]dy + o(\epsilon) \\
&\quad= -\delta_{l1}\cdot\left\{a(\xi_0)a_2\left(\frac{d_1}{2t}\right)^{n-2}\right\}\cdot\epsilon + o(\epsilon)
\end{aligned} \tag{4.18}$$

where δ_{ij} is the Kronecker delta (cf. [1, Lemma C.2(64)]).

Finally, for $l < i$, we get

$$\begin{aligned}
&\int_{A_l} aU_l^p U_i \\
&\quad= a(\xi_0)\int_{A_l} U_l^p U_i + \int_{A_l}\big(a(x)-a(\xi_0)\big)U_l^p U_i\,dx
\end{aligned}$$

$$= \left(\frac{\delta_i}{\delta_l}\right)^{\frac{n-2}{2}} \int\limits_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})\setminus B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})} \frac{a(\xi_0)\alpha_n^{p+1}}{(1+|y-s_l\nu(\xi_0)|^2)^{\frac{n+2}{2}}} \frac{dy}{[(\delta_i/\delta_l)^2+|y-(\delta_i/\delta_l)s_i\nu(\xi_0)|^2]^{\frac{n-2}{2}}}$$
$$+ o(\epsilon)$$
$$= \delta_{i(l+1)} a(\xi_0)\left(\frac{d_{l+1}}{d_l}\right)^{\frac{n-2}{2}} F(s_l)\epsilon + o(\epsilon). \tag{4.19}$$

Here

$$F(s) := \alpha_n^{p+1} \int\limits_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} \frac{1}{|y+s\nu(\xi_0)|^{n-2}}\,dy = \alpha_n^{p+1}|B^n|\frac{1}{(1+s^2)^{\frac{n-2}{2}}}. \tag{4.20}$$

The last equality follows from the fact that $U = U_{1,0}$ solves the equation $-\Delta U = U^p$ in $\mathbb{R}^n$ and so it can be rewritten using the Green's representation formula

$$U(x) = \frac{1}{n(n-2)|B^n|} \int\limits_{\mathbb{R}^n} U^p(y)\frac{1}{|y-x|^{n-2}}\,dy,$$

which implies $F(s) = \alpha_n^p |B^n| U(s\nu(\xi_0))$.

By combining (4.17), (4.18) and (4.19) with (4.16), estimate (4.5) follows. □

Lemma 4.3. *It holds true that*

$$\frac{1}{p+1}\int\limits_\Omega a|V|^{p+1} - \frac{1}{p+1-\epsilon}\int\limits_\Omega a|V|^{p+1-\epsilon}$$
$$= -a(\xi_0)\frac{k(n+k-2)}{2(p+1)} \cdot a_1\epsilon\log\epsilon$$
$$+ a(\xi_0)\left[\frac{ka_3}{p+1} - \frac{ka_1}{(p+1)^2} - \frac{(n-2)^2}{4n}\cdot a_1\sum_{i=1}^k \log d_i\right]\epsilon + o(\epsilon). \tag{4.21}$$

Proof. By the Taylor expansion we deduce

$$\frac{1}{p+1}\int\limits_\Omega a|V|^{p+1} - \frac{1}{p+1-\epsilon}\int\limits_\Omega a|V|^{p+1-\epsilon}$$
$$= \left[\frac{1}{p+1}\int\limits_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|\right.$$
$$\left. - \frac{1}{(p+1)^2}\int\limits_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1}\right]\epsilon + o(\epsilon). \tag{4.22}$$

Arguing as in the proof of the previous lemma we get

$$\int_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1} = a(\xi_0)ka_1 + o(1). \tag{4.23}$$

Moreover, we have

$$\begin{aligned}
&\int_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right| \\
&\quad= \sum_{j=1}^k\int_{A_j} a\left|\sum_{i=1}^k(-1)^{i+1}U_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}U_i\right| + o(1) \\
&\quad= -a(\xi_0)\frac{k(n+k-2)}{2(p+1)}\cdot a_1\log\epsilon + a(\xi_0)\left[\frac{ka_3}{p+1} - \frac{(n-2)^2}{4n}\cdot a_1\sum_{i=1}^k\log d_i\right] + o(1).
\end{aligned} \tag{4.24}$$

By combining (4.22), (4.23) and (4.24), (4.21) follows.

Let us prove (4.24).

To get the first equality, it is sufficient to show that

$$\begin{aligned}
&\int_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}PU_i\right| \\
&\quad= \int_\Omega a\left|\sum_{i=1}^k(-1)^{i+1}U_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}U_i\right| + o(1)
\end{aligned} \tag{4.25}$$

and

$$\int_{\Omega\setminus B(\xi_k,\rho\epsilon)} a\left|\sum_{i=1}^k(-1)^{i+1}U_i\right|^{p+1}\log\left|\sum_{i=1}^k(-1)^{i+1}U_i\right| = o(1). \tag{4.26}$$

If we write

$$V := \sum_{i=1}^k(-1)^{i+1}PU_i, \qquad E := \sum_{i=1}^k(-1)^{i+1}(U_i - PU_i) \quad \text{and} \quad g(s) := |s|^{p+1}\log|s| \quad \text{for } s\neq 0,$$

then we see that

$$\begin{aligned}
&\int_\Omega a\cdot\big|g(V+E)-g(V)\big|\,dx \\
&\quad\leqslant C\int_\Omega\int_0^1\left(|V+\theta E|^{p+\sigma} + |V+\theta E|^{p-\sigma} + |V+\theta E|^p\right)\cdot|E|\,d\theta\,dx \quad \big(\text{by (a1) and (3.2)}\big)
\end{aligned}$$

$$\leqslant C\left[\int_\Omega \left(|V|^{p+\sigma}+|V|^{p-\sigma}+|V|^p\right)\cdot|E|\,dx+\int_\Omega \left(|E|^{p+\sigma}+|E|^{p-\sigma}+|E|^p\right)dx\right]$$
$$=o(1)\quad \text{(by the Hölder inequality and Lemma A.3)}$$

for some constant $C>0$. This proves (4.25).

Furthermore, denoting $\widetilde{V}:=\sum_{i=1}^k(-1)^{i+1}U_i$, we have

$$\left|\int_{\Omega\setminus B(\xi_k,\rho\epsilon)} ag(\widetilde{V})\right|\leqslant C\int_{\Omega\setminus B(\xi_k,\rho\epsilon)}\left(|\widetilde{V}|^{p+\sigma}+|\widetilde{V}|^{p-\sigma}\right)\leqslant C\sum_{i=1}^k\int_{\Omega\setminus B(\xi_k,\rho\epsilon)}\left(U_i^{p+\sigma}+U_i^{p-\sigma}\right)$$
$$\leqslant C\sum_{i=1}^k\left(\frac{\delta_i}{\epsilon}\right)^n\left[\left(\frac{\delta_i}{\epsilon^2}\right)^{\frac{n-2}{2}\sigma}+\left(\frac{\delta_i}{\epsilon^2}\right)^{-\frac{n-2}{2}\sigma}\right]=o(1),$$

which implies (4.26).

Finally, the second equality can be obtained as in (6.39) in [19]. □

From Lemmas 4.1, 4.2 and 4.3, we conclude that estimate (2.17) is true in the C^0-sense.

5. Energy expansion: The C^1-estimates

In this section, we will deduce that (2.17) holds C^1-uniformly on compact subsets of the admissible set Λ.

Let us denote again $V=V_{\mathbf{d},\mathbf{t}}$ and $\phi=\phi^\epsilon_{\mathbf{d},\mathbf{t}}$ for the sake of simplicity. We need to prove that for $\mathbf{d}:=(d_1,\dots,d_k)\in(0,+\infty)^k$ and $\mathbf{t}:=(t,s_1,\dots,s_{k-1})\in(0,+\infty)\times\mathbb{R}^{k-1}$,

$$\partial_r\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t})=\partial_r\Phi(\mathbf{d},\mathbf{t})\epsilon+o(\epsilon)\tag{5.1}$$

C^0-uniformly on compact sets of Λ where $\widetilde{J}_\epsilon$ and Φ are defined in (2.16) and (2.18), respectively, and r is one of $d_1,\dots,d_k,t,s_1,\dots,s_{k-2}$ and s_{k-1}.

5.1. The case $r=d_l$ $(l=1,\dots,k)$ or $r=s_l$ $(l=1,\dots,k-1)$

We decompose $\partial_r\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t})$ into

$$\partial_r\widetilde{J}_\epsilon(\mathbf{d},\mathbf{t})=J'_\epsilon(V)(\partial_rV)+\left[J'_\epsilon(V+\phi)-J'_\epsilon(V)\right]\partial_rV+J'_\epsilon(V+\phi)(\partial_r\phi)$$

and estimate each term.

Lemma 5.1. *It is satisfied that*

$$J'_\epsilon(V)(\partial_rV)=\partial_r\Phi(\mathbf{d},\mathbf{t})\epsilon+o(\epsilon)\quad \textit{for } r=d_1,\dots,d_k,s_1,\dots,s_{k-1}.\tag{5.2}$$

Proof. Set $p=(n+2)/(n-2)$. We split $J'_\epsilon(V)(\partial_rV)$ as

$$J'_\epsilon(V)(\partial_rV)=\int_\Omega a\nabla V\cdot\nabla(\partial_rV)-\int_\Omega a|V|^{p-1-\epsilon}V(\partial_rV)$$

$$= \left[\sum_{i=1}^{k}(-1)^{i+1}\int_\Omega a\nabla PU_i\cdot\nabla(\partial_r V) - \int_\Omega a|V|^{p-1}V(\partial_r V)\right]$$
$$+\left[\int_\Omega a|V|^{p-1}V(\partial_r V) - \int_\Omega a|V|^{p-1-\epsilon}V(\partial_r V)\right]$$
$$= \int_\Omega a\left(\sum_{i=1}^{k}(-1)^{i+1}U_i^p - |V|^{p-1}V\right)\cdot(\partial_r V) + \sum_{i=1}^{k}(-1)^i\int_\Omega (\nabla a\cdot\nabla PU_i)(\partial_r V)$$
$$+\left[\int_\Omega a\big(|V|^{p-1}V - |V|^{p-1-\epsilon}V\big)\cdot(\partial_r V)\right]$$
$$=: T_r^1 + T_r^2 + T_r^3$$

and estimate each T_r^i $(i=1,2,3)$.

Suppose that $r=d_l$ for some $l=1,\dots,k$. Note that in this case

$$\partial_r V = \partial_{d_l}V = (-1)^{l+1}\partial_{d_l}PU_l = (-1)^{l+1}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\cdot P\big(\psi_l^0 + s_l\psi_l^n\big)$$

where $P: D^{1,2}(\mathbb{R}^n)\to H_0^1(\Omega)$ is the projection operator given by (2.6) and $\psi_l^j := \psi_{\delta_l,\xi_l}^j$ $(j=0,n)$ are functions defined as (2.3) and (2.4). By simple manipulation, we get

$$T_{d_l}^1 = \int_{A_l} a\left(\sum_{i=1}^{k}(-1)^{i+1}U_i^p - |V|^{p-1}V\right)\cdot(-1)^{l+1}\partial_{d_l}PU_l + o(\epsilon)$$
$$= \int_{A_l} a\big(\big|(-1)^{l+1}U_l\big|^{p-1}(-1)^{l+1}U_l - |V|^{p-1}V\big)\cdot(-1)^{l+1}\partial_{d_l}PU_l + o(\epsilon).$$

On the other hand, by adapting the way to estimate I in the C^0-estimation and using (A.4), we can deduce that

$$\left|\epsilon^{\frac{n-1+2(l-1)}{n-2}}\int_{A_l} a\big(\big|(-1)^{l+1}U_l\big|^{p-1}(-1)^{l+1}U_l - |V|^{p-1}V\big)\cdot(-1)^{l+1}\big(P\psi_l^j - \psi_l^j\big)\right| = O\big(\epsilon^{\frac{n+4}{n+2}}\big).$$

Thus by the mean value theorem

$$T_{d_l}^1 = \int_{A_l} a\big(\big|(-1)^{l+1}U_l\big|^{p-1}(-1)^{l+1}U_l - |V|^{p-1}V\big)\cdot(-1)^{l+1}\partial_{d_l}U_l + o(\epsilon)$$
$$= p\int_{A_l} aU_l^{p-1}(U_l - PU_l)\partial_{d_l}U_l + \sum_{i\neq l}(-1)^{i+l+1}p\int_{A_l} aU_l^{p-1}PU_i\cdot\partial_{d_l}U_l + o(\epsilon).$$

From Lemmas A.1 and A.2, it follows that

$$p\int_{A_l} aU_l^{p-1}(U_l-PU_l)\partial_{d_l}U_l = p\int_{A_l} aU_l^{p-1}(U_l-PU_l)d_l^{-1}\delta_l\psi_l^0 + p\int_{A_l} aU_l^{p-1}(U_l-PU_l)d_l^{-1}s_l\delta_l\psi_l^n$$
$$= \left[\delta_{l1}a(\xi_0)\frac{a_2}{2}\partial_{d_1}\left(\frac{d_1}{2t}\right)^{n-2}\epsilon + o(\epsilon)\right] + o(\epsilon).$$

Furthermore, for $l<i$, we obtain by applying Lemma A.12 in particular that

$$\begin{aligned}
&p\int_{A_l} aU_l^{p-1}PU_i\cdot\partial_{d_l}U_l \\
&= p\int_{A_l} aU_l^{p-1}U_i\partial_{d_l}U_l + o(\epsilon) = \int_{A_l} a\big(\partial_{d_l}U_l^p\big)U_i + o(\epsilon) \\
&= \partial_{d_l}\left(\int_{A_l} aU_l^pU_i\right) - \partial_{d_m}\left(\int_{A_m} aU_l^pU_i\right)\bigg|_{l=m} + o(\epsilon) \\
&= \delta_{i(l+1)}a(\xi_0)\alpha_n^{p+1}\epsilon\cdot\partial_{d_l}\Bigg[\left(\frac{d_{l+1}}{d_l}\right)^{\frac{n-2}{2}}\int_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})\setminus B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})} \frac{1}{(1+|y-s_l\nu(\xi_0)|^2)^{\frac{n+2}{2}}} \\
&\quad\times\frac{1}{[(d_{l+1}/d_l)^2\cdot\epsilon^{\frac{4}{n-2}} + |y-(d_{l+1}/d_l)\epsilon^{\frac{2}{n-2}}s_{l+1}\nu(\xi_0)|^2]^{\frac{n-2}{2}}}\,dy\Bigg] + o(\epsilon) \\
&= \delta_{i(l+1)}a(\xi_0)\partial_{d_l}\left(\frac{d_{l+1}}{d_l}\right)^{\frac{n-2}{2}}F(s_l)\epsilon + o(\epsilon)
\end{aligned}$$

where we set $d_{k+1}=0$ and the function F is defined in (4.20). If $l>i$, through the procedure changing the order of i and l that was conducted in computing (4.7) (see (4.9) and the following computations), we can see

$$p\int_{A_l} aU_l^{p-1}PU_i\cdot\partial_{d_l}U_l = a(\xi_0)\delta_{i(l-1)}\partial_{d_l}\left(\frac{d_l}{d_{l-1}}\right)^{\frac{n-2}{2}}F(s_{l-1})\epsilon + o(\epsilon),$$

letting $F(s_0)=0$. As a result, it holds that

$$T_{d_l}^1 = a(\xi_0)\left[\delta_{l1}\frac{a_2}{2}\partial_{d_1}\left(\frac{d_1}{2t}\right)^{n-2} + \partial_{d_l}\left(\frac{d_{l+1}}{d_l}\right)^{\frac{n-2}{2}}F(s_l) + \partial_{d_l}\left(\frac{d_l}{d_{l-1}}\right)^{\frac{n-2}{2}}F(s_{l-1})\right]\epsilon + o(\epsilon). \tag{5.3}$$

Employing Lemma A.9, we can easily show that

$$T_{d_l}^2 = o(\epsilon), \tag{5.4}$$

so it suffices to compute $T_{d_l}^3$. Clearly

$$T_{d_l}^3 = \epsilon\int_{\Omega} a|V|^{p-1}V\log|V|\cdot(-1)^{l+1}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\big(\psi_l^0 + s_l\psi_l^n\big) + o(\epsilon).$$

Also, utilizing

$$\int_{\mathbb{R}^n} \frac{|y|^2-1}{(1+|y|^2)^{n+1}}\,dy = \int_{\mathbb{R}^n} \frac{y_n}{(1+|y|^2)^{n+1}}\,dy = 0, \tag{5.5}$$

Lemma A.12 and performing a similar computation to the derivation of (4.24), we find

$$\begin{aligned}
&\epsilon \int_{\Omega} a|V|^{p-1}V\log|V|\cdot(-1)^{l+1}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\psi_l^0 \\
&= \epsilon\sum_{j=1}^k \int_{A_j} a\left|\sum_{i=1}^k(-1)^{i+1}U_i\right|^{p-1}\left(\sum_{i=1}^k(-1)^{i+1}U_i\right)\log\left|\sum_{i=1}^k(-1)^{i+1}U_i\right| \\
&\quad \cdot(-1)^{l+1}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\psi_l^0 + o(\epsilon) \\
&= \epsilon\sum_{j=1}^k \int_{A_j} aU_j^p\log U_j\cdot(-1)^{l+j}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\psi_l^0 + o(\epsilon) \\
&= \frac{1}{p+1}a(\xi_0)\epsilon d_l^{-1}\delta_l\int_{A_l}\left(\partial_{\delta_l}U_l^{p+1}\right)\log U_l + o(\epsilon) \\
&= \frac{1}{p+1}a(\xi_0)\epsilon d_l^{-1}\delta_l\cdot\left[\partial_{\delta_l}\left(\int_{A_l}U_l^{p+1}\log U_l\right) - \int_{A_l}U_l^p\psi_l^0 - \partial_{\delta_m}\left(\int_{A_m}U_l^{p+1}\log U_l\right)\bigg|_{m=l}\right] + o(\epsilon) \\
&= -\frac{1}{p+1}\cdot\frac{n-2}{2}\cdot a(\xi_0)d_l^{-1}\epsilon\int_{B(0,\sqrt{\frac{\delta_{l-1}}{\delta_l}})\setminus B(0,\sqrt{\frac{\delta_{l+1}}{\delta_l}})} U_l^{p+1}\log U_l + o(\epsilon) \\
&= -\frac{(n-2)^2}{4n}a(\xi_0)d_l^{-1}a_1\epsilon + o(\epsilon)
\end{aligned}$$

and

$$\epsilon\int_{\Omega} a|V|^{p-1}V\log|V|\cdot(-1)^{l+1}\epsilon^{\frac{n-1+2(l-1)}{n-2}}\psi_l^n = a(\xi_0)\epsilon\int_{A_l}U_l^p\log U_l\cdot\delta_l\psi_l^n + o(\epsilon) = o(\epsilon).$$

Thus

$$T_{d_l}^3 = -\frac{(n-2)^2}{4n}a(\xi_0)d_l^{-1}a_1\epsilon + o(\epsilon). \tag{5.6}$$

Combining (5.3), (5.4) and (5.6), we see that

$$\begin{aligned}
&J_\epsilon'(V)(\partial_r V) \\
&= a(\xi_0)\left[\delta_{l1}\frac{a_2}{2}\cdot\partial_{d_1}\left(\frac{d_1}{2t}\right)^{n-2} + \left\{\partial_{d_l}\left(\frac{d_{l+1}}{d_l}\right)^{\frac{n-2}{2}}\frac{\alpha_n^{p+1}|B^n|}{(1+s_l^2)^{\frac{n-2}{2}}} + \partial_{d_l}\left(\frac{d_l}{d_{l-1}}\right)^{\frac{n-2}{2}}\frac{\alpha_n^{p+1}|B^n|}{(1+s_{l-1}^2)^{\frac{n-2}{2}}}\right\}\right]\epsilon
\end{aligned}$$

$$-\frac{(n-2)^2}{4n}a(\xi_0)a_1(\partial_{d_l}\log d_l)\epsilon+o(\epsilon)$$

and hence (5.2) is valid if $r=d_l$.

The case $r=s_l$ for some $l=1,\dots,k-1$ can be dealt with in a similar way to the case $r=d_l$. Hence the proof follows. □

Lemma 5.2. *For any* $r=d_1,\dots,d_k,s_1,\dots,s_{k-1}$, *the following holds*:

$$\big[J'_\epsilon(V+\phi)-J'_\epsilon(V)\big]\partial_r V=o(\epsilon).$$

Proof. We consider only when $r=d_l$ here. The case $r=s_l$ is similar. Expand

$$\begin{aligned}
&\big[J'_\epsilon(V+\phi)-J'_\epsilon(V)\big]\partial_{d_l}V\\
&\quad=\Big[\int_\Omega a\nabla\phi\cdot\nabla\partial_{d_l}V-ap|V|^{p-1}\phi\partial_{d_l}V\Big]\\
&\qquad-\Big[\int_\Omega a\big\{|V+\phi|^{p-1-\epsilon}(V+\phi)-|V|^{p-1-\epsilon}V-(p-\epsilon)|V|^{p-1-\epsilon}\phi\big\}\partial_{d_l}V\Big]\\
&\qquad+\Big[\int_\Omega a\big\{p|V|^{p-1}-(p-\epsilon)|V|^{p-1-\epsilon}\big\}\phi\partial_{d_l}V\Big]\\
&\quad=:I_1+I_2+I_3
\end{aligned}$$

and study each summand.

Let us estimate I_1. We have

$$I_1=\sum_{i=1}^k\Big(\int_\Omega a\nabla\phi\cdot\nabla\delta_i\big(P\psi_i^0+s_iP\psi_i^n\big)-p\int_\Omega a|V|^{p-1}\phi\delta_i\big(P\psi_i^0+s_iP\psi_i^n\big)\Big).$$

By (2.5) and (2.6),

$$\begin{aligned}
&\int_\Omega a\nabla\phi\cdot\nabla\big(\delta_iP\psi_i^j\big)-p\int_\Omega a|V|^{p-1}\phi\big(\delta_iP\psi_i^j\big)\\
&\quad=p\int_\Omega a\phi\big(U_i^{p-1}-|V|^{p-1}\big)\delta_i\psi_i^j-p\int_\Omega a\phi|V|^{p-1}\delta_i\big(P\psi_i^j-\psi_i^j\big)-\int_\Omega\nabla a\cdot\nabla\big(\delta_iP\psi_i^j\big)\phi
\end{aligned}$$

for $j=0,n$, so it suffices to estimate three terms in the right-hand side of the above equality. Notice that by (2.15) and (4.13), we have

$$\begin{aligned}
\int_{\Omega\setminus A_i}|\phi|\big|U_i^{p-1}-|V|^{p-1}\big|\big|\delta_i\psi_i^j\big|&\leqslant\|\phi\|\cdot\Big(\sum_{l=1}^k\big\|U_l^{p-1}\big\|_{L^{\frac{n}{2}}(\Omega\setminus A_i)}\Big)\cdot\big\|\delta_i\psi_i^j\big\|_{L^{p+1}(\Omega\setminus A_i)}\\
&=o(\sqrt{\epsilon})\cdot O(1)\cdot O(\sqrt{\epsilon})=o(\epsilon)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_i} |\phi|\big|U_i^{p-1} - |V|^{p-1}\big|\big|\delta_i \psi_i^j\big| \\
&\quad \leqslant \chi \cdot C \int_{A_i} |\phi| U_i \Big(|PU_i - U_i|^{p-1} + \sum_{l\neq i} U_l^{p-1} \Big) + C \int_{A_i} |\phi| U_i^{p-1} \Big(|PU_i - U_i| + \sum_{l\neq i} U_l \Big) \\
&\quad \leqslant \chi \cdot C\|\phi\| \cdot \|U_i\|_{L^{p+1}(A_i)} \Big(\big\| |PU_i - U_i|^{p-1} \big\|_{L^{\frac{n}{2}}(A_i)} + \sum_{l\neq i} \big\| U_l^{p-1} \big\|_{L^{\frac{n}{2}}(A_i)} \Big) \\
&\qquad + C\|\phi\| \cdot \big\| U_i^{p-1} \big\|_{L^{\frac{n}{2}}(A_i)} \Big(\|PU_i - U_i\|_{L^{p+1}(A_i)} + \sum_{l\neq i} \|U_l\|_{L^{p+1}(A_i)} \Big) \\
&\quad = \chi \cdot o(\sqrt{\epsilon}) \cdot O(1) \cdot O\big(\epsilon^{\frac{2}{n-2}}\big) + o(\sqrt{\epsilon}) \cdot O(1) \cdot O(\sqrt{\epsilon}) = o(\epsilon)
\end{aligned}$$

for some $C > 0$ (see [19, Lemma A.1]), where χ is a function such that $\chi = 0$ if $n \geqslant 6$ and $\chi = 1$ if $n \leqslant 5$. Furthermore, Lemma A.5 implies

$$\begin{aligned}
\int_{\Omega} |\phi||V|^{p-1}\delta_i \big|P\psi_i^j - \psi_i^j\big| &\leqslant \|\phi\| \cdot \big\|V^{p-1}\big\|_{L^{\frac{n}{2}}(\Omega)} \cdot \big\|\delta_i\big(P\psi_i^j - \psi_i^j\big)\big\|_{L^{p+1}(\Omega)} \\
&= o(\sqrt{\epsilon}) \cdot O(1) \cdot O(\sqrt{\epsilon}) = o(\epsilon).
\end{aligned}$$

Finally, by applying Young's inequality (see Appendix A.3) and (2.15), we observe that

$$\left| \int_{\Omega} \nabla a \cdot \nabla\big(\delta_i P\psi_i^j\big)\phi \right| \leqslant C\delta_i \big\| U_i^{p-1}\psi_i^j \big\|_{L^{\frac{2n}{n+4-\sigma}}(\Omega)} \cdot \|\phi\|_{L^{\frac{2n}{n-2}}(\Omega)} = O\big(\delta_i^{1-\frac{\sigma}{2}}\big) \cdot o(\sqrt{\epsilon}) = o(\epsilon) \tag{5.7}$$

where $\sigma > 0$ is a sufficiently small parameter. Therefore $I_1 = o(\epsilon)$.

Likewise, we can check that $I_2, I_3 = o(\epsilon)$ holds. (Refer to pages 29–31 in [19].) □

Lemma 5.3. *We have*

$$J'_\epsilon(V + \phi)(\partial_r \phi) = o(\epsilon) \quad \textit{for } r = d_1, \ldots, d_k, s_1, \ldots, s_{k-1}.$$

Proof. We can argue as in the derivation of (7.6) in [19]. Since we need a by-product that is derived during the proof of the lemma in the next subsection, we briefly sketch the proof.

Eq. (2.12) reads as

$$\begin{aligned}
S(V + \phi) &:= -\operatorname{div}\big(a\nabla(V + \phi)\big) - a|V + \phi|^{p-1-\epsilon}(V + \phi) \\
&= -\sum_{i=1}^{k} \big[c_{i0} \cdot \operatorname{div}\big(a\nabla P\psi_i^0\big) + c_{in} \cdot \operatorname{div}\big(a\nabla P\psi_i^n\big)\big].
\end{aligned} \tag{5.8}$$

Testing (5.8) with the function $\partial_r \phi$ and using the fact $\phi \in K^\perp$ where $K^\perp$ is defined in (2.11), we get

$$J'_\epsilon(V + \phi)(\partial_r \phi) = \sum_{i,j} c_{ij} \int_{\Omega} a\nabla P\psi_i^n \cdot \nabla(\partial_r \phi) = -\sum_{i,j} c_{ij} \int_{\Omega} a\nabla\big(\partial_r P\psi_i^n\big) \cdot \nabla\phi. \tag{5.9}$$

On the other hand, testing (5.8) with the function $P\psi_l^m$ for any fixed $m = 1, \dots, k$ and $l = 0, n$ and applying Lemmas A.7 and A.9, we can check that

$$c_{ij} = o(\delta_i \sqrt{\epsilon}). \tag{5.10}$$

Since Lemma A.8 and (2.15) imply that

$$\left| \int_\Omega a\nabla\big(\partial_r P\psi_i^n\big) \cdot \nabla\phi \right| \leqslant C \big\| \partial_r P\psi_i^j \big\| \cdot \|\phi\| = o\big(\delta_i^{-1}\sqrt{\epsilon}\big)$$

for some $C > 0$, we get the result. □

To sum up, we deduce (5.1) from Lemmas 5.1, 5.2 and 5.3 if $r = d_l$ ($l = 1, \dots, k$) or $r = s_l$ ($l = 1, \dots, k-1$).

5.2. The case $r = t$

When $r = t$, we have

$$\partial_r V = \partial_t V = \sum_{i=1}^k (-1)^{i+1} \partial_t PU_i = \sum_{i=1}^k (-1)^{i+1} \epsilon \cdot P\psi_l^n.$$

Thus, unlike the previous case $r = d_l$ or s_l where $\partial_r U_i = O(\delta_i(\psi_i^0 + \psi_i^n)) = O(U_i)$ holds, $\partial_t U_i = O(U_i)$ is not true anymore. In fact, it turns out that this difference makes it hard to obtain (5.1) in a direct way in this case. Fortunately, we can borrow the idea from [12] to overcome this problem, where the authors replaced the term, in our setting, $\partial_t V(x) = \epsilon \sum_{i=1}^k (-1)^{i+1} \partial_{(\xi_i)_n} V(x)$ with $\epsilon \sum_{i=1}^k (-1)^{i+1} \partial_{x_n} V(x)$ ($x \in \Omega$) in the expansion of the reduced energy functional $\partial_t \widetilde{J}_\epsilon$ and used a Pohožaev-type identity to estimate it. Such an approach was also applied in [19] successfully.

Lemma 5.4. *We have*

$$J'_\epsilon(V + \phi)(\partial_t V + \partial_t \phi) = \partial_t \Phi(\mathbf{d}, \mathbf{t})\epsilon + o(\epsilon).$$

Proof. As the first step, let us compute $J'_\epsilon(V + \phi)(\partial_t V)$. By utilizing (A.3) and (A.5), we get

$$\epsilon |c_{ij}| \int_\Omega a U_i^{p-1} \big|\psi_i^j\big| \big|P\psi_l^n - \psi_l^n\big|, \epsilon |c_{ij}| \int_\Omega a U_i^{p-1} \big|\psi_i^j\big| \big|\partial_{x_n}(PU_l - U_l)\big| = o\big(\epsilon^{\frac{3}{2}}\big).$$

Also, the application of (5.10), the proof of Lemma A.6 and Young's inequality (see Appendix A.3) gives

$$\begin{aligned}
\epsilon |c_{ij}| \int_\Omega \big|\nabla P\psi_i^j\big| \cdot |\partial_{(\xi_l)_n} PU_l + \partial_{x_n} PU_l| &\leqslant o\big(\delta_i \epsilon^{\frac{3}{2}}\big) \cdot O\left(\frac{\delta_l^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right) \int_\Omega \big|\nabla P\psi_i^j\big| \\
&\leqslant o\big(\delta_i \epsilon^{\frac{3}{2}}\big) \cdot O\left(\frac{\delta_l^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right) \int_\Omega \int_\Omega \frac{1}{|x-y|^{n-1}} \big(U_i^{p-1}\psi_i^j\big)(y)\, dy\, dx
\end{aligned}$$

$$\leqslant o\big(\epsilon^{\frac{3}{2}}\big) \cdot O\left(\frac{\delta_l^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right) \big\| U_i^p \big\|_{L^1(\Omega)}$$
$$= o(\sqrt{\epsilon}) \cdot O\left(\frac{\delta_i^{\frac{n-2}{2}} \delta_l^{\frac{n-2}{2}}}{\epsilon^{n-2}}\right) = o\big(\epsilon^{\frac{3}{2}}\big).$$

Hence

$$J'_\epsilon(V+\phi)(\partial_t V)$$
$$= p\epsilon \sum_{i,j} c_{ij} \int_\Omega a U_i^{p-1} \psi_i^j \left(\sum_{l=1}^k (-1)^{l+1} \partial_{(\xi_l)_n} P U_l \right) - \epsilon \sum_{i,j} c_{ij} \int_\Omega \nabla a \cdot \nabla P \psi_i^j \left(\sum_{l=1}^k (-1)^{l+1} \partial_{(\xi_l)_n} P U_l \right)$$
$$= p\epsilon \sum_{i,j,l} (-1)^{l+1} c_{ij} \int_\Omega a U_i^{p-1} \psi_i^j \big[\big(P\psi_l^n - \psi_l^n \big) + \partial_{x_n} (P U_l - U_l) - \partial_{x_n} P U_l \big]$$
$$- \epsilon \sum_{i,j} c_{ij} \int_\Omega \nabla a \cdot \nabla P \psi_i^j \left[\sum_{l=1}^k (-1)^{l+1} \big\{ (\partial_{(\xi_l)_n} P U_l + \partial_{x_n} P U_l) - \partial_{x_n} P U_l \big\} \right]$$
$$= -\int_\Omega S(V+\phi)(\partial_{x_n} V)\epsilon + o(\epsilon).$$

To estimate $J'_\epsilon(V+\phi)(\partial_t \phi)$, we observe that (5.9) implies

$$J'_\epsilon(V+\phi)(\partial_t \phi) = \sum_{i,j} c_{ij} \int_\Omega a\big(\Delta \partial_t P \psi_i^j\big)\phi + \sum_{i,j} c_{ij} \int_\Omega \nabla a \cdot \nabla\big(\partial_t P \psi_i^j\big)\phi.$$

Since it holds that

$$\int_\Omega a \partial_t \big(\Delta P \psi_i^j\big)\phi = -\int_\Omega p a \partial_t \big(U_i^{p-1} \psi_i^j\big)\phi = -p\epsilon \int_\Omega a \partial_{(\xi_i)_n} \big(U_i^{p-1} \psi_i^j\big)\phi = p\epsilon \int_\Omega a \partial_{x_n} \big(U_i^{p-1} \psi_i^j\big)\phi$$
$$= -p\epsilon \int_\Omega \partial_{x_n} a \cdot U_i^{p-1} \psi_i^j \phi - p\epsilon \int_\Omega a U_i^{p-1} \psi_i^j (\partial_{x_n} \phi)$$
$$= -p\epsilon \int_\Omega a U_i^{p-1} \psi_i^j (\partial_{x_n} \phi) + o\big(\delta_i^{-1} \epsilon^{\frac{3}{2}}\big),$$

and Eqs. (5.10), (2.15) and Lemma A.10 assert that

$$\left| c_{ij} \int_\Omega \nabla a \cdot \nabla\big(\partial_t P \psi_i^j\big)\phi \right| \leqslant |c_{ij}| \cdot \|\nabla a\|_{L^\infty(\Omega)} \cdot \big\| \nabla \partial_t P \psi_i^j \big\|_{L^{\frac{2n}{n+2}}(\Omega)} \cdot \|\phi\| = o(\epsilon) \tag{5.11}$$

(in fact, this is the only part we use the assumption $n \geqslant 4$ substantially; see Remark A.11), we deduce

$$J'_\epsilon(V+\phi)(\partial_t \phi) = -p\epsilon \sum_{i,j} c_{ij} \int_\Omega a U_i^{p-1} \psi_i^j (\partial_{x_n} \phi) + o(\epsilon).$$

On the other hand, by multiplying (5.8) by $\partial_{x_n}\phi$ and integrating the result over Ω, we get

$$\int_\Omega S(V+\phi)(\partial_{x_n}\phi) = p\sum_{i,j} c_{ij}\int_\Omega aU_i^{p-1}\psi_i^j(\partial_{x_n}\phi) + O\left(\sum_{i,j}|c_{ij}|\cdot\|P\psi_i^j\|\cdot\|\phi\|\right).$$

Thus using (5.10), (2.15) and Lemma A.7, we conclude that

$$J'_\epsilon(V+\phi)(\partial_t\phi) = -\int_\Omega S(V+\phi)(\partial_{x_n}\phi)\epsilon + o(\epsilon).$$

Accordingly, if we set $u = V+\phi$,

$$J'_\epsilon(u)(\partial_t u) = -S(u)(\partial_{x_n}u)\epsilon + o(\epsilon) = \left(\int_\Omega \operatorname{div}(a\nabla u)\partial_{x_n}u + \int_\Omega a|u|^{p-1-\epsilon}u\partial_{x_n}u\right)\epsilon + o(\epsilon)$$
$$=: (K_1+K_2)\epsilon + o(\epsilon).$$

Let us estimate the term K_2: From (2.15), the proof of Lemma 4.2 and (a3) (which implies $\partial_{x_n}a(\xi_0) = \partial_\nu a(\xi_0)$), we find

$$\begin{aligned}
K_2 &= \frac{1}{p+1-\epsilon}\int_\Omega a\left(\partial_{x_n}|u|^{p+1-\epsilon}\right) = -\frac{1}{p+1-\epsilon}\int_\Omega (\partial_{x_n}a)|V+\phi|^{p+1-\epsilon}\\
&= -\frac{1}{p+1-\epsilon}\int_\Omega (\partial_{x_n}a)|V|^{p+1-\epsilon} + o(1) = -\frac{1}{p+1}\int_\Omega (\partial_{x_n}a)|V|^{p+1} + o(1)\\
&= -\frac{1}{p+1}ka_1\partial_{x_n}a(\xi_0) + o(1) = -\frac{1}{p+1}ka_1\partial_\nu a(\xi_0) + o(1)
\end{aligned}$$

where a_1 is the quantity defined in (4.2).

Next, we consider K_1: Write

$$K_1 = \int_\Omega (\nabla a\cdot\nabla u)(\partial_{x_n}u) + \int_\Omega a\Delta u(\partial_{x_n}u) = \frac{1}{2}\int_\Omega (\partial_{x_n}a)|\nabla u|^2 - \frac{1}{2}\int_{\partial\Omega} a|\nabla u|^2\nu_n\,dS =: K_{11}+K_{12}$$

where ν_n is the n-th component of the inward unit normal vector to $\partial\Omega$ and dS is the surface measure on $\partial\Omega$ (see the proof of Step 1 on page 5 in [20]). We compute each term. Firstly, as for K_2, we have

$$K_{11} = \frac{1}{2}\int_\Omega (\partial_{x_n}a)|\nabla V|^2 + o(1) = \frac{1}{2}ka_1\partial_\nu a(\xi_0) + o(1).$$

On the other hand, (2.10) of [20] gives

$$\int_{\partial\Omega} |\nabla PU_i|^2\,dS = O\left(\frac{\delta_i^{n-2}}{\epsilon^{n-1}}\right)$$

and by mimicking the proof of [19, Lemma 7.2] or (2.12) in [20], one can prove that

$$\int_{\partial\Omega} |\nabla\phi|^2\,dS = o(1).$$

Thus

$$\begin{aligned}
K_{12} &= -\frac{1}{2}\int_{\partial\Omega} a|\nabla V|^2\nu_n\,dS + o(1) = -\frac{1}{2}\int_{\partial\Omega} a|\nabla PU_1|^2\nu_n\,dS + o(1) \\
&= -\int_\Omega aU_1^p(\partial_{x_n}PU_1) + \left\{\int_\Omega (\nabla a\cdot\nabla PU_1)\partial_{x_n}PU_1 - \frac{1}{2}\int_\Omega (\partial_{x_n}a)|\nabla PU_1|^2\right\} + o(1) \\
&= -p\int_\Omega aU_1^{p-1}\psi_1^n PU_1 \\
&\quad + \left\{\int_\Omega (\nabla a\cdot\nabla PU_1)\partial_{x_n}PU_1 + \int_\Omega (\partial_{x_n}a)U_1^p PU_1 - \frac{1}{2}\int_\Omega (\partial_{x_n}a)|\nabla PU_1|^2\right\} + o(1)
\end{aligned}$$

(see the proof of Step 2 on page 5 in [20]). However, we have

$$p\int_\Omega aU_1^{p-1}\psi_1^n PU_1 = \left(\frac{n+2}{2n}\right)a_1\partial_\nu a(\xi_0) - \frac{1}{2}a(\xi_0)a_2\partial_t\left(\frac{d_1}{2t}\right)^{n-2} + o(1) \tag{5.12}$$

and

$$\int_\Omega (\partial_{x_n}a)|\nabla PU_1|^2,\ \int_\Omega (\partial_{x_n}a)U_1^p PU_1,\ n\int_\Omega (\nabla a\cdot\nabla PU_1)\partial_{x_n}PU_1 = a_1\partial_\nu a(\xi_0) + o(1) \tag{5.13}$$

whose detailed proofs are given below. As a result, we obtain

$$K_{12} = \frac{1}{2}a(\xi_0)a_2\partial_t\left(\frac{d_1}{2t}\right)^{n-2} + o(1)$$

where a_2 is given in (4.3).

Proof of (5.12). We write

$$p\int_\Omega aU_1^{p-1}\psi_1^n PU_1 = p\int_\Omega aU_1^p\psi_1^n + p\int_\Omega aU_1^{p-1}\psi_1^n(PU_1 - U_1) \tag{5.14}$$

and we estimate the first term in the right-hand side of (5.14). By applying (5.5), (a3) (in particular, $\langle\nabla a(\xi_k), y\rangle = \partial_\nu a(\xi_k)\cdot y_n$) and Taylor's theorem,

$$\begin{aligned}
p\int_\Omega aU_1^p\psi_1^n &= p\int_{B(\xi_1,\rho\epsilon)} aU_1^p\psi_1^n + o(1) \\
&= \left[\frac{(n+2)\alpha_n^{p+1}}{\delta_1}\right]\cdot\left[-a(\xi_k)\int_{B(0,\delta^{-1}\rho\epsilon)^c}\frac{y_n}{(1+|y|^2)^{n+1}}\,dy\right.
\end{aligned}$$

$$+ \int\limits_{B(0,\delta_1^{-1}\rho\epsilon)} \left\{a\left(\delta_1 y + \delta_1 s_1 \nu(\xi_0) + \xi_k\right) - a(\xi_k)\right\} \frac{y_n}{(1+|y|^2)^{n+1}}\, dy \Bigg] + o(1)$$

$$= \partial_\nu a(\xi_k) \cdot (n+2)\alpha_n^{p+1} \int\limits_{B(0,\delta_1^{-1}\rho\epsilon)} \frac{y_n^2}{(1+|y|^2)^{n+1}}\, dy + o(1)$$

$$= \left(\frac{n+2}{2n}\right) \partial_\nu a(\xi_0) a_1 + o(1).$$

To estimate the second term in the right-hand side of (5.14), we need

$$\left| (\partial_{x,n} H)(\delta_i y + \xi_i, \xi_j) + (n-2) \frac{(\delta_i y + \xi_i - \xi_j^*)_n}{|\delta_i y + \xi_i - \xi_j^*|^n} \right| = O\left(\frac{1}{\epsilon^{n-2}}\right) \quad \text{for } |y| \leqslant \delta_i^{-1}\rho\epsilon \tag{5.15}$$

where $\nabla H(x,\xi) = (\nabla_x H(x,\xi), \nabla_\xi H(x,\xi)) = (\partial_{x,1} H(x,\xi), \dots, \partial_{x,n} H(x,\xi), \partial_{\xi,1} H(x,\xi), \dots, \partial_{\xi,n} H(x,\xi))$ and $i, j = 1, \dots, k$. Now, by Lemmas A.2, A.1 and A.13, (5.15) and the mean value theorem,

$$\int\limits_{\Omega} a\left(\partial_{(\xi_1)_n} U_1^p\right)(PU_1 - U_1)$$

$$= \int\limits_{B(\xi_1,\rho\epsilon)} a\left(\partial_{(\xi_1)_n} U_1^p\right) \cdot \alpha_n \delta_1^{\frac{n-2}{2}} H(\cdot, \xi_1) + o(1)$$

$$= \int\limits_{B(\xi_1,\rho\epsilon)} a U_1^p \cdot \alpha_n \delta_1^{\frac{n-2}{2}} \partial_{(\xi_1)_n}\left(H(\cdot,\xi_1)\right) + \partial_{x_n}\left(\int\limits_{B(x,\rho\epsilon)} a U_1^p \cdot \alpha_n \delta_1^{\frac{n-2}{2}} H(\cdot,\xi_1)\right)\Bigg|_{x=\xi_1}$$

$$- \alpha_n^{p+1} \partial_{(\xi_1)_n}\left(\int\limits_{B(0,\delta_1^{-1}\rho\epsilon)} a(\delta_1 y + \xi_1) \frac{\delta_1^{n-2}}{(1+|y|^2)^{\frac{n+2}{2}}} H(\delta_1 y + \xi_1, \xi_1)\, dy \right) + o(1)$$

$$= -\alpha_n^{p+1} \int\limits_{B(0,\delta_1^{-1}\rho\epsilon)} a(\delta_1 y + \xi_1) \frac{\delta_1^{n-2}}{(1+|y|^2)^{\frac{n+2}{2}}} (\partial_{x,n} H)(\delta_1 y + \xi_1, \xi_1)\, dy + o(1)$$

$$= \alpha_n^{p+1}(n-2) \int\limits_{B(0,\delta_1^{-1}\rho\epsilon)} a(\delta_1 y + \xi_1) \frac{\delta_1^{n-2}}{(1+|y|^2)^{\frac{n+2}{2}}} \frac{(2\epsilon t \nu(\xi_0) + \delta_1(y + 2s_1\nu(\xi_0)))_n}{|2\epsilon t \nu(\xi_0) + \delta_1(y + 2s_1\nu(\xi_0))|^n}\, dy + o(1)$$

$$= -\frac{1}{2} a(\xi_0) a_2 \partial_t \left(\frac{d_1}{2t}\right)^{n-2} + o(1).$$

Hence (5.12) is proved. □

Derivation of (5.13). By the argument in Section 4, we immediately get

$$\int\limits_{\Omega} (\partial_{x_n} a)|\nabla PU_1|^2, \int\limits_{\Omega} (\partial_{x_n} a) U_1^p PU_1 = a_1 \partial_\nu a(\xi_0) + o(1).$$

On the other hand, by Lemma A.4,

$$n\int_{\Omega}(\nabla a\cdot\nabla PU_1)\partial_{x_n}PU_1=n\int_{\Omega}(\nabla a\cdot\nabla U_1)\partial_{x_n}U_1+o(1).$$

Since (a3) implies $\partial_{x_n}a(\xi_0)=\partial_\nu a(\xi_0)$ and

$$\begin{aligned}n\int_{\Omega}(\partial_{x_i}a)\cdot(\partial_{x_i}U_1)\cdot(\partial_{x_n}U_1)&=\delta_{in}\cdot\partial_{x_n}a(\xi_0)\cdot\alpha_n^2(n-2)^2\int_{\mathbb{R}^n}\frac{|y|^2}{(1+|y|^2)^n}+o(1)\\&=\delta_{in}\cdot\partial_\nu a(\xi_0)a_1+o(1)\end{aligned}$$

for $i=1,\dots,n$, (5.13) follows. □

In conclusion,

$$J_\epsilon'(u)(\partial_t u)=\left(\frac{1}{2}-\frac{1}{p+1}\right)ka_1\partial_\nu a(\xi_0)\cdot\epsilon+\frac{1}{2}a(\xi_0)a_2\cdot\partial_t\left(\frac{d_1}{2t}\right)^{n-2}\epsilon+o(\epsilon)$$

as desired. □

Consequently, (5.1) for $s=t$ is valid and the proof of Proposition 2.5 is finished.

Appendix A

In this appendix, we study functions $PU_{\delta,\xi}$ and $P\psi^j_{\delta,\xi}$ $(j=0,n)$ defined through (2.1), (2.3), (2.4) and (2.6).

A.1. Comparison between $U_{\delta,\xi}$ and $PU_{\delta,\xi}$

Denote by $G(x,y)$ the Green function associated to $-\Delta$ with Dirichlet boundary condition and $H(x,y)$ its regular part: Namely,

$$\begin{cases}-\Delta_x G(x,y)=\delta_y(x) & \text{for } x\in\Omega,\\ G(x,y)=0 & \text{for } x\in\partial\Omega,\end{cases}$$

and

$$G(x,y)=\gamma_n\left(\frac{1}{|x-y|^{n-2}}-H(x,y)\right)\quad\text{where } \gamma_n=\frac{1}{(n-2)|S^{n-1}|}.$$

Since Ω is smooth, we can choose small $d_0>0$ such that, for every $x\in\Omega$ with $d(x,\partial\Omega)\leqslant d_0$, there is a unique point $x_\nu\in\partial\Omega$ satisfying $d(x,\partial\Omega)=|x-x_\nu|$. For such $x\in\Omega$, we define $x^*=2x_\nu-x$ the reflection point of x with respect to $\partial\Omega$.

The following two lemmas are proved in [1, Appendix A] under the assumption that Ω is of class C^2.

Lemma A.1. *There exists a constant $C>0$ such that*

$$\left|H(x,\xi)-\frac{1}{|x-\xi^*|^{n-2}}\right|\leqslant\frac{Cd(\xi,\partial\Omega)}{|x-\xi^*|^{n-2}},\qquad\left|\nabla_\xi\left(H(x,\xi)-\frac{1}{|x-\xi^*|^{n-2}}\right)\right|\leqslant\frac{C}{|x-\xi^*|^{n-2}}$$

and

$$0 \leqslant H(x,\xi) \leqslant \frac{C}{|x-\xi^*|^{n-2}}, \qquad \left|\nabla_\xi H(x,\xi)\right| \leqslant \frac{C}{|x-\xi^*|^{n-1}},$$

for any $x \in \Omega$ and $\xi \in \{y \in \Omega\colon d(y,\partial\Omega) \leqslant d_0\}$. In particular, we obtain

$$H(x,\xi) \leqslant \frac{C}{|x-\xi|^{n-2}} \quad \text{and} \quad \left|\nabla_\xi H(x,\xi)\right| \leqslant \frac{C}{|x-\xi|^{n-1}} \quad \text{for any } x,\xi \in \Omega,$$

by taking $C > 0$ larger if necessary.

Lemma A.2. *If $\xi \in \{y \in \Omega\colon d(y,\partial\Omega) \leqslant d_0\}$, then there exists a constant $C > 0$ such that*

$$0 \leqslant U_{\delta,\xi}(x) - PU_{\delta,\xi}(x) \leqslant \alpha_n \delta^{\frac{n-2}{2}} H(x,\xi) \leqslant \frac{C\delta^{\frac{n-2}{2}}}{|x-\xi^*|^{n-2}} \quad \text{for all } x \in \Omega. \tag{A.1}$$

Moreover, it holds true that

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \alpha_n \delta^{\frac{n-2}{2}} H(x,\xi) + O\left(\frac{\delta^{\frac{n+2}{2}}}{d(\xi,\partial\Omega)^n}\right), \quad x \in \Omega.$$

From the previous lemmas, we can show that

Lemma A.3. *Denote $PU_i = PU_{\delta_i,\xi_i}$. Then*

$$\|U_i - PU_i\|_{L^q(\Omega)} = o(1) \quad \text{if } q \in \left(\frac{n}{n-2}, \frac{2n}{n-3}\right) \text{ if } n \geqslant 4 \text{ or } q \in \left(\frac{n}{n-2}, +\infty\right) \text{ if } n = 3.$$

Proof. By (A.1) and (2.7), we have

$$\begin{aligned}
\|U_i - PU_i\|_{L^q(\Omega)}^q &\leqslant \int_\Omega \frac{\delta_i^{\frac{(n-2)q}{2}}}{|x-\xi_i^*|^{(n-2)q}} = \delta_i^{n-\frac{(n-2)q}{2}} \int_{\frac{\Omega-\xi_i}{\delta_i}} \frac{dy}{|y + 2((\epsilon/\delta_i)t + s_i)\nu(\xi_0)|^{(n-2)q}} \\
&\leqslant C\delta_i^{n-\frac{(n-2)q}{2}} \int_{\epsilon\delta_i^{-1}}^{C\delta_i^{-1}} \frac{s^{n-1}}{s^{(n-2)q}}\,ds \leqslant C\delta_i^{\frac{(n-2)q}{2}} \epsilon^{n-(n-2)q} \leqslant C\epsilon^{\frac{(n-1)q}{2}} \cdot \epsilon^{n-(n-2)q} \\
&= O\left(\epsilon^{n-\frac{(n-3)q}{2}}\right) = o(1)
\end{aligned}$$

for some $C > 0$. $\square$

In addition, we can estimate the $H^1(\Omega)$-norm of $U_i - PU_i$ as follows.

Lemma A.4. *It holds true that*

$$\|U_i - PU_i\|_{H^1(\Omega)} = O(\sqrt{\epsilon}).$$

Proof. From the definition (2.1) of U_i and the fact $\alpha_n^{p-1} = n(n-2)$, we get

$$\begin{aligned}
\|U_i - PU_i\|^2_{H^1(\Omega)} &= \left(\int_\Omega |\nabla PU_i|^2 - 2\int_\Omega \nabla PU_i \cdot \nabla U_i\right) + \int_\Omega |\nabla U_i|^2 \\
&= \left(\int_\Omega U_i^{\frac{n+2}{n-2}} PU_i - 2\int_\Omega U_i^{\frac{n+2}{n-2}} PU_i\right) + \alpha_n^2 (n-2)^2 \delta_i^{n-2} \int_\Omega \frac{|x-\xi_i|^2}{(\delta_i^2 + |x-\xi_i|^2)^n} \\
&= \left(-\alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} + O(\epsilon)\right) + \left(\alpha_n^2 (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} + O(\epsilon)\right) \\
&= O(\epsilon). \qquad \square
\end{aligned}$$

A.2. Estimates of ψ_i^j's

First, we want to establish a result similar to the ones proved in Lemma A.2 and Lemma A.3.

Lemma A.5. *For any $i = 1, \ldots, k$, we have*

$$P\psi_i^0 = \psi_i^0 - \alpha_n \left(\frac{n-2}{2}\right) \delta_i^{\frac{n-4}{2}} H(\cdot, \xi_i) + O\left(\frac{\delta_i^{\frac{n}{2}}}{\epsilon^n}\right) \quad \text{in } \Omega \tag{A.2}$$

and

$$P\psi_i^n = \psi_i^n - \alpha_n \delta_i^{\frac{n-2}{2}} (\partial_{\xi,n} H)(\cdot, \xi_i) + O\left(\frac{\delta_i^{\frac{n+2}{2}}}{\epsilon^{n+1}}\right) \quad \text{in } \Omega \tag{A.3}$$

where $(\partial_{\xi,n} H)(x, \xi)$ is the n-th component of $\nabla_\xi H(x, \xi)$. Moreover,

$$\left\|\delta_i\left(P\psi_i^j - \psi_i^j\right)\right\|_{L^{\frac{2n}{n-2}}(\Omega)} = O\left(\epsilon^{\frac{n}{n-2}}\right) \tag{A.4}$$

for $j = 0, n$.

Proof. From the comparison principle, we easily deduce (A.2) and (A.3). Arguing exactly as in Lemma A.3 and taking into account Lemma A.1, we can prove (A.4). $\square$

The above lemma enables to estimate the difference between $\partial_{x_n} PU_i$ and $\partial_{x_n} U_i$ for $i = 1, \ldots, k$. Let $p = (n+2)/(n-2)$.

Lemma A.6. *For $i = 1, \ldots, k$,*

$$\partial_{x_n} PU_i(x) = \partial_{x_n} U_i(x) + \alpha_n \delta_i^{\frac{n-2}{2}} (\partial_{\xi,n} H)(x, \xi_i) + O\left(\frac{\delta_i^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right). \tag{A.5}$$

Proof. Let $w = \partial_{x_n} PU_i + P\psi_i^n$. Then it solves $\Delta w = 0$ in Ω and $w = \partial_{x_n} PU_i$ on $\partial\Omega$. Thus, by the maximum principle, $\|w\|_{L^\infty(\Omega)} \leqslant \|\partial_{x_n} PU_i\|_{L^\infty(\partial\Omega)}$. Recalling $H(x, y) = H(y, x)$ and applying Lemma A.1, we observe that there is a constant $C > 0$ such that

$$\left|\partial_{x_n} PU_i(x)\right| \leqslant \int\limits_\Omega \left|\partial_{x_n} G(x,y)\right| \cdot U_i^p(y)\,dy = \gamma_n(n-2)\int\limits_\Omega \left|\frac{(x-y)_n}{|x-y|^n} - (\partial_{\xi,n}H)(y,x)\right| \cdot U_i^p(y)\,dy$$
$$\leqslant C\int\limits_\Omega \frac{1}{|x-y|^{n-1}} U_i^p(y)\,dy.$$

Now we choose $\rho > 0$ sufficiently small so that $B(x,\rho\epsilon) \cap B(\xi_i,\rho\epsilon) = \emptyset$ for any $x \in \partial\Omega$. Then for $x \in \partial\Omega$,

$$\int\limits_{\Omega\cap B(x,\rho\epsilon)} \frac{1}{|x-y|^{n-1}} U_i^p(y)\,dy \leqslant C\left(\frac{\delta_i^{\frac{n+2}{2}}}{\epsilon^{n+2}}\right)\int\limits_{B(x,\rho\epsilon)} \frac{1}{|x-y|^{n-1}}\,dy = O\left(\frac{\delta_i^{\frac{n+2}{2}}}{\epsilon^{n+1}}\right)$$

and

$$\int\limits_{\Omega\setminus B(x,\rho\epsilon)} \frac{1}{|x-y|^{n-1}} U_i^p(y)\,dy \leqslant C\left(\frac{1}{\epsilon^{n-1}}\right)\int\limits_{\mathbb{R}^n} \frac{\delta_i^{\frac{n-2}{2}}}{(1+|z|^2)^{\frac{n+2}{2}}}\,dz = O\left(\frac{\delta_i^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right).$$

Therefore we deduce

$$\|\partial_{x_n} PU_i\|_{L^\infty(\Omega)} = O\left(\frac{\delta_i^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right).$$

Consequently, by (A.3), we obtain

$$\partial_{x_n} PU_i(x) = -P\psi_i^n(x) + O\left(\frac{\delta_i^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right) = \partial_{x_n} U_i(x) + \alpha_n \delta_i^{\frac{n-2}{2}}(\partial_{\xi,n}H)(x,\xi_i) + O\left(\frac{\delta_i^{\frac{n-2}{2}}}{\epsilon^{n-1}}\right).$$

Hence (A.5) holds. □

The next lemma is crucial for the proof of Proposition 2.1.

Lemma A.7. *For $i,l = 1,\dots,k$, $i \leqslant l$ and $j,m = 0,n$, it holds that*

$$\langle P\psi_i^j, P\psi_l^m\rangle = \begin{cases} a(\xi_0)c_j\frac{1}{\delta_i^2} + o(\frac{1}{\delta_i^2}) & \text{if } i = l \text{ and } j = m,\\ o(\frac{1}{\delta_i^2}) & \text{otherwise,}\end{cases}$$

where c_0 and c_n are positive constants.

Proof. By (2.5) we get

$$\langle P\psi_i^j, P\psi_l^m\rangle = p\int\limits_\Omega aU_i^{p-1}\psi_i^j\psi_l^m + p\int\limits_\Omega aU_i^{p-1}\psi_i^j\left(P\psi_l^m - \psi_l^m\right) - \int\limits_\Omega \left(\nabla a\cdot\nabla P\psi_i^j\right)P\psi_l^m$$
$$=: M_1 + M_2 + M_3.$$

We will estimate M_1, M_2 and M_3 respectively.

To estimate M_1, note that $\delta_{i_1} \ll |\xi_{i_2} - \xi_0|$ for any $i_1, i_2 = 1, \dots, k$. Then arguing as in the proof of [19, Lemma A.5], we get

$$M_1 = \begin{cases} a(\xi_0)\frac{c_j}{\delta_i^2} + o(\frac{1}{\delta_i^2}) & \text{if } i = l \text{ and } j = m, \\ o(\frac{1}{\delta_i^2}) & \text{otherwise,} \end{cases}$$

with positive constants c_0 and c_n.

Let us estimate M_2 when $j = m = n$. By (A.3), assumption (a1) and Lemma A.1, we deduce that

$$M_2 = -\alpha_n \delta_l^{\frac{n-2}{2}} p \int\limits_{B(\xi_i, \rho\epsilon)} a U_i^{p-1} \psi_i^n (\partial_{\xi,n} H)(\cdot, \xi_l) + O\left(\frac{\epsilon^{\frac{n+1}{n-2}}}{\delta_i^2}\right) = o\left(\frac{1}{\delta_i^2}\right),$$

where $\rho > 0$ is chosen sufficiently small, since by (2.7), assumption (a1), Lemma A.1 and (A.3) we get

$$\begin{aligned} \left| \delta_l^{\frac{n-2}{2}} \int\limits_{B(\xi_i, \rho\epsilon)} a U_i^{p-1} \psi_i^n (\partial_{\xi,n} H)(\cdot, \xi_l) \right| &\leqslant C \delta_l^{\frac{n-2}{2}} \int\limits_{B(\xi_i, \rho\epsilon)} \frac{\delta_i^{\frac{n+2}{2}} |x - \xi_i|}{(\delta_i^2 + |x - \xi_i|^2)^{\frac{n+4}{2}}} \cdot \frac{1}{|x - \xi_i^*|^{n-1}} \, dx \\ &\leqslant C \delta_l^{\frac{n-2}{2}} \delta_i^{\frac{n-4}{2}} \int\limits_{B(0, \rho\epsilon\delta_i^{-1})} \frac{|y|}{(1 + |y|^2)^{\frac{n+4}{2}}} \frac{1}{\epsilon^{n-1}} \, dy \\ &= O\left(\frac{\epsilon^{\frac{n-1}{n-2}}}{\delta_i^2}\right) \end{aligned}$$

where ξ_i^* is the reflection of ξ_i with respect to $\partial\Omega$ defined in the previous subsection and $C > 0$ is some constant. The cases when either j or m is 0 can be carried out in a similar way using (A.2).

Finally, M_3 is estimated using Lemma A.9, which yields to $M_3 = o(1/\delta_i^2)$.

This concludes the proof. □

Finally, we need

Lemma A.8. *For $i = 1, \dots, k$, and $j = 0, n$, there hold*

$$\|\partial_r P\psi_i^j\| = \begin{cases} 0 & \text{if } r = d_l \ (l = 1, \dots, k), \ s_l \ (l = 1, \dots, k-1), \ l \neq i, \\ O(\delta_l^{-1}) & \text{if } r = d_i \text{ or } s_i, \end{cases}$$

and

$$\|\partial_t P\psi_i^j\| = O\left(\epsilon\delta_l^{-2}\right).$$

Proof. For $r = d_1, \dots, d_k, t, s_1, \dots, s_{k-1}$,

$$-\Delta\left(\partial_r P\psi_i^j\right) = p\left(\partial_r U_i^{p-1}\right)\psi_i^j + pU_i^{p-1}\left(\partial_r \psi_i^j\right) \quad \text{in } \Omega, \qquad \partial_r P\psi_i^j = 0 \quad \text{on } \partial\Omega.$$

Therefore

$$\|\partial_r P\psi_i^j\| \leqslant C\left\{ \left\| \left(\partial_r U_i^{p-1}\right)\psi_i^j \right\|_{L^{\frac{2n}{n+2}}(\Omega)} + \left\| U_i^{p-1}\left(\partial_r \psi_i^j\right) \right\|_{L^{\frac{2n}{n+2}}(\Omega)} \right\}$$

for some $C > 0$. Now estimate the right-hand side. □

A.3. Application of Young's inequality

In this subsection, we gather estimations which can be obtained by Young's inequality. We again denote $p = (n+2)/(n-2)$.

Lemma A.9. *Assume that $i, l = 1, \dots, k$ and $j, m = 0, n$. Then we have*

$$\int_\Omega |\nabla PU_i| PU_l = o(\epsilon) \tag{A.6}$$

and

$$\int_\Omega |\nabla PU_i| P\psi_l^m = o\left(\frac{\epsilon}{\delta_l}\right) \quad \text{and} \quad \int_\Omega \left|\nabla P\psi_i^j\right| P\psi_l^m = o\left(\frac{1}{\delta_i^2}\right).$$

Proof. The proof is essentially given in the proof of [1, Lemma A.2]. For the sake of reader's convenience, we reprove (A.6). Observe that Lemma A.1 tells us that

$$\left|\nabla PU_i(x)\right| = \left|\int_\Omega \nabla_x G(x,y) U_i^p(y)\,dy\right| \leqslant C \int_\Omega \frac{1}{|x-y|^{n-1}} U_i^p(y)\,dy$$

for some constant $C > 0$. Hence, by Young's inequality [17, Theorem 4.2],

$$\begin{aligned}\int_\Omega \left|\nabla PU_i(x)\right| PU_l(x)\,dx &\leqslant C \int_\Omega \int_\Omega U_l(x) \frac{1}{|x-y|^{n-1}} U_i^p(y)\,dy\,dx \\ &\leqslant C \|U_l\|_{L^q(\Omega)} \|f\|_{L^r(B(0,M))} \left\|U_i^p\right\|_{L^s(\Omega)}\end{aligned}$$

for any $q, r, s \geqslant 1$ which satisfies $1/q + 1/r + 1/s = 2$, where $f(x) = |x|^{1-n}$ and M is the diameter of Ω.

Fixing $\sigma > 0$ small enough, we choose

$$q = \frac{n}{1-(n-1)\sigma} > \frac{n}{n-2}, \qquad r = \frac{n}{(n-1)(1+\sigma)}, \qquad s = 1.$$

Since

$$\|U_l\|_{L^q(\Omega)} = O\left(\delta_l^{\frac{n}{q} - \frac{n-2}{2}}\right) \quad \text{for } q > \frac{n}{n-2}, \qquad \left\|U_i^p\right\|_{L^s(\Omega)} = O\left(\delta_i^{\frac{n}{s} - \frac{n+2}{2}}\right) \quad \text{for } s \geqslant 1$$

and $\|f\|_{L^r(B(0,M))} = O(1)$ for $r \in [1, n/(n-1))$, it then follows that

$$\|U_l\|_{L^q(\Omega)} \|f\|_{L^r(B(0,M))} \left\|U_i^p\right\|_{L^s(\Omega)} = O\left(\delta_1^{n(\frac{1}{q}+\frac{1}{s}-1)}\right) = O\left(\delta_1^{1-(n-1)\sigma}\right) = O\left(\epsilon^{\frac{n-1}{n-2}\cdot(1-(n-1)\sigma)}\right) = o(\epsilon),$$

which gives (A.6). □

Lemma A.10. *For $i = 1, \dots, k$ and $j = 0, n$,*

$$\|\nabla PU_i\|_{L^{\frac{2n}{n+2}}(\Omega)} = o(\epsilon) \quad \text{and} \quad \left\|\nabla \partial_t P\psi_i^j\right\|_{L^{\frac{2n}{n+2}}(\Omega)} = O\left(\epsilon^{1-\sigma}\delta_i^{-1}\right) \quad \text{if } n \geqslant 4.$$

Proof. We take into account only $\|\nabla PU_i\|_{L^{\frac{2n}{n+2}}(\Omega)}$. The other thing can be checked similarly.

Denote $\tilde{p} = \frac{2n}{n+2}$ and as the proof of the previous lemma, we compute

$$\|\nabla PU_i\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} \leqslant C \int_\Omega \left|\nabla PU_i(x)\right|^{\tilde{p}} dx \leqslant C \int_\Omega \int_\Omega \left|\nabla PU_i(x)\right|^{\tilde{p}-1} \frac{1}{|x-y|^{n-1}} U_i^p(y)\, dy\, dx$$
$$\leqslant C \left\| |\nabla PU_i|^{\tilde{p}-1} \right\|_{L^{\frac{\tilde{p}}{\tilde{p}-1}}(\Omega)} \|f\|_{L^r(B(0,M))} \left\| U_i^p \right\|_{L^s(\Omega)}$$

where $f(x) = |x|^{1-n}$ and M is the diameter of Ω again. Hence, if $n \geqslant 4$ the choice

$$r = \frac{n}{(n-1)(1+\sigma)} \quad \text{and} \quad s = \frac{2n}{n+4-2(n-1)\sigma} > 1$$

for any sufficiently small $\sigma > 0$ gives

$$\|\nabla PU_i\|_{L^{\tilde{p}}(\Omega)} \leqslant C \left\| U_i^p \right\|_{L^s(\Omega)} = O\left(\delta_i^{1-(n-1)\sigma}\right) = o(\epsilon). \qquad \square$$

Remark A.11. We point out that the assumption $n \geqslant 4$ is used in a crucial way in the proof of estimate (5.11). All the results necessary to the proof of the main theorem remain true for $n = 3$ except Lemma 5.4. In particular, the proofs of Proposition 2.3 and Lemma 5.2 can be slightly modified when for $n = 3$. Indeed, in the proof of Lemma A.10, we choose $r = 6/5$ and $s = 1$ to get $\|\nabla PU_i\|_{L^{6/5}(\Omega)} = O(\delta_i^{\frac{1}{2}}) = O(\epsilon)$. This implies $\|R_4\| = O(\epsilon)$ in the proof of Proposition 2.3, which is sufficient to conclude the validity of the proposition. Moreover,

$$\left| \int_\Omega \nabla a \cdot \nabla\left(\delta_i P\psi_i^j\right)\phi \right| \leqslant C\delta_i \left\| U_i^{p-1} \psi_i^j \right\|_{L^1(\Omega)} \cdot \|\phi\|_{L^6(\Omega)} = O\left(\delta_i^{\frac{1}{2}}\right) \cdot o(\sqrt{\epsilon}) = o(\epsilon),$$

so (5.7) holds to be true and the conclusion of Lemma 5.2 is true.

However, when $n = 3$ the argument of Lemma A.10 only guarantees $\|\nabla \partial_t P\psi_i^j\|_{L^{\frac{6}{5}}(\Omega)} = O(\delta_i^{-\frac{3}{2}})$ which does not allow to get the estimate (5.11), since

$$|c_{ij}| \cdot \left\| \nabla \partial_t P\psi_i^j \right\|_{L^{\frac{6}{5}}(\Omega)} \cdot \|\phi\| = o\left(\delta_i^{-\frac{1}{2}} \epsilon^2\right) \neq o(\epsilon) \quad \text{for } i \geqslant 2.$$

A.4. Differentiation under the integral sign

Here we recall some useful operations from elementary calculus. (See [13, Appendix C].)

Lemma A.12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and integrable. Then*

$$\frac{d}{dr} \int_{B(x_0,r)} f(x)\, dx = \int_{\partial B(x_0,r)} f\, dS$$

for any $x_0 \in \mathbb{R}^n$ and $r > 0$.

Lemma A.13. *Suppose $\{U(t)\}_{t\in\mathbb{R}}$ is a family of smooth bounded domains in $\mathbb{R}^n$ which depends on t smoothly. Denote $\mathbf{v}$ as the velocity of the moving boundary $\partial U(t)$ and ν as the inner unit normal vector to $\partial U(t)$. If $f:\mathbb{R}^n\to\mathbb{R}$ is smooth, then*

$$\frac{d}{dt}\int\limits_{U(t)} f(x)\,dx = -\int\limits_{\partial U(t)} f\mathbf{v}\cdot\nu\,dS.$$

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