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# Clustered boundary layer sign-changing solutions for a supercritical problem

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Abstract

We study the existence and profile of sign-changing solutions of the supercritical problem

$$-\Delta u = |u|^{p-1}u \text{ in } \mathcal{D}, \quad u = 0 \text{ on } \partial\mathcal{D},$$

where $\mathcal{D}$ is a smooth open bounded domain in $\mathbb{R}^n$ and $p > 1$. In particular, for suitable domains $\mathcal{D}$, we prove that, for any integer $m$, if $p$ is large enough, such a problem has a sign-changing solution which concentrates positively and negatively along $m$ different $(n-2)$-dimensional submanifolds of the boundary of $\mathcal{D}$ that collapse to a suitable submanifold of the boundary as $p \to +\infty$.

## 1. *Introduction*

We are interested in the classical Lane–Emden–Fowler problem

$$\Delta v + |v|^{p-1}v = 0 \text{ in } \mathcal{D}, \quad v = 0 \text{ on } \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is a smooth open bounded domain in $\mathbb{R}^n$ and $p > 1$.

As is well known, the existence of positive and sign-changing solutions to problem (1.1) when $p$ is above the critical exponent $p_1^* := (n+2)/(n-2)$ is a difficult issue.

When $p \in (1, (n+2)/(n-2))$ compactness of Sobolev's embedding ensures the existence of at least one positive solution and infinitely many sign-changing solutions.

When $p \geqslant (n+2)/(n-2)$, the situation is much more delicate. Pohožaev [**25**] proved that problem (1.1) does not have any solutions if the domain $\mathcal{D}$ is star-shaped. On the other hand, Kazdan and Warner [**15**] proved that if the domain $\mathcal{D}$ is an annulus, then problem (1.1) has infinitely many radial solutions. In the critical case, that is, $p = (n+2)/(n-2)$, Bahri and Coron proved that a positive solution of (1.1) exists if the domain $\mathcal{D}$ is topologically nontrivial. Moreover, in [**14**, **17**] it was proved that if the domain $\mathcal{D}$ has a small hole, then the number of sign-changing solutions of problem (1.1) increases as the size of the hole decreases. In the supercritical case, that is, $p > (n+2)/(n-2)$, the presence of topology does not guarantee the existence of solutions of (1.1). In fact, Passaseo [**19**, **20**] proved that if $n \geqslant 4$ and $1 \leqslant k \leqslant n-3$ is a given integer, then there exists a smooth bounded domain in $\mathbb{R}^n$ homotopically equivalent to a $k$-sphere such that problem (1.1) does not have any solutions if $p \geqslant (n-k+2)/(n-k-2)$.

Let us define $p_{k+1}^* := (n-k+2)/(n-k-2)$, $0 \leqslant k \leqslant n-3$ as the *$(k+1)$th critical exponent* and set $p_{n-1}^* := +\infty$.

The *almost first critical case*, that is, $p = p_1^* \pm \epsilon$, when $\epsilon$ is positive and small enough, has been widely studied. The slightly subcritical case, that is, $p = p_1^* - \epsilon$, was considered in [**3**, **28**], where the existence of positive solutions which blow up at one or more points of $\mathcal{D}$ as $\epsilon \to 0$

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was established. The existence of sign-changing solutions was studied in [**6**, **18**, **24**], where the authors constructed a large number of solutions with simple or multiple positive and negative blow-up points. The existence and nonexistence of positive solutions with one or more blow-up points in the slightly supercritical case, that is, $p = p_1^* + \epsilon$, was established in [**8**, **21**, **23**].

In [**22**], the authors considered the *almost second critical case*, that is, $p = p_2^* - \epsilon$. They proved that, for some suitable domains $\mathcal{D}$, if $\epsilon$ is positive, small enough and different from an explicit set of values, then problem (1.1) has a positive solution which concentrates along a one-dimensional submanifold of the boundary $\partial\mathcal{D}$. In the same paper, the authors ask the question whether one can find concentration results for larger critical exponents. More precisely,

(Q1) *for any integer $k = 2, \ldots, n-2$, if $p$ approaches from below $p_k^*$ and $p$ is (possibly) different from an explicit set of values, for some suitable domains $\mathcal{D}$ does problem* (1.1) *have a positive solution which concentrates along a $k$-dimensional submanifold of the boundary $\partial\mathcal{D}$?*

Having in mind that when $p$ approaches from below the first critical exponent $p_1^*$ a large number of sign-changing solutions exist, another question naturally arises:

(Q2) *for any integer $k = 1, \ldots, n-2$, if $p$ approaches from below $p_k^*$ and $p$ is (possibly) different from an explicit set of values, for some suitable domains $\mathcal{D}$ does problem* (1.1) *have a sign-changing solution which concentrates along a $k$-dimensional submanifold of the boundary $\partial\mathcal{D}$?*

In [**1**, **16**], we afford both questions in the *almost $(k+1)$th critical case* for $1 \leqslant k \leqslant n-3$.

In the present paper, we consider the *almost $(n-1)$th critical case*, that is, $k = n-2$ and we give a positive answer when $p$ goes to $+\infty$. More precisely, we exhibit some domains $\mathcal{D}$ such that if $p$ is large enough, then problem (1.1) has a positive solution which concentrates along an $(n-2)$-dimensional submanifold $\Gamma_0$ of the boundary $\partial\mathcal{D}$ as $p$ goes to $+\infty$. Moreover, we exhibit some domains $\mathcal{D}$ such that, for any integer $m$, if $p$ is large enough, then problem (1.1) has many sign-changing solutions which concentrate positively and negatively along $(n-2)$-dimensional submanifolds $\Gamma_{1p}, \ldots, \Gamma_{mp}$ of the boundary $\partial\mathcal{D}$ such that the $\Gamma_{ip}$'s accumulate to $\Gamma_0$ as the exponent $p$ goes to $+\infty$.

Let us state our main result more precisely.

Let $\Omega$ be a smooth open bounded domain in $\mathbb{R}^2$ such that

$$\bar{\Omega} \subset \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 > 0\} \quad \text{or} \quad \bar{\Omega} \subset \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1, x_2 > 0\}.$$

Let $h = 1$ or $h = 2$ be fixed. Let $M = M_1 + M_h$ with $M_i \geqslant 2$ and set

$$\mathcal{D} := \{(y_1, y_h, x') \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_h} \times \mathbb{R}^{2-h} \,:\, (|y_1|, |y_h|, x') \in \Omega\}.$$

Then $\mathcal{D}$ is a smooth open bounded domain in $\mathbb{R}^n$ with $n := M + 2 - h$.

As mentioned, we are interested in finding sign-changing solutions for the supercritical problem (1.1) when $p$ is large enough.

The solutions we are looking for are $\mathcal{G}$-invariant for the action of the group $\mathcal{G} := \mathcal{O}(M_1) \times \mathcal{O}(M_h)$ on $\mathbb{R}^N$ given by

$$(g_1, g_h)(y_1, y_h, x') := (g_1 y_1, g_h y_h, x').$$

Here, $\mathcal{O}(M_i)$ denotes the group of linear isometries of $\mathbb{R}^{M_i}$. More precisely, we look for solutions that satisfy

$$v(y_1, y_h, x') = v(g_1 y_1, g_h y_h, x') \quad \text{for all } g_i \in \mathcal{O}(M_i), \;\; y_i \in \mathbb{R}^{M_i}, \;\; x' \in \mathbb{R}^{2-h}.$$

A simple calculation shows that a function $v$ of the form $v(y_1, y_h, x') = u(|y_1|, |y_h|, x')$ solves problem (1.1) if and only if $u$ solves

$$-\Delta u - \sum_{i=1}^{h} \frac{M_i - 1}{x_i} \frac{\partial u}{\partial x_i} = |u|^{p-1} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

which can be rewritten as

$$-\mathrm{div}(a(x)\nabla u) = a(x)|u|^{p-1} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where

$$a(x_1, x_2) = x_1^{n-2} \quad \text{if } h = 1, \tag{1.2}$$

$$a(x_1, x_2) = x_1^{M_1 - 1} x_2^{M_2 - 1} \quad \text{if } h = 2 \quad (\text{here } n = M_1 + M_2). \tag{1.3}$$

Thus, we are led to study the more general anisotropic Lane–Emden–Fowler equation

$$\Delta_a u + |u|^{p-1} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $p$ is a large exponent, $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $a : \Omega \to \mathbb{R}$ is a smooth function over $\bar{\Omega}$ such that

$$0 < a_1 \leqslant a(x) \leqslant a_2 < +\infty,$$

and $\Delta_a$ is an operator defined as

$$\Delta_a u = \frac{1}{a(x)} \nabla(a(x)\nabla u) = \Delta u + \nabla \log a \cdot \nabla u \quad \text{for any } u \in H_0^1(\Omega).$$

We will assume that

(A1) $\bar{x} \in \partial\Omega$ is a strict local minimum of $a$;
(A2) $\partial_\nu a(\bar{x}) := \langle \nabla a(\bar{x}), \nu(\bar{x}) \rangle > 0$ where $\nu = \nu(\bar{x})$ is the inward unit normal at $\bar{x} \in \partial\Omega$.

Our goal is to construct solutions to problem (1.4) with positive and negative bubbles which accumulate to $\bar{x}$ as $p$ goes to $+\infty$. It corresponds to construct solutions to problem (1.1) such that they concentrate positively and negatively along $(n-2)$-dimensional submanifolds of the boundary of $\mathcal{D}$ which accumulate to the $\mathcal{G}$-orbit of $\bar{x}$ on the boundary of $\mathcal{D}$ as $p$ goes to $+\infty$. We point out that the $\mathcal{G}$-orbit of $\bar{x}$ on the boundary of $\mathcal{D}$ is a $(n-2)$-dimensional submanifold of the boundary of $\mathcal{D}$ diffeomorphic to $\mathbb{S}^{M_1 - 1} \times \mathbb{S}^{M_h - 1}$ (recall that $M - h = n - 2$), where $\mathbb{S}^{M_i - 1}$ is the unit sphere in $\mathbb{R}^{M_i}$.

Our first result concerns the existence of a single bubble solution for problem (1.4).

THEOREM 1.1. *Assume* (A1)–(A2). *There is $p_0$ such that, for $p > p_0$, there is a family of positive solutions $u_p$ for problem* (1.4) *with one positive bubble at $\bar{x}$ as $p \to +\infty$. More precisely,*

$$u_p(x) = \frac{1}{\gamma \mu^{2/(p-1)}} \log \frac{8\delta^2}{(\delta^2 + |x - \xi^p|^2)^2} + O\left(\frac{1}{p}\right),$$

*where the parameters $\gamma$, $\delta$ and $\mu$ satisfy*

$$\gamma = p^{p/(p-1)} \, e^{-p/2(p-1)}, \quad \delta = \mu \, e^{-p/4}, \quad \frac{1}{Cp} \leqslant \mu \leqslant \frac{C}{p},$$

*for some $C > 0$ and $\xi^p$ satisfies*

$$\xi^p \to \bar{x}, \quad \frac{\tilde{C}_1}{p} \leqslant d(\xi^p, \partial\Omega) \leqslant \frac{\tilde{C}_2}{p},$$

*for some* $\tilde{C}_1, \tilde{C}_2 > 0$. *In particular, for any* $\rho > 0$, *as* $p \to \infty$

$$p|u_p|^p u_p \rightharpoonup 8\pi e \delta_{\bar{x}} \quad \textit{weakly in } \mathcal{D}'(\mathbb{R}^2), \tag{1.5}$$

$$u_p \longrightarrow 0 \quad \textit{uniformly in } \Omega \setminus B_\rho(\bar{x}) \tag{1.6}$$

*and*

$$\sup_{x \in B_\rho(\bar{x})} u_p(x) \longrightarrow \sqrt{e}. \tag{1.7}$$

The corresponding result for problem (1.1) reads as follows.

THEOREM 1.2. *Let* $a$ *be as in* (1.2) *or in* (1.3). *Assume* (A1)–(A2). *There exists* $p_0$ *such that, for any* $p \geqslant p_0$, *problem* (1.1) *has a positive solution* $v_p$ *which concentrates positively along a* $(n-2)$*-dimensional submanifold of the boundary of* $\mathcal{D}$, *namely the* $\mathcal{G}$*-orbit of* $\bar{x}$, *as* $p \to +\infty$.

Our second result concerns the existence of a sign-changing solution for (1.4) with one positive and one negative bubble which accumulates to the same point.

THEOREM 1.3. *Assume* (A1)–(A2). *There is* $p_0$ *such that, for* $p > p_0$, *there is a family of sign-changing solutions* $u_p$ *with one positive bubble and one negative bubble which accumulate to* $\bar{x}$ *as* $p \to +\infty$. *More precisely,*

$$u_p(x) = \frac{1}{\gamma \mu_1^{2/(p-1)}} \log \frac{8\delta_1^2}{(\delta_1^2 + |x - \xi_1^p|^2)^2} - \frac{1}{\gamma \mu_2^{2/(p-1)}} \log \frac{8\delta_2^2}{(\delta_2^2 + |x - \xi_2^p|^2)^2} + O\left(\frac{1}{p}\right),$$

*where the parameters* $\gamma$, $\delta_i$ *and* $\mu_i$ *satisfy, for* $i = 1, 2$,

$$\gamma = p^{p/(p-1)} e^{-p/2(p-1)}, \quad \delta_i = \mu_i e^{-p/4}, \quad \frac{1}{Cp \log p} \leqslant \mu_i \leqslant \frac{C \log p}{p}, \tag{1.8}$$

*for some* $C > 0$ *and* $\xi_i^p$ *satisfies*

$$\xi_i^p \to \bar{x}, \quad \frac{\tilde{C}_1}{p} \leqslant d(\xi_i^p, \partial\Omega) \leqslant \frac{\tilde{C}_2}{p} \quad \textit{and} \quad |\xi_1^p - \xi_2^p| \geqslant \frac{1}{p \log p},$$

*for some* $\tilde{C}_1, \tilde{C}_2 > 0$. *In particular, for any* $\rho > 0$, *as* $p \to \infty$

$$p|u_p|^p u_p \rightharpoonup 0 \quad \textit{weakly in } \mathcal{D}'(\mathbb{R}^2), \tag{1.9}$$

$$u_p \longrightarrow 0 \quad \textit{uniformly in } \Omega \setminus B_\rho(\bar{x}) \tag{1.10}$$

*and*

$$\sup_{x \in B_{\rho/p^2}(\xi_1^p)} u_p(x) \longrightarrow \sqrt{e} \quad \textit{and} \quad \inf_{x \in B_{\rho/p^2}(\xi_2^p)} u_p(x) \longrightarrow -\sqrt{e}. \tag{1.11}$$

The corresponding result for problem (1.1) reads as follows.

THEOREM 1.4. *Let* $a$ *be as in* (1.2) *or in* (1.3). *Assume* (A1)–(A2). *There exists* $p_0$ *such that, for any* $p \geqslant p_0$, *problem* (1.1) *has a sign-changing solution* $v_p$ *which concentrates positively and negatively along two different* $(n-2)$*-dimensional submanifolds of the boundary of* $\mathcal{D}$ *that accumulate to the* $\mathcal{G}$*-orbit of* $\bar{x}$, *as* $p \to +\infty$.

We would like to point out that the solutions found in Theorem 1.3 are generated by a minimum point of the reduced energy (see Section 4). In fact, in that case the interaction between the two peaks is negative and that allows one to minimize the reduced energy in the configuration space where the concentration points lie (see (2.1)). When the number of peaks $m \geqslant 3$, the interaction between the peaks is no longer positive and so the reduced energy has no longer a minimum point. Actually, in this case the reduced energy should have a critical point of min–max type, which seems to be very difficult to catch. However, if we assume some symmetry conditions which allow one to reduce the configuration space so that we can again minimize the reduced energy, a solution with an arbitrary number of positive and negative bubbles near a local minimum of $a$ can be found. More precisely, we assume that:

(A3) $\Omega$ is symmetric with respect to the line $\mathfrak{L} := \{\bar{x} + \ell\nu(\bar{x}) : \ell \in \mathbb{R}\}$, that is,

$$\bar{x} + (x,\nu)\nu + (x,\tau)\tau \in \Omega \;\Leftrightarrow\; \bar{x} + (x,\nu)\nu - (x,\tau)\tau \in \Omega,$$

and $a$ is even with respect to the line $\mathfrak{L}$, that is,

$$a(\bar{x} + (x,\nu)\nu + (x,\tau)\tau) = a(\bar{x} + (x,\nu)\nu - (x,\tau)\tau),$$

where $\tau = \tau(\bar{x})$ is a unit tangent vector at $\bar{x} \in \partial\Omega$.

THEOREM 1.5. *Assume* (A2)–(A3)*. Then, for any $m \in \mathbb{N}$, there is $p_0 = p_0(m)$ such that, for $p > p_0$, there is a family of sign-changing solutions $u_p^m$ for problem* (1.4) *with $m$ alternating positive and negative bubbles which accumulate to $\bar{x}$ as $p \to +\infty$. More precisely,*

$$u_p^m(x) = \sum_{i=1}^{m} \frac{(-1)^{i+1}}{\gamma\mu_i^{2/(p-1)}} \log \frac{8\delta_i^2}{(\delta_i^2 + |x - \xi_i^p|^2)^2} + O\left(\frac{1}{p}\right),$$

*where the parameters $\gamma$, $\delta_i$ and $\mu_i$ are as in* (1.8) *and every $\xi_i^p = \bar{x} + t_i^p/p\nu$ is aligned on the line $\mathfrak{L}$ and satisfies*

$$t_i^p \to t_i,\; i = 1, \dots, m \quad \text{and} \quad 0 < t_1 < \cdots < t_m.$$

*Moreover, $u_p^m$ is even with respect to the line $\mathfrak{L}$. In particular, for any $\rho > 0$, as $p \to \infty$*

$$p|u_p|^p u_p \rightharpoonup 8\pi e \sum_{i=1}^{m} (-1)^{i+1}\delta_{\bar{x}} \quad \text{weakly in } \mathcal{D}'(\mathbb{R}^2),$$
$$u_p \longrightarrow 0 \quad \text{uniformly in } \Omega \setminus B_\rho(\bar{x})$$

*and*

$$\sup_{x \in B_{\rho/p^2}(\xi_i^p)} u_p(x) \longrightarrow \sqrt{e} \quad (i \text{ is odd}) \quad \text{and} \quad \inf_{x \in B_{\rho/p^2}(\xi_i^p)} u_p(x) \longrightarrow -\sqrt{e} \quad (i \text{ is even}).$$

We point out that if we assume (A3), then we do not require that $\bar{x}$ is a minimum point of $a$. The corresponding results for problem (1.1) read as follows.

THEOREM 1.6. (i) *Let $a$ be as in* (1.2)*. Assume* (A2)–(A3) *and assume also that $\Omega$ is symmetric with respect to the $x_1$-axis, that is, $(x_1, x_2) \in \Omega$ if and only if $(x_1, -x_2) \in \Omega$.*

*For any integer $m$, there exists $p_m$ such that, for any $p \geqslant p_m$, problem* (1.1) *has a sign-changing solution $v_p$ concentrating positively and negatively along $m$ different $(n-2)$-dimensional submanifolds of the boundary of $\mathcal{D}$ which accumulate to the $\mathcal{G}$-orbit of $\bar{x}$ as $p \to +\infty$.*

(ii) *Let $a$ be as in (1.3), with $n \geqslant 4$ even and $M_1 = M_2 = n/2$. Assume* (A2)–(A3) *and also that $\Omega$ is symmetric with respect to the line $\mathfrak{L} := \{(x_1, x_1) : x_1 \in \mathbb{R}\}$, that is, $(x_1, x_2) \in \Omega$ if and only if $(x_2, x_1) \in \Omega$.*

*For any integer $m$, there exists $p_m$ such that, for any $p \geqslant p_m$ problem (1.1) has a sign-changing solution $v_p$ concentrating positively and negatively along $m$ different $(n-2)$-dimensional submanifolds of the boundary of $\mathcal{D}$ which accumulate to the $\mathcal{G}$-orbit of $\bar{x}$ as $p \to +\infty$.*

We point out that problem (1.4) was first studied in [**2**, **11**, **26**, **27**] in the nonanisotropic case, that is, $a(x) \equiv 1$. Successively, in [**12**, **13**] the authors constructed for $p$ large enough positive and sign-changing solutions with simple positive and negative blow-up points. In [**29**], the authors studied the effect of the anisotropic coefficient $a$. In particular, they constructed positive solutions with an arbitrary large number of positive blow-up points which accumulate to a strict local maximum point of $a$ in $\Omega$. We also quote the papers [**31**, **32**] where the authors studied the existence of positive solutions for an anisotropic Emden–Fowler equation and the paper [**30**], where sign-changing solutions with multiple blow-up points are found for an anisotropic sinh-Poisson equation.

The proof of our results rely on a very well-known Ljapunov–Schmidt reduction. In Section 2, we write the approximate solution. In Section 3, we study the linear problem. In Section 4, we reduce the problem to a finite-dimensional one, we study the reduced energy and we prove Theorems 1.1 and 1.3. In Section 5, we treat the symmetric case and we prove Theorem 1.5. In the Appendix, we recall some important estimates on Green's function $G(x, y)$ and Robin's function $H_R(x) = H(x, x)$, $x, y \in \Omega$, where $H$ is the regular part of Green's function.

## 2. *An approximation for the solution*

A key ingredient to define an approximate solution to (1.4) is given by the standard bubble:

$$U_{\delta,\xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x-\xi|^2)^2}, \quad \xi, x \in \mathbb{R}^2, \ \delta > 0.$$

It is well known (see [**10**]) that those are all the solutions of the problem

$$-\Delta u = e^u \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < +\infty.$$

Let us introduce the configuration space in which the concentration points belong to:

$$\Lambda := \Bigg\{ \xi = (\xi_1, \dots, \xi_m) \in (B_\rho(\bar{x}) \cap \Omega)^m \, : \, \frac{\tilde{C}_1}{p} \leqslant d(\xi_i, \partial\Omega) \leqslant \frac{\tilde{C}_2}{p},$$
$$|\xi_i - \xi_j| \geqslant \frac{1}{p \log p}, \ i, j = 1, \dots, m, \ i \neq j \Bigg\}, \tag{2.1}$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ will be chosen in Lemma 4.3.

Next, we set, for $i \in \{1, \dots, m\}$,

$$\gamma = p^{p/(p-1)} \, e^{-p/2(p-1)}, \quad \delta_i = \mu_i \, e^{-p/4}, \quad \frac{1}{Cp \log^{m-1} p} \leqslant \mu_i \leqslant \frac{C \log^{m-1} p}{p},$$

for some $C > 0$ where the exact description for $\mu_i$ will be given later, and introduce a function that was introduced in [**13**],

$$\tilde{U}_i(x) = \frac{a_i}{\gamma \mu_i^{2/(p-1)}} \left[ U_{\delta_i,\xi_i}(x) + \frac{1}{p} w_0 \left( \frac{x - \xi_i}{\delta_i} \right) + \frac{1}{p^2} w_1 \left( \frac{x - \xi_i}{\delta_i} \right) \right], \tag{2.2}$$

with $a_i \in \{-1, 1\}$. Here, $w_0$ and $w_1$ are radial solutions of

$$\Delta w_i + \frac{8}{(1+|y|^2)^2} w_i = \frac{1}{(1+|y|^2)^2} f_i(|y|) \quad \text{in } \mathbb{R}^2, \tag{2.3}$$

for $i = 0, 1$, respectively, with

$$f_0 = 4U_{1,0}, \quad f_1 = -8\left(\frac{U_{1,0}^4}{8} + \frac{U_{1,0}^3}{3} - \frac{w_0 U_{1,0}^2}{2} - w_0 U_{1,0} + \frac{w_0^2}{2}\right)$$

having an asymptotic expansion

$$w_i(r) = C_i \log r + O\left(\frac{1}{r}\right), \quad \partial_r w_i(r) = \frac{C_i}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as } r \longrightarrow \infty, \ r = |y|, \tag{2.4}$$

for $i = 0, 1$ where

$$C_i = \int_0^\infty t \frac{t^2 - 1}{(t^2+1)^3} f_i(t)\, dt \tag{2.5}$$

(see also [**9**]).

We now approximate the solution by

$$U = \sum_{i=1}^m U_i = \sum_{i=1}^m (\tilde{U}_i + H_i^p), \tag{2.6}$$

where $H_i^p$ is a correction term defined as a solution of

$$\Delta_a H_i^p + \nabla \log a \cdot \nabla \tilde{U}_i = 0 \text{in } \Omega, \quad H_i^p = -\tilde{U}_i \text{on } \partial\Omega. \tag{2.7}$$

The following estimation was derived in [**29**].

Lemma 2.1. *For any $\gamma \in (0, 1)$, $p$ large,*

$$H_i^p(x) = \frac{a_i}{\gamma \mu_i^{2/(p-1)}} \left[\left(1 - \frac{C_0}{4p} - \frac{C_1}{4p^2}\right) H(x, \xi_i) - \log(8\delta_i^2) + \frac{\log \delta_i}{p}\left(C_0 + \frac{C_1}{p}\right) + O(\delta_i^\gamma)\right]$$

*uniformly in $\Omega$.*

Furthermore, we define $\mu_i$ $(i = 1, \ldots, m)$ as a solution of

$$\log(8\mu_i^4) = H(\xi_i, \xi_i)\left(1 - \frac{C_0}{4p} - \frac{C_1}{4p^2}\right) + \frac{\log \delta_i}{p}\left(C_0 + \frac{C_1}{p}\right) + \sum_{\substack{j=1 \\ j \neq i}}^m (a_i a_j)\left(\frac{\mu_i}{\mu_j}\right)^{2/(p-1)} G(\xi_i, \xi_j)\left(1 - \frac{C_0}{4p} - \frac{C_1}{4p^2}\right). \tag{2.8}$$

As in [**29**, Lemma 2.4], we can check that (2.8) is solvable and $\mu_i$ satisfies

$$\mu_i = \exp\left\{-\frac{3}{4} + \frac{1}{4}\left(H(\xi_i, \xi_i) + \sum_{j \neq i} a_i a_j G(\xi_i, \xi_j)\right)\right\} \cdot \left(1 + O\left(\frac{\log(\log p)}{p}\right)\right). \tag{2.9}$$

In particular, by our construction (2.1) of the configuration spaces, there exists a constant $C > 0$ independent of $p$ and $i$ such that $\mu_i \in [(Cp)^{-1} \log^{1-m} p, \, Cp^{-1} \log^{m-1} p]$ for any $p$ sufficiently large. Also, such a choice of $(\mu_1,\dots,\mu_m)$ gives us that if we write $x=\delta_i y+\xi_i$ for fixed $i\in\{1,\dots,m\}$, then

$$\begin{aligned} U(x) &= U_i(x)+\sum_{j\neq i}U_j(x)\\ &= \frac{a_i}{\gamma\mu_i^{2/(p-1)}}\Bigg[p+U_{1,0}(y)+\frac{1}{p}w_0(y)+\frac{1}{p^2}w_1(y)\\ &\quad+\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)(H(\delta y+\xi_i,\xi_i)-H(\xi_i,\xi_i))+O(\delta_i^\gamma)\Bigg]\\ &\quad+\sum_{j\neq i}\left[U_j(x)-\frac{a_j}{\gamma\mu_j^{2/(p-1)}}G(\xi_i,\xi_j)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)\right] \end{aligned} \tag{2.10}$$

uniformly in $(\Omega-\xi_i)/\delta_i$ owing to the previous lemma. In particular, from Lemma A.2 we have

$$\begin{aligned} U(x) &= \frac{a_i}{\gamma\mu_i^{2/(p-1)}}\left[p+U_{1,0}(y)+\frac{1}{p}w_0(y)+\frac{1}{p^2}w_1(y)+O\left(\delta_i^\gamma\left(1+\frac{|y|^\gamma}{d(\xi_i,\partial\Omega)}\right)\right)\right]\\ &\quad+\sum_{j\neq i}\frac{a_j}{\gamma\mu_j^{2/(p-1)}}\left[O(\delta_j^\gamma)+O\left(\frac{\delta_i^\gamma|y|^\gamma}{d(\xi_j,\partial\Omega)}\right)+O(\delta_i^{\tilde\gamma}|y|^{\tilde\gamma})\right], \end{aligned} \tag{2.11}$$

in $|x-\xi_i|<p^{-2}$ for any $\gamma\in(0,1)$, $\tilde\gamma\in(0,1/2)$, $x\in\Omega$ and $\xi\in\Lambda$. This justifies our selection of $U$ as an *ansatz* for the solution of problem (1.4).

We will seek for a solution of (1.4) in the form $u=U+\phi$. If we write $W=p|U|^{p-1}$, then (1.4) is equivalent to an equation of $\phi$ given by

$$L(\phi)=-(R+N(\phi)) \text{ in } \Omega,\quad \phi=0 \text{ on } \partial\Omega,$$

where

$$L(\phi)=\Delta_a\phi+W\phi,\quad R=\Delta_a U+|U|^{p-1}U$$

and

$$N(\phi)=|U+\phi|^{p-1}(U+\phi)-|U|^{p-1}U-W\phi.$$

Define the norm

$$\|h\|_*=\sup_{x\in\Omega}\left|\left(\sum_{i=1}^m\frac{\delta_i}{(\delta_i^2+|x-\xi_i|^2)^{3/2}}\right)^{-1}h(x)\right|,$$

for any $h\in L^\infty(\Omega)$. We then have the following estimation for the remainder term $R$.

LEMMA 2.2. *There exists $C>0$ such that, for any $\xi\in\Lambda$ and $p$ large,*

$$\|R\|_*\leqslant\frac{C}{p^4}.$$

*Proof.* Since its proof is similar to the proof of [**12**, Proposition 2.1] (cf. [**13**, Lemma 2.1, **29**, Proposition 2.1]), we give only a sketch. However, we stress that here we have to take care of the effect due to the boundary of the domain.

If $|x-\xi_i|\geqslant p^{-2}$ for all $i=1,\dots,m$, then the direct computation using (2.10) and (A.2) yields that

$$|\Delta_a U(x)|=|\Delta\tilde U(x)|\leqslant Cp^5\log^{2(m-1)}p\,e^{-p/2}\quad\text{and}\quad|U(x)|\leqslant C\frac{\log p}{p}, \tag{2.12}$$

hence

$$\|\Delta_a U + |U|^{p-1}U\|_* \leqslant Cp\log^{m-1} p\, e^{p/4}\left[p^5 \log^{2(m-1)} p\, e^{-p/2} + \left(\frac{\log p}{p}\right)^p\right]$$
$$\leqslant Cp^6 \log^{3(m-1)} p\, e^{-p/4}.$$

On the other hand, if $|x-\xi_i| \leqslant p^{-2}\sqrt{\delta_i}$ for some $i$, then, by (2.11), the relation

$$\left(\frac{p}{\gamma \mu_i^{2/(p-1)}}\right)^p = \frac{1}{\gamma \delta_i^2 \mu_i^{2/(p-1)}},$$

and the expansion

$$\left(1+\frac{A}{p}+\frac{B}{p^2}+\frac{C}{p^3}\right)^p = e^A\left[1+\frac{1}{p}\left(B-\frac{A^2}{2}\right)+\frac{1}{p^2}\left(C-AB+\frac{A^3}{3}+\frac{B^2}{2}+\frac{A^4}{8}-\frac{A^2B}{2}\right)\right.$$
$$\left. + O\left(\frac{\log^6(|y|+2)}{p^3}\right)\right], \tag{2.13}$$

which holds for $|y| \leqslant C\, e^{p/8}$ provided $-4\log(|y|+2) \leqslant A(y) \leqslant C$ and $|B(y)|+|C(y)| \leqslant C\log(|y|+2)$, we have

$$\|\Delta_a U + |U|^{p-1}U\|_* \leqslant \frac{1}{p^4}.$$

Finally, for $p^{-2}\sqrt{\delta_i} \leqslant |x-\xi_i| \leqslant p^{-2}$, since $(1+sp^{-1})^p \leqslant e^s$, we get, from (2.11),

$$\|U^p\|_*, \quad \|\Delta U\|_* \leqslant Cp^{1/2}\log^{1/2(m-1)} p\, e^{-p/8}.$$

This concludes the proof. □

## 3. *The linearized problem*

For fixed $\delta_i > 0$, $\xi_i = ((\xi_i)_1, (\xi_i)_2)$ and any $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$Z_{ij}(x) := \begin{cases} \dfrac{\delta_i}{2}\cdot \dfrac{\partial U_{\delta_i,\xi_i}}{\partial \delta_i}(x) = \dfrac{-\delta_i^2 + |x-\xi_i|^2}{\delta_i^2+|x-\xi_i|^2} & \text{if } j=0,\\[2mm] \delta_i \cdot \dfrac{\partial U_{\delta_i,\xi_i}}{\partial (\xi_i)_j}(x) = \dfrac{4\delta_i (x-\xi_i)_j}{\delta_i^2+|x-\xi_i|^2} & \text{if } j=1,2, \end{cases} \tag{3.1}$$

for $i = 1, \ldots, m$.

In this section, we will solve the following linear problem: *find $\phi \in W^{2,2}(\Omega)$ and $c_{ij} \in \mathbb{R}$ which solve*

$$\begin{cases} L(\phi) = h + \displaystyle\sum_{i=1}^m \sum_{j=1}^2 c_{ij}\, e^{U_{\delta_i,\xi_i}} Z_{ij} & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij}\phi = 0 & \text{for } i=1,\ldots,m,\ j=1,2, \end{cases} \tag{3.2}$$

if $h \in C(\Omega)$.

Proposition 3.1. *For $p$ large enough, $\xi \in \Lambda$ and $h \in C(\Omega)$, (3.2) admits a unique solution $\phi$ and $c_{ij}$ which satisfies*

$$\|\phi\|_{L^\infty(\Omega)} \leqslant Cp\|h\|_*,$$

*where $C > 0$ is independent of $p$.*

The proof of Proposition 3.1 consists of a series of lemmas.

LEMMA 3.1. *For sufficiently large $p$, $\xi \in \Lambda$ and $h \in C^{0,\alpha}(\Omega)$ ($\alpha \in (0,1)$ fixed), a solution for*

$$\begin{cases} L(\psi) = h & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij}\psi = 0 & \text{for } i = 1,\dots,m,\ j = 0,1,2 \end{cases}$$

*satisfies*

$$\|\psi\|_{L^\infty(\Omega)} \leqslant C\|h\|_*, \tag{3.3}$$

*where $C > 0$ is independent of $p$.*

*Proof.* The proof of this result consists of three steps.

*Step* 1. For sufficient large $p$, there is a large number $R > 0$ independent of $p$ such that the operator $L$ satisfies the maximum principle in $\tilde\Omega := \Omega \setminus \bigcup_{i=1}^m B_{R\delta_i}(\xi_i)$. In other words, if $L\varphi \leqslant 0$ in $\tilde\Omega$ and $\varphi \geqslant 0$ in $\partial\tilde\Omega$, then $\varphi \geqslant 0$ in $\tilde\Omega$.

To prove it, it suffices to construct a barrier $Z$ in $\tilde\Omega$, that is, $Z > 0$ and $LZ < 0$ in $\tilde\Omega$. The barrier which we introduce here is the one used in [**31**]. Let $\Phi_0$ be a solution of

$$-\Delta_a\Phi_0 = 1 \text{ in } \Omega, \quad \Phi_0 = 2 \text{ on } \partial\Omega$$

and

$$Z(x) = \sum_{i=1}^m \left\{ \Phi_0(x) - \left(\frac{\delta}{|x-\xi_i|}\right)^\alpha \right\},$$

where $\alpha$ is any number in $(0,1)$.

For any $R \geqslant 1$ fixed, $1 \leqslant Z(x) \leqslant M$ in $\tilde\Omega$ for some $M > 0$. Moreover, since

$$W \leqslant C\sum_{i=1}^m e^{U_{\delta_i,\xi_i}} \quad \text{for some constant } C > 0 \tag{3.4}$$

(see [**12**, Lemma 3.1] or [**29**, Lemma 3.1]), we have

$$\begin{aligned} LZ(x) &\leqslant \sum_{i=1}^m \left( -1 - \frac{\alpha(\alpha+1)\delta_i^\alpha}{|x-\xi_i|^{\alpha+2}} + \nabla\log a(x)\cdot\frac{\alpha\delta_i^\alpha(x-\xi_i)}{|x-\xi_i|^{\alpha+2}} + \frac{8CM\delta_i^2}{(\delta_i^2+|x-\xi_i|^2)^2} \right) \\ &:= \sum_{i=1}^m \hat Z_i(x). \end{aligned}$$

Now fix $i \in \{1,\dots,m\}$. In $R\delta_i \leqslant |x-\xi_i| \leqslant p^{-2}$, for sufficiently large $p$ and $R$,

$$\begin{aligned} \hat Z_i(x) &\leqslant -1 - \frac{\alpha\delta_i^\alpha}{|x-\xi_i|^{\alpha+2}}\left(\alpha + 1 - \|\nabla\log a\|_{L^\infty(\Omega)}\cdot\frac{1}{p^2}\right) + \frac{8CM\delta_i^2}{(\delta_i^2+|x-\xi_i|^2)^2} \\ &\leqslant -1 - \frac{\alpha^2\delta_i^\alpha}{2|x-\xi_i|^{\alpha+2}} + \frac{8CM\delta_i^2}{|x-\xi_i|^4} < -1. \end{aligned}$$

If $|x-\xi_i| \geqslant p^{-2}$, then, for any large $p$,

$$\hat Z_i(x) \leqslant -1 + \|\nabla\log a\|_{L^\infty(\Omega)}\alpha\delta_i^\alpha p^{2(\alpha+1)} + 8CM\delta_i^2 p^8 \leqslant -\tfrac12.$$

Summing up, $LZ(x) \leqslant -m/2$ in $\tilde\Omega$.

*Step* 2. Define the inner norm of $\phi$ by

$$\|\phi\|_i = \sup_{x \in \bigcup_{i=1}^m B_{R\delta_i}(\xi_i)} |\phi(x)|.$$

Then there exists $C > 0$ such that every solution $\phi$ of $L\phi = h$ in $\Omega$, $\phi = 0$ on $\partial\Omega$ with some $h \in C^{0,\alpha}(\Omega)$ satisfies

$$\|\phi\|_{L^\infty(\Omega)} \leqslant C(\|\phi\|_i + \|h\|_*). \tag{3.5}$$

This follows from the application of the maximum principle in $\tilde{\Omega}$ obtained in Step 1 to the function $\tilde{\phi} := (\|\phi\|_i + \|h\|_*)Z$. In particular, we can check that $\tilde{\phi} \geqslant |\phi|$ in $\tilde{\Omega}$, which gives rise to (3.5).

*Step* 3. We conclude the proof. Suppose, on the contrary, that there are sequences of parameters $p_n \to \infty$, $m$-tuples of points $\xi_n = (\xi_1^n, \dots, \xi_m^n) \in \Lambda$, functions $h_n \in C^{0,\alpha}(\Omega)$ and corresponding solutions $\psi_n$ such that

$$\|h_n\|_* \longrightarrow 0, \quad \|\psi_n\|_{L^\infty(\Omega)} = 1 \quad \text{as } n \longrightarrow \infty. \tag{3.6}$$

We consider the expansion of $W^n = p_n|U^n|^{p_n-1}$ where $U^n$ is defined by (2.6) with $\xi_n \in \Lambda$: for $|x - \xi_i^n| \leqslant p_n^{-2}\sqrt{\delta_i^n}$,

$$\begin{aligned} W^n(x) &= W^n(\delta_i^n y + \xi_i^n) \\ &= \frac{1}{(\delta_i^n)^2} \cdot \frac{8}{(1+|y|^2)^2} \left[ 1 + \frac{1}{p_n} \left( w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2} \right) \right. \\ &\quad \left. + O\left( \frac{\log^4(|y|+2)}{p_n^2} \right) \right], \end{aligned} \tag{3.7}$$

which can be derived from (2.11), (2.13) and (A.2). Then employing (3.7) and elliptic regularity, we can deduce that $\hat{\psi}_i^n := \psi_n(\delta_i^n \cdot + \xi_i^n)$ where $\xi_n = (\xi_1^n, \dots, \xi_m^n) \in \Lambda$, $\mu_n = (\mu_1^n, \dots, \mu_m^n)$ and $\delta_i^n = \mu_i^n e^{-p_n/4}$ for $i \in \{1, \dots, m\}$ converges uniformly over compact sets to a solution $\hat{\psi}_i^\infty$ of

$$\Delta\phi + \frac{8}{(1+|y|^2)^2}\phi = 0 \quad \text{in } \mathbb{R}^2, \tag{3.8}$$

which satisfies

$$8a(\xi_i^\infty) \int_{\mathbb{R}^2} \frac{-1+|y|^2}{(1+|y|^2)^3} \hat{\psi}_i^\infty(y)\, dy = 0 \tag{3.9}$$

and

$$32a(\xi_i^\infty) \int_{\mathbb{R}^2} \frac{y_j}{(1+|y|^2)^3} \hat{\psi}_i^\infty(y)\, dy = 0, \quad y = (y_1, y_2) \in \mathbb{R}^2, \tag{3.10}$$

for each $i = 1, \dots, m$ and $j = 1, 2$. Here, $\xi_i^\infty \in \partial\Omega$ is an accumulation point of the sequence $(\xi_i^n)_n$. However, by the result of [**5**], any bounded solution of (3.8) can be expressed as a linear combination of

$$\frac{-1+|y|^2}{1+|y|^2}, \quad \frac{4y_1}{1+|y|^2} \quad \text{and} \quad \frac{4y_2}{1+|y|^2}$$

(that is, $Z_{1j}$ defined in (3.1) with $\delta_1 = 1$ and $\xi_1 = 0$ for $j = 0, 1, 2$). Therefore, (3.9) and (3.10) imply $\hat{\psi}_i^\infty = 0$ or $\lim_{n\to\infty} \|\psi_n\|_i = 0$. However, by (3.5) and (3.6), $\liminf_{n\to\infty} \|\psi_n\|_i > 0$, which is a contradiction. $\square$

Lemma 3.2. *For sufficiently large $p$, $\xi \in \Lambda$ and $h \in C^{0,\alpha}(\Omega)$, a solution for*

$$\begin{cases} L(\phi) = h & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij}\phi = 0 & \text{for } i = 1,\dots,m,\ j = 1,2 \end{cases}$$

*satisfies*

$$\|\phi\|_{L^\infty(\Omega)} \leqslant Cp\|h\|_*, \tag{3.11}$$

*where $C > 0$ is independent of $p$.*

*Proof.* Suppose, by contradiction, that there are sequences of parameters $p_n \to \infty$, $m$-tuples of points $\xi_n = (\xi_1^n, \dots, \xi_m^n) \in \Lambda$, functions $h_n \in C^{0,\alpha}(\Omega)$ and corresponding solutions $\phi_n$ such that

$$p_n\|h_n\|_* \longrightarrow 0, \quad \|\phi_n\|_{L^\infty(\Omega)} = 1 \text{ as } n \longrightarrow \infty. \tag{3.12}$$

For each $i \in \{1, \dots, m\}$, define a sequence $\{\hat\phi_i^n\}_{n\in\mathbb{N}}$ of scaled functions by

$$\hat\phi_i^n(y) := \phi_n(\delta_i^n y + \xi_i^n) \quad \text{for all } y \in \Omega_i^n := (\Omega - \xi_i^n)/\delta_i^n,$$

with $\xi_n = (\xi_1^n, \dots, \xi_m^n) \in \Lambda$, $\mu_n = (\mu_1^n, \dots, \mu_m^n)$ and $\delta_i^n = \mu_i^n\, e^{-p_n/4}$. Then, by (3.7) and elliptic regularity, $\hat\phi_i^n$ converges to $\hat\phi_i^\infty$ uniformly over compact subsets in $\mathbb{R}^2$ and consequently $\hat\phi_i^\infty$ solves

$$\Delta\hat\phi_i^\infty + \frac{8}{(1+|y|^2)^2}\hat\phi_i^\infty = 0 \quad \text{in } \mathbb{R}^2,$$

and satisfies

$$32a(\xi_i^\infty)\int_{\mathbb{R}^2} \frac{y_j}{(1+|y|^2)^3}\hat\phi_i^\infty(y)\,dy = 0,$$

for each $i = 1, \dots, m$ and $j = 1, 2$ where $\xi_i^\infty \in \partial\Omega$ is an accumulation point of the sequence $(\xi_i^n)_n$. By the nondegeneracy result on the linearized Liouville equation (3.8) in [**5**], it follows that, as in the proof of Lemma 3.1,

$$\hat\phi_i^n \to \hat\phi_i^\infty = \hat C_i \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C_{\text{loc}}(\mathbb{R}^2), \tag{3.13}$$

for some constant $\hat C_i \in \mathbb{R}$.

To obtain a contradiction, we will show that $\hat C_i = 0$ for all $i = 1, \dots, m$ and, for this aim, we consider smooth solutions $w, t$ of

$$\Delta w + \frac{8}{(1+|y|^2)^2}w = \frac{8}{(1+|y|^2)^2}\cdot\frac{-1+|y|^2}{1+|y|^2},$$
$$\Delta t + \frac{8}{(1+|y|^2)^2}t = \frac{8}{(1+|y|^2)^2} \quad \text{in } \mathbb{R}^2,$$

such that

$$w(y) = \frac{4}{3}\log|y| + O\left(\frac{1}{|y|}\right), \quad t(y) = O\left(\frac{1}{|y|}\right) \text{ as } |y| \longrightarrow \infty \tag{3.14}$$

and

$$\nabla w(y) = \frac{4}{3}\cdot\frac{y}{1+|y|^2} + O\left(\frac{1}{1+|y|^2}\right), \quad \nabla t(y) = O\left(\frac{1}{1+|y|^2}\right) \text{ for all } y \in \mathbb{R}^2, \tag{3.15}$$

whose existence are shown in [12, Lemma 2.1]. Using these functions, for any $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, set

$$u_i^n(x) = w\left(\frac{x-\xi_i^n}{\delta_i^n}\right) + \frac{4}{3}\log\delta_i^n \cdot Z_{i0}(x) + \frac{1}{3}H(\xi_i^n,\xi_i^n)t\left(\frac{x-\xi_i^n}{\delta_i^n}\right)$$

and

$$\tilde{u}_i^n = u_i^n + h_i^n,$$

where $h_i^n$ is a solution of

$$\Delta_a h_i^n + \nabla\log a \cdot \nabla u_i^n = 0 \text{ in } \Omega, \quad h_i^n = -u_i^n \text{ on } \partial\Omega.$$

By means of (3.14), (3.15), (A.2), (2.1) and elliptic regularity, it can be estimated that

$$\|h_i^n + \tfrac{1}{3}H(x,\xi_i^n)\|_{C^0(\bar{\Omega})} = O(p_n\delta_i^n) \tag{3.16}$$

and

$$\|h_i^n + u_i^n + \tfrac{1}{3}G(x,\xi_i^n)\|_{C^0(\{|x-\xi_j^n|\leqslant p^{-2}\sqrt{\delta_j^n}\})} = O(p_n\log^2 p_n\delta_i^n) \quad \text{for } j\neq i. \tag{3.17}$$

Observe that $\tilde{u}_i^n$ solves

$$\Delta_a\tilde{u}_i^n + W^n\tilde{u}_i^n = (W^n - e^{U_{\delta_i^n,\xi_i^n}})\tilde{u}_i^n + e^{U_{\delta_i^n,\xi_i^n}}Z_{i0} + e^{U_{\delta_i^n,\xi_i^n}}(\tilde{u}_i^n - u_i^n + \tfrac{1}{3}H(\xi_i^n,\xi_i^n)).$$

Thus, it follows that

$$\begin{aligned}\int_\Omega a[e^{U_{\delta_i^n,\xi_i^n}}Z_{i0}\phi_n + (W^n - e^{U_{\delta_i^n,\xi_i^n}})\tilde{u}_i^n\phi_n]\\ = \int_\Omega a\left[\tilde{u}_i^n h_n - e^{U_{\delta_i^n,\xi_i^n}}\left(\tilde{u}_i^n - u_i^n + \frac{1}{3}H(\xi_i^n,\xi_i^n)\right)\phi_n\right].\end{aligned} \tag{3.18}$$

We estimate each term of (3.18). First of all, by (3.13),

$$\begin{aligned}\int_\Omega a\,e^{U_{\delta_i^n,\xi_i^n}}Z_{i0}\phi_n &= \int_{(\Omega-\xi_i^n)/\delta_i^n} a(\delta_i^n y+\xi_i^n)\frac{8}{(1+|y|^2)^2}\cdot\frac{-1+|y|^2}{1+|y|^2}\hat{\phi}_i^n(y)\,dy\\ &\longrightarrow \hat{C}_i a(\xi_i^\infty)\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(|y|^2+1)^4}\,dy = \frac{8\pi}{3}\hat{C}_i a(\xi_i^\infty).\end{aligned}$$

Moreover, the second term of the left-hand side of (3.18) can be estimated as

$$\begin{aligned}&\int_\Omega a(W^n - e^{U_{\delta_i^n,\xi_i^n}})\tilde{u}_i^n\phi_n\\ &= \int_{|x-\xi_i^n|\leqslant p_n^{-2}\sqrt{\delta_i^n}} a(W^n - e^{U_{\delta_i^n,\xi_i^n}})\tilde{u}_i^n\phi_n\,dx\\ &\quad - \frac{1}{3}\sum_{j\neq i}G(\xi_j,\xi_i)\int_{|x-\xi_j^n|\leqslant p_n^{-2}\sqrt{\delta_j^n}} aW^n\phi_n\,dx + O(e^{-(p_n/8)(1-\epsilon)})\\ &= \int_{|y|\leqslant Cp_n^{-3/2}\log^{(1/2)(m-1)}p_n\,e^{p_n/8}}\left[\frac{8a(\delta_i^n y+\xi_i^n)}{(1+|y|^2)^2}\cdot\frac{1}{p_n}\cdot\left(w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2}\right)\right.\\ &\quad \left.\cdot\left\{w(y) + \frac{4}{3}\log\delta_n\frac{|y|^2-1}{|y|^2+1} + \frac{1}{3}H(\xi_n,\xi_n)t(y) - \frac{1}{3}H(\delta_n y+\xi_n,\xi_n) + O(\delta_n)\right\}\hat{\phi}_i^n(y)\right]dy\\ &\quad - \frac{1}{3}\sum_{j\neq i}G(\xi_j,\xi_i)\int_{|y|\leqslant Cp_n^{-3/2}\log^{(1/2)(m-1)}p_n\,e^{p_n/8}} a(\delta_j^n y+\xi_j^n)\frac{8}{(1+|y|^2)^2}\hat{\phi}_j^n(y)\,dy + O\left(\frac{1}{p}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{3}\hat{C}_i \frac{\log \delta_i^n}{p_n} \int_{|y|\leqslant Cp_n^{-3/2}\log^{(1/2)(m-1)} p_n\, e^{p_n/8}} \left[\frac{8a(\delta_i^n y+\xi_i^n)(|y|^2-1)^2}{(|y|^2+1)^4}\right.\\
&\quad \left.\times \left(w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2}\right)\right] dy + o(1)\\
&\longrightarrow \frac{8\pi}{3}\hat{C}_i a(\xi_i^\infty),
\end{aligned}
$$

with constants $C > 0$ and $\epsilon \in (0,1)$. In this chain of equalities, the first equality is due to (3.17), and the second equality comes from (3.16) and (3.7). Besides, we used (A.2), the logarithmic growth of $U_{1,0}$ and $w_0$ (see (2.4)) and

$$
\int_{\mathbb{R}^2} \frac{|y|^2-1}{(1+|y|^2)^3}\, dy = 0, \quad \int_{\mathbb{R}^2} \frac{8(|y|^2-1)^2}{(|y|^2+1)^4}\left(w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2}\right) dy = -8\pi,
$$

to obtain the third equality and the last implication.

On the other hand, $\int_\Omega a\tilde{u}_i^n h_n$ is bounded by a constant multiple of $p_n\|h_n\|_*$, since

$$
\left|\int_\Omega a\tilde{u}_i^n h_n\right| \leqslant C\|h_n\|_* \left(\int_{(\Omega-\xi_i^n)/\delta_i^n} |\tilde{u}_i^n(\delta_n y+\xi_n)| \frac{dy}{(1+|y|^2)^{3/2}}\right),
$$

and the estimates (3.16) and (A.2) imply that $\|\tilde{u}_i^n\|_{L^\infty} = O(\log \delta_i^n) = O(p_n)$. By (3.16) and (A.2) again, we see that

$$
\begin{aligned}
\tilde{u}_i^n - u_i^n + \tfrac{1}{3}H(\xi_i^n,\xi_i^n) &= \tfrac{1}{3}(H(\xi_i^n,\xi_i^n) - H(x,\xi_i^n)) + O(p_n\delta_i^n)\\
&= O(p_n)|x-\xi_i^n|^\gamma + O(p_n\delta_i^n),
\end{aligned}
$$

for any $\gamma \in (0,1)$, and so

$$
\begin{aligned}
&\left|\int_\Omega a\, e^{U_{\delta_i^n,\xi_i^n}} \left(\tilde{u}_i^n - u_i^n + \frac{1}{3}H(\xi_i^n,\xi_i^n)\right)\phi_n\right|\\
&\quad \leqslant C\int_\Omega [e^{U_{\delta_i^n,\xi_i^n}}(x)\{O(p_n)|x-\xi_i^n|^\gamma + O(p_n\delta_i^n)\}]\, dx = O((\delta_i^n)^\gamma).
\end{aligned}
$$

Summing up all of the estimates, we conclude that

$$
\frac{16\pi}{3}\hat{C}_i a(\xi_i^\infty) = 0,
$$

or $\hat{C}_i = 0$, which implies, in particular, that the inner norm $\|\phi_n\|_i \to 0$ as $n \to \infty$. However, (3.12) and (3.5) tells us $\liminf_{n\to\infty} \|\phi_n\|_i > 0$, so a contradiction arises. The proof is completed. □

LEMMA 3.3. *For $p$ sufficiently large, $\xi \in \Lambda$ and $h \in C^{0,\alpha}(\Omega)$, a solution for (3.2) satisfies*

$$
\|\phi\|_{L^\infty(\Omega)} \leqslant Cp\|h\|_*,
$$

*with $C > 0$ independent of $p$.*

*Proof.* As before, we assume the existence of sequences of parameters $p_n$ diverging to $\infty$, $m$-tuples of points $\xi_n = (\xi_1^n, \ldots, \xi_m^n) \in \Lambda$, functions $h_n \in C^{0,\alpha}(\Omega)$ and corresponding solutions $\phi_n$, $c_{ij}^n$ satisfying (3.12).

By Lemma 3.2, any solution $\phi$ of (3.2) must satisfy

$$
\|\phi\|_{L^\infty(\Omega)} \leqslant Cp\left(\|h\|_* + \sum_{i=1}^m\sum_{j=1}^2 |c_{ij}|\cdot\|e^{U_{\delta_i,\xi_i}}Z_{ij}\|_*\right) \leqslant Cp\left(\|h\|_* + \sum_{i=1}^m\sum_{j=1}^2 |c_{ij}|\right).
$$

Hence, our assumption gives a small positive number $\epsilon_0$ such that

$$p_n \sum_{i=1}^{m} \sum_{j=1}^{2} |c_{ij}^n| \geqslant \epsilon_0 > 0. \tag{3.19}$$

We will show that $p_n \sum_{i=1}^{m} \sum_{j=1}^{2} |c_{ij}^n| \to 0$ as $n \to \infty$ to obtain a contradiction.

For $i = 1, \ldots, m$ and $j = 1, 2$, let $h_i^n$ be a correction term defined as a solution of

$$\Delta_a h_{ij}^n + \nabla \log a \cdot \nabla Z_{ij}^n = 0 \text{ in } \Omega, \quad h_{ij}^n = -Z_{ij}^n \text{on } \partial\Omega,$$

where $Z_{ij}^n$ is equal to $Z_{ij}$ with $\xi_n \in \Lambda$. Through $W^{2,p}$-estimation, we can check that

$$\|h_{ij}^n\|_{H^{1,2}(\Omega)} = O((\delta_i^n)^{1-\epsilon}) \quad \text{uniformly in } n \text{ for any } \epsilon > 0. \tag{3.20}$$

Define then

$$\bar{Z}_{ij}^n = Z_{ij}^n + h_{ij}^n,$$

so that it satisfies

$$\Delta_a \bar{Z}_{ij}^n = \Delta Z_{ij}^n = -e^{U_{\delta_i^n, \xi_i^n}} Z_{ij}^n \text{ in } \Omega, \quad \bar{Z}_{ij}^n = 0 \text{ on } \partial\Omega.$$

Multiplying $a\bar{Z}_{kl}^n$ ($k = 1, \ldots, m$ and $l = 1, 2$) on both sides of $L(\phi_n) = h_n + \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij}^n \times e^{U_{\delta_i^n, \xi_i^n}} Z_{ij}$ and integrating, this yields

$$\sum_{i=1}^{m} \sum_{j=1}^{2} \left( c_{ij}^n \int_\Omega a \nabla \bar{Z}_{ij}^n \cdot \nabla \bar{Z}_{kl}^n \right) + \int_\Omega a h_n \bar{Z}_{kl}^n = \int_\Omega a W^n \phi_n \bar{Z}_{kl}^n - \int_\Omega a\, e^{U_{\delta_k^n, \xi_k^n}} \phi_n Z_{kl}^n. \tag{3.21}$$

For the first term of the left-hand side of (3.21), the correction term estimation (3.20) shows

$$\begin{aligned}
\int_\Omega a \nabla \bar{Z}_{ij}^n \cdot \nabla \bar{Z}_{il}^n &= \int_\Omega a\, e^{U_{\delta_i^n, \xi_i^n}} Z_{ij}^n \bar{Z}_{il}^n = \int_\Omega a\, e^{U_{\delta_i^n, \xi_i^n}} Z_{ij}^n Z_{il}^n + O((\delta_i^n)^{1-\epsilon}) \\
&= 128 \int_{(\Omega - \xi_i^n)/\delta_i^n} a(\delta_i^n y + \xi_i^n) \frac{y_j y_l}{(1+|y|^2)^4}\, dy + O((\delta_i^n)^{1-\epsilon}) \\
&= \frac{32\pi}{3} a(\xi_i^\infty) \delta_{jl} + O((\delta_i^n)^{1-\epsilon})
\end{aligned} \tag{3.22}$$

and

$$\int_\Omega a \nabla \bar{Z}_{ij}^n \cdot \nabla \bar{Z}_{kl}^n = O(\delta_i^n p_n \log p_n) + O((\delta_k^n)^{1-\epsilon}) \quad \text{for } i \neq k, \tag{3.23}$$

where $\delta_{ij}$ denotes the Kronecker delta and $\xi_i^n \to \xi_i^\infty \in \partial\Omega$ along a subsequence. Besides, one can immediately check

$$\left| \int_\Omega a h_n \bar{Z}_{kl}^n \right| \leqslant C \|h_n\|_*.$$

Next, the right-hand side of (3.21) can be estimated as follows, owing to (3.7):

$$\begin{aligned}
&\int_\Omega a W^n \phi_n \bar{Z}_{kl}^n - \int_\Omega a\, e^{U_{\delta_k^n, \xi_k^n}} \phi_n Z_{kl}^n \\
&= \int_\Omega a (W^n - e^{U_{\delta_k^n, \xi_k^n}}) \phi_n (Z_{kl}^n + h_{kl}^n) + \int_\Omega a\, e^{U_{\delta_k^n, \xi_k^n}} h_{kl}^n \phi_n \\
&= \int_{|x - \xi_k^n| \leqslant p_n^{-2} \sqrt{\delta_k^n}} a (W^n - e^{U_{\delta_k^n, \xi_k^n}}) \phi_n Z_{kl}^n + O((\delta_k^n)^{1-\epsilon}) \\
&= \frac{a(\xi_k^\infty)}{p_n} \int_{\mathbb{R}^2} \frac{32 y_l}{(1+|y|^2)^3} \left( w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2} \right) \hat{\phi}_k^n(y)\, dy + O((\delta_k^n)^{1-\epsilon}).
\end{aligned}$$

Accordingly, (3.21) can be reduced to

$$\frac{32\pi}{3}a(\xi_k^\infty)c_{kl}^n + o(1)\sum_{i=1}^m\sum_{j=1}^2 c_{ij}^n = O\left(\|h_n\|_* + \frac{1}{p_n}\right),$$

or equivalently

$$\sum_{i=1}^m\sum_{j=1}^2 |c_{ij}^n| = O\left(\|h_n\|_* + \frac{1}{p_n}\right).$$

This in particular implies that

$$\hat{\phi}_i^n \longrightarrow \bar{C}_i\frac{|y|^2-1}{|y|^2+1} \quad \text{in } C_{\text{loc}}(\mathbb{R}^2),$$

for some constant $\bar{C}_i \in \mathbb{R}$ and thus

$$\int_{\mathbb{R}^2}\frac{32y_j}{(1+|y|^2)^3}\left(w_0(y) - U_{1,0}(y) - \frac{U_{1,0}^2(y)}{2}\right)\hat{\phi}_i^n(y)\,dy \longrightarrow 0.$$

Therefore,

$$\sum_{i=1}^m\sum_{j=1}^2 |c_{ij}^n| = O(\|h_n\|_*) + o\left(\frac{1}{p_n}\right),$$

which is impossible because of (3.19). □

*Proof of Proposition* 3.1. Step 6 of the proof of [**12**, Proposition 3.1], based on a well-known argument which utilizes the Fredholm alternative, can be adapted to our case with some minor modifications. More precisely, in this case we have

$$K_\xi = \left\{\sum_{i=1}^m\sum_{j=1}^2 c_{ij}\bar{Z}_{ij} \,:\, c_{ij}\in\mathbb{R}\right\},$$
$$K_\xi^\perp = \left\{\phi\in L^2(\Omega) : \int_\Omega a\,e^{U_{\delta_i,\xi_i}}Z_{ij}\phi = 0 \text{ for } i=1,\dots,m \text{ and } j=1,2\right\}$$

and

$$\Pi_\xi\phi = \sum_{i=1}^m\sum_{j=1}^2 c_{ij}\bar{Z}_{ij},$$

where the coefficients $c_{ij}$ are uniquely determined by the system of linear equations

$$\int_\Omega a\,e^{U_{\delta_i,\xi_i}}Z_{ij}\left(\phi - \sum_{k=1}^m\sum_{l=1}^2 c_{kl}\bar{Z}_{kl}\right) = 0 \quad \text{for } i=1,\dots,m \text{ and } j=1,2,$$

owing to (3.22). Moreover, the Hilbert space $K_\xi^\perp \cap H_0^1(\Omega)$ is equipped with the inner product

$$(\phi,\psi)_{H_0^1(\Omega)} = \int_\Omega a\nabla\phi\cdot\nabla\psi.$$

We omit the details. □

## 4. *The reduced problem*

First, we want to solve the nonlinear problem: *find* $\phi \in W^{2,2}(\Omega)$ *and* $c_{ij} \in \mathbb{R}$ *which solve*

$$\begin{cases} L(\phi) = -(R + N(\phi)) + \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{2} c_{ij}\, e^{U_{\delta_i,\xi_i}} Z_{ij} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij}\phi = 0 & \text{for } i = 1, \dots, m \text{ and } j = 1, 2. \end{cases} \tag{4.1}$$

Proposition 4.1. *For $p$ large enough and $\xi \in \Lambda$, there is a unique solution $\phi_\xi$ and $c_{ij,\xi}$ for (4.1) with*

$$\|\phi_\xi\|_{L^\infty(\Omega)} \leqslant \frac{C}{p^3}, \quad \|\phi_\xi\|_{H^1(\Omega)} \leqslant \frac{C}{p^3} \quad \textit{and} \quad |c_{ij,\xi}| \leqslant \frac{C}{p^4}. \tag{4.2}$$

*Proof.* Proposition 3.1 allows us to apply the contraction mapping principle to find a solution for the problem (4.1) satisfying (4.2). Since it is a standard procedure, we shall not present the detailed proof here. See the proof of [**12**, Lemma 4.1]. □

Let $I_p$ be the energy functional whose critical points are solution to the problem (1.4), namely

$$I_p(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega a(x)|u|^{p+1}\,dx \quad \text{for } u \in H_0^1(\Omega).$$

Using Proposition 3.1, we can introduce the *reduced energy*

$$F_p(\xi) = I_p(U_\xi + \phi_\xi), \quad \xi \in \Lambda, \tag{4.3}$$

where the subscript $\xi$ in $U_\xi$ is used to emphasize the dependence of $U$ on $\xi \in \Lambda$.

Lemma 4.1. *The function $F_p : \Lambda \to \mathbb{R}$ is of class $C^1$. Furthermore, if $F_p'(\xi) = 0$, then $c_i(\xi) = 0$ for $i = 1, 2$ and, in particular, $U_\xi + \phi_\xi$ is a solution of (1.4).*

*Proof.* To obtain $F_p \in C^1(\Lambda)$, it suffices to check that $\xi \mapsto \phi_\xi$ is $C^1$. In fact, it follows from the implicit function theorem with Proposition 3.1 (we refer the reader to [**12**, pp. 57–58] for details).

Suppose now that $\xi \in \Lambda$ is a point such that $F_p'(\xi) = 0$. Then we have

$$\begin{aligned} 0 &= I_p'(U_\xi + \phi_\xi)(DU_\xi + D\phi_\xi) \\ &= -\sum_{i=1}^{m}\sum_{j=1}^{2} c_{ij}(\xi)\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij} DU_\xi + \sum_{i=1}^{m}\sum_{j=1}^{2} c_{ij}(\xi)\int_\Omega aD(e^{U_{\delta_i,\xi_i}} Z_{ij})\phi_\xi. \end{aligned} \tag{4.4}$$

To obtain the second equality, we took advantage of the relation $\int_\Omega a\, e^{U_{\delta_i,\xi_i}} Z_{ij}\phi_\xi = 0$. Also, $D$ denotes the differentiation with respect to the parameter $\xi = (\xi_1, \dots, \xi_m)$, that is,

$$D = \left(\frac{\partial}{\partial(\xi_1)_1}, \frac{\partial}{\partial(\xi_1)_2}, \dots, \frac{\partial}{\partial(\xi_m)_1}, \frac{\partial}{\partial(\xi_m)_2}\right).$$

As the first step of estimation for (4.4), we assert

$$\frac{\partial(U_\xi)_i}{\partial(\xi_i)_j}(x) = \frac{1}{\gamma\mu_i^{2/(p-1)}\delta_i}\left(Z_{ij}(x) - \frac{1}{p}\frac{\partial w_0}{\partial(\xi_i)_j}\left(\frac{x-\xi_i}{\delta_i}\right) - \frac{1}{p^2}\frac{\partial w_1}{\partial(\xi_i)_j}\left(\frac{x-\xi_i}{\delta_i}\right) + O\left(\delta_i^{(2-q)/q}\right)\right), \tag{4.5}$$

for arbitrary $q \in (1, 2)$. To see this, we first decompose $DU_\xi = D\tilde{U}_\xi + D(H_p)_\xi$ and estimate each term. Since $\partial\mu/\partial\xi_j = O(\log^m p)$ by definition (2.8) of $\mu$ and $\|U_{\delta_i,\xi_i}\|_{L^\infty(\Omega)} = O(p)$, we obtain that

$$\frac{\partial(\tilde{U}_\xi)_i}{\partial(\xi_i)_j}(x) = \frac{1}{\gamma\mu_i^{2/(p-1)}\delta_i}\left(Z_{ij}(x) - \frac{1}{p}\frac{\partial w_0}{\partial(\xi_i)_j}\left(\frac{x-\xi_i}{\delta_i}\right) - \frac{1}{p^2}\frac{\partial w_1}{\partial(\xi_i)_j}\left(\frac{x-\xi_i}{\delta_i}\right)\right) + O(\log^{2m-1} p).$$

Furthermore, by the elliptic regularity of the equation that $(\partial/\partial(\xi_i)_j)(H_i^p)_\xi$ satisfies, we obtain $\|D(H_i^p)_\xi\|_{L^\infty(\Omega)} = O(p^{-1}\delta_i^{(2-2q)/q})$ for any $q \in (1, 2)$. Hence, putting them together, we derive (4.5).

On the other hand, it can be shown that $\|D(e^{U_{\delta_i,\xi_i}}Z_{ij})\|_{L^\infty(\Omega)} = O(\delta_i^{-1})$ by computing directly. Consequently, (4.4) can be written as, for each $k = 1, \ldots, m$ and $l = 1, 2$,

$$\begin{aligned} 0 &= -\sum_{i=1}^m\sum_{j=1}^2 c_{ij}(\xi)\int_\Omega a\, e^{U_{\delta,\xi}} Z_{ij}Z_{kl} + \sum_{i=1}^m\sum_{j=1}^2 c_{ij}(\xi) O\left(\frac{1}{p}\right) \\ &= 64a(\xi)c_{kl}(\xi)\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}\,dy + \sum_{i=1}^m\sum_{j=1}^2 c_{ij}(\xi)O\left(\frac{1}{p}\right) \\ &= \frac{16\pi}{3}a(\xi)c_{kl}(\xi) + \sum_{i=1}^m\sum_{j=1}^2 c_{ij}(\xi)O\left(\frac{1}{p}\right), \end{aligned}$$

which implies $c_{kl}(\xi) = 0$. This proves the lemma. □

Moreover, the following energy expansion holds.

LEMMA 4.2. *For sufficiently large $p$, it holds that*

$$F_p(\xi) = \sum_{i=1}^m \frac{4\pi e}{p}a(\xi_i)\left[1 - 2\frac{\log p}{p} + \frac{c}{p} - \frac{1}{p}\Psi_i(\xi)\right] + O\left(\frac{1}{p^{3-\epsilon}}\right) \tag{4.6}$$

*uniformly for $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$ where*

$$\Psi_i(\xi) = H(\xi_i, \xi_i) + \sum_{j\neq i} a_i a_j G(\xi_i, \xi_j), \quad i = 1, \ldots, m, \tag{4.7}$$

*the constant $c$ is defined as*

$$c := 6 + \frac{1}{8\pi}\int_{\mathbb{R}^2}\left(\frac{8U_{1,0}(y)}{(1+|y|^2)^2} - \Delta w_0(y)\right)dy, \tag{4.8}$$

*and $\epsilon > 0$ arbitrarily small.*

*Proof.* By multiplying with $a(U_\xi + \phi_\xi)$ on both sides of

$$\Delta_a(U_\xi + \phi_\xi) + |U_\xi + \phi_\xi|^{p-1}(U_\xi + \phi_\xi) = \sum_{i=1}^m\sum_{j=1}^2 c_{ij}\, e^{U_{\delta_i,\xi_i}} Z_{ij} \quad \text{in } \Omega, \tag{4.9}$$

and applying (4.2), we get

$$F_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega a(x)\left(|\nabla U_\xi|^2 + 2\nabla U_\xi\cdot\nabla\phi_\xi + |\nabla\phi_\xi|^2\right)dx + O\left(\frac{1}{p^4}\right).$$

We consider $\int_\Omega a(x)|\nabla U_\xi|^2\,dx$ first. In fact, by (2.1), (2.2) and (2.11), we have

$$\begin{aligned}
&\int_\Omega a(x)|\nabla U_\xi|^2\,dx\\
&=\sum_{i=1}^m \int_\Omega a(x)\cdot\frac{a_i}{\gamma\mu_i^{2/(p-1)}}\left[e^{U_{\delta_i,\xi_i}(x)}+\frac{1}{p}(\Delta w_0)\left(\frac{x-\xi_i}{\delta_i}\right)+\frac{1}{p^2}(\Delta w_1)\left(\frac{x-\xi_i}{\delta_i}\right)\right]U_\xi(x)\,dx\\
&=\sum_{i=1}^m \int_{|x-\xi_i|\leqslant p^{-2}} a(x)\cdot\frac{a_i}{\gamma\mu_i^{2/(p-1)}}\left[e^{U_{\delta_i,\xi_i}(x)}+\frac{1}{p}(\Delta w_0)\left(\frac{x-\xi_i}{\delta_i}\right)\right.\\
&\quad\left.+\frac{1}{p^2}(\Delta w_1)\left(\frac{x-\xi_i}{\delta_i}\right)+O(e^{-p/2+\epsilon})\right]U_\xi(x)\,dx\\
&=\sum_{i=1}^m \int_{|y|\leqslant p^{-2}\delta_i^{-1}} \frac{a(\delta_i y+\xi_i)}{(\gamma\mu_i^{2/(p-1)})^2}\left[\frac{8}{(1+|y|^2)^2}+\frac{1}{p}(\Delta w_0)(y)+\frac{1}{p^2}(\Delta w_1)(y)+O(e^{-p+\epsilon})\right]\\
&\quad\cdot\left[p+U_{1,0}(y)+\frac{1}{p}w_0(y)+\frac{1}{p^2}w_1(y)+O\left(\frac{1}{p^{1-\epsilon}}\right)\right]dy\\
&=\sum_{i=1}^m \frac{1}{(\gamma\mu_i^{2/(p-1)})^2}\left\{\int_{|y|\leqslant p^{-2}\delta_i^{-1}} a(\delta_i y+\xi_i)\frac{8p}{(1+|y|^2)^2}\,dy\right.\\
&\quad\left.+\int_{|y|\leqslant p^{-2}\delta_i^{-1}} a(\delta_i y+\xi_i)\left(\frac{8U_{1,0}(y)}{(1+|y|^2)^2}-\Delta w_0(y)\right)dy+O\left(\frac{1}{p^{1-\epsilon}}\right)\right\},
\end{aligned}$$

where $\epsilon>0$ is sufficiently small. Furthermore, from the expansion

$$\gamma^{-2}=\frac{e}{p^2}+\frac{-2e\log p+e}{p^3}+O\left(\frac{1}{p^{4-\epsilon}}\right),\quad \mu_i^{-4/(p-1)}=1-\frac{4}{p}\log\mu_i+O\left(\frac{1}{p^2}\right),$$

and (2.8), it follows that

$$\begin{aligned}
&\int_\Omega a(x)|\nabla U_\xi|^2\,dx\\
&=\sum_{i=1}^m \frac{e}{p^2}a(\xi_i)\left[1+\frac{1}{p}-\frac{2\log p}{p}+O\left(\frac{1}{p^{2-\epsilon}}\right)\right]\cdot\left[1-\frac{4}{p}\log\mu_i+O\left(\frac{1}{p^2}\right)\right]\\
&\quad\cdot\left[8\pi p+\int_{\mathbb{R}^2}\left(\frac{8U_{1,0}(y)}{(1+|y|^2)^2}-\Delta w_0(y)\right)dy+O\left(\frac{1}{p^{1-\epsilon}}\right)\right]\\
&=\sum_{i=1}^m \frac{8\pi e}{p}a(\xi_i)\left[1-2\frac{\log p}{p}-\frac{1}{p}\Psi_i(\xi)+\frac{1}{p}\left\{4+\frac{1}{8\pi}\int_{\mathbb{R}^2}\left(\frac{8U_{1,0}(y)}{(1+|y|^2)^2}-\Delta w_0(y)\right)dy\right\}\right]\\
&\quad+O\left(\frac{1}{p^{3-\epsilon}}\right).
\end{aligned}\tag{4.10}$$

On the other hand, by virtue of (4.2), we deduce that

$$\int_\Omega a(x)(2\nabla U_\xi\cdot\nabla\phi_\xi+|\nabla\phi_\xi|^2)\,dx=O\left(\frac{1}{p^4}\right).\tag{4.11}$$

Consequently, putting (4.10) and (4.11) together, we obtain (4.6). □

Now, we look for critical points of the reduced energy (4.3).

Lemma 4.3. *Assume either $m=1$ and $a_1=+1$ or $m=2$, $a_1=+1$ and $a_2=-1$. For sufficiently large $p$, the minimum of $F_p$ in $\Lambda$ is attained by a point in the interior of $\Lambda$.*

*Proof.* We consider the case when $m = 2$ only, since the case $m = 1$ is a little bit easier and can be handled just as in the case $m = 2$.

Let $\xi^p = (\xi_1^p, \xi_2^p) \in \Lambda$ be a minimizer of $F_p$ in $\Lambda$. We will show that it is contained in the interior of $\Lambda$.

Define $\xi_1^0 = \bar{x} + \nu(\bar{x})/p$ and $\xi_2^0 = \bar{x} + 2(\nu(\bar{x})/p)$. Then the point $\xi^0 := (\xi_1^0, \xi_2^0) \in \Lambda$ provided $\tilde{C}_1 < 1$ and $2 < \tilde{C}_2$. By utilizing (A.4), we obtain an upper energy estimation:

$$\min_{\xi \in \Lambda} F_p(\xi) \leqslant F_p(\xi^0) = \frac{8\pi e}{p} a(\bar{x}) \left(1 + \frac{2\log p}{p}\right) + \frac{\alpha}{p^2} + O\left(\frac{1}{p^{3-\epsilon}}\right), \tag{4.12}$$

where $\alpha$ is a positive constant.

Suppose now that $\xi^p \in \partial\Lambda$. Then one of the three alternatives holds:

$$\xi_i^p \in \partial B_\rho(\bar{x}), \quad d(\xi_i^p, \partial\Omega) = \frac{\tilde{C}_1}{p} \text{ or } \frac{\tilde{C}_2}{p}, \quad \text{or} \quad |\xi_1^p - \xi_2^p| = \frac{1}{p\log p},$$

for some $i = 1, 2$.

First, if $\xi_i^p \in \partial B_\rho(\bar{x})$, by (A1) we deduce that there exists a positive constant $C$ such that

$$a(\xi_i^p) \geqslant a(\bar{x}) + C.$$

Thus,

$$\min_{\xi \in \Lambda} F_p(\xi) = F_p(\xi^p) \geqslant \frac{8\pi e}{p} a(\bar{x}) \left(1 + \frac{2\log p}{p}\right) + \frac{4\pi e C}{p} + O\left(\frac{1}{p^2}\right),$$

but it contradicts (4.12). Note that this argument also shows that $\xi_p \to \bar{x}$ as $p \to \infty$. Here, we used in a crucial way that the interaction between the peaks $G(\xi_1, \xi_2)$ is positive.

Next, if $d(\xi_i^p, \partial\Omega) = \tilde{C}_j/p$ $(j = 1, 2)$ for some $i = 1, 2$, we denote by $x_i^p \in \partial\Omega$ the orthogonal projection of $\xi_i^p$ into $\partial\Omega$ and by (A2) we can select $C_0 > 0$ such that

$$a(\xi_i^p) \geqslant a(x_i^p) + C_0 \frac{\tilde{C}_j}{p} \geqslant a(\bar{x}) + C_0 \frac{\tilde{C}_j}{p}.$$

Hence,

$$\min_{\xi \in \Lambda} F_p(\xi) = F_p(\xi^p) \geqslant \frac{8\pi e}{p} a(\bar{x}) \left(1 + \frac{2\log p}{p}\right) + \frac{1}{p^2}(\beta_1 \tilde{C}_j - \beta_2 \log \tilde{C}_j - \gamma),$$

where $\beta_1$, $\beta_2$ and $\gamma$ are positive constants. This and (4.12) imply again a contradiction if we choose $\tilde{C}_1$ sufficiently small and $\tilde{C}_2$ sufficiently large.

Finally, if $|\xi_1^p - \xi_2^p| = 1/p\log p$, then we have $G(\xi_1^p, \xi_2^p) \geqslant C \log\log p$ for some positive constant $C$. Therefore,

$$\min_{\xi \in \Lambda} F_p(\xi) = F_p(\xi^p) \geqslant \frac{8\pi e}{p} a(\bar{x}) \left(1 + \frac{2\log p}{p} + C\frac{\log\log p}{p}\right) + O\left(\frac{1}{p^2}\right),$$

and yet it is again impossible in view of (4.12).

This proves that a minimum point $\xi_p$ of $F_p$ should be contained in the interior of $\Lambda$. □

*Proof of Theorems* 1.1 *and* 1.3. Lemma 4.3 shows that $F_p$ has a minimum point in $\Lambda$, which is stable under $C^0$-perturbations. Therefore, by Lemma 4.1, we have a solution to (1.4) of the form $U_{\xi^p} + \phi_{\xi^p}$ with some $\xi^p = (\xi_1^p, \dots, \xi_m^p) \in \Lambda^p$ such that $\xi_i^p \to \bar{x}$ as $p \to \infty$, given $p$ is sufficiently large. This proves the existence part in the statement of Theorems 1.1 and 1.3. The properties (1.5)–(1.7) in Theorem 1.1 and (1.9)–(1.11) in Theorem 1.3 follow from (4.2), (2.12), (2.11) and (3.7). □

## 5. *The symmetric case and the proof of Theorem* 1.5

We will look for a solution of (1.4) as in (2.6) where the concentration points $\xi_1, \dots, \xi_m$ are aligned on the line $\mathfrak{L}$ (see (A3)):

$$\xi_i = \xi_i(\mathbf{t}) = \bar{x} + \frac{t_i}{p}\nu(\bar{x}) \quad \text{with } \mathbf{t} = (t_1, \dots, t_m),\ \ 0 < t_1 < \cdots < t_m. \tag{5.1}$$

We search for solutions in the space $H_e$ of functions which are even with respect to the line $\mathfrak{L}$ (see (A3)). We point out that the approximate solution (2.6) belongs to $H_e$ when $\xi_i$ satisfies (5.1).

A standard argument (see, for example, [7]) proves that if $\mathbf{t}$ is a critical point of the reduced energy $\hat{F}_p(\mathbf{t}) := F_p(\xi(\mathbf{t}))$ defined as in (4.3), then the function $U_{\xi(t)} + \phi_{\xi(t)} \in H_e$ is a solution of problem (1.4). Therefore, we are led to compute the expansion of the reduced energy and, with the aid of (A.4) and (A.1), we obtain that, given $a_i = (-1)^i$,

$$\hat{F}(\mathbf{t}) = m\frac{4\pi e}{p}a(\bar{x})\left[1 + \frac{2\log p}{p} + \frac{c}{p}\right] + \frac{4\pi e}{p^2}\Gamma(\mathbf{t}) + o\left(\frac{1}{p^2}\right),$$

where the constant $c$ is defined in (4.8) and

$$\Gamma(\mathbf{t}) := \sum_{i=1}^{m}\left[(\partial_\nu a(\bar{x})\cdot t_i - 4a(\bar{x})\log t_i) + 4a(\bar{x})\sum_{j\neq i}(-1)^{i+j+1}\log\frac{|t_i + t_j|}{|t_i - t_j|}\right].$$

Finally, Theorem 1.5 will follow by the next lemma, where we prove that $\Gamma$ has a minimum point which is stable under uniform perturbations.

Lemma 5.1. *The set $\Gamma$ has a minimizer in $\hat{\Lambda} := \{(t_1, \dots, t_m) : 0 < t_1 < \cdots < t_m\}$.*

*Proof.* We claim that $\Gamma(t_1, \dots, t_m) \to +\infty$ if there are $i, j \in \{1, \dots, m\}$, $i < j$ such that either $t_i \to 0$ or $+\infty$, or $t_j - t_i \to 0$.

To show that $t_i \to 0$ or $+\infty$ for some $i$ implies $\Gamma(t_1, \dots, t_m) \to \infty$, it suffices to prove

$$\sum_{i=1}^{m}\sum_{j\neq i}(-1)^{i+j+1}\log\frac{|t_i + t_j|}{|t_i - t_j|} = 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(-1)^{i+j+1}\log\frac{t_j + t_i}{t_j - t_i} \geqslant 0,$$

since $\partial_\nu a(0), a(0) > 0$ by our hypothesis. If $m - i$ is even, then we have

$$\sum_{j=i+1}^{m}(-1)^{i+j+1}\log\frac{t_j + t_i}{t_j - t_i} = \log\frac{(t_{i+1} + t_i)(t_{i+2} - t_i)}{(t_{i+1} - t_i)(t_{i+2} + t_i)} + \cdots + \log\frac{(t_{m-1} + t_i)(t_m - t_i)}{(t_{m-1} - t_i)(t_m + t_i)},$$

and each summand in the right-hand side of the above identity should be nonnegative, since it holds

$$\frac{(t_k + t_i)(t_{k+1} - t_i)}{(t_k - t_i)(t_{k+1} + t_i)} \geqslant 1,$$

whenever $k \in \{i+1, i+3, \dots, m-1\}$. If $m - i$ is odd, then

$$\sum_{j=i+1}^{m}(-1)^{i+j+1}\log\frac{t_j + t_i}{t_j - t_i}$$
$$= \log\frac{(t_{i+1} + t_i)(t_{i+2} - t_i)}{(t_{i+1} - t_i)(t_{i+2} + t_i)} + \cdots + \log\frac{(t_{m-2} + t_i)(t_{m-1} - t_i)}{(t_{m-2} - t_i)(t_{m-1} + t_i)} + \log\frac{t_m + t_i}{t_m - t_i},$$

which is nonnegative, too.

Finally, the claim follows arguing exactly as in [7, Proposition 3.1]. □

*Appendix. Estimates of Green's functions and Robin's functions*

This appendix lists some interior and boundary estimates of Green's function and Robin's function defined as follows.

Let $G_D(x, y)$ be Green's function associated to $-\Delta$ with Dirichlet boundary condition, namely,

$$\begin{aligned} -\Delta_x G_D(x, y) &= \delta_y(x) \quad \text{for } x \in \Omega, \\ G_D(x, y) &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

and $H_D(x, y)$ be its regular part defined as

$$H_D(x, y) = G_D(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

Also, denote Green's function of $-\Delta_a$ with Dirichlet boundary condition as $G(x, y)$, that is, let $G(x, y)$ be a function satisfying the following equation:

$$\begin{aligned} -(\Delta_a)_x G(x, y) &= 8\pi\delta_y(x) \quad \text{for } x \in \Omega, \\ G(x, y) &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

and define

$$H(x, y) = G(x, y) - 4 \log \frac{1}{|x - y|}.$$

Then we have the following estimations.

Lemma A.1. *Let $d_0 > 0$ be so small that there is a unique point $x_\nu \in \partial\Omega$ satisfying $d(x, \partial\Omega) = |x - x_\nu|$ for any $x \in \Omega$, $d(x, \partial\Omega) \leqslant d_0$. For such $x \in \Omega$, let $x^* = 2x_\nu - x$ denote the reflection point of $x$ with respect to $\partial\Omega$. Then*

$$G(\cdot, x) = 4 \log \frac{|\cdot - x^*|}{|\cdot - x|} + o(1), \tag{A.1}$$

*where $o(1) \to 0$ as $d(x, \partial\Omega) \to 0$ uniformly in $\Omega$.*

Lemma A.2. *For any $y \in \Omega$, $y \mapsto H(\cdot, y)$ is a continuous map from $\Omega$ into $C^{0,\gamma}(\bar{\Omega})$, $\forall\gamma \in (0, 1)$ and*

$$\|H(\cdot, y)\|_{L^\infty(\bar{\Omega})} = O(|\log d(y, \partial\Omega)|), \quad \|H(\cdot, y)\|_{C^{0,\gamma}(\bar{\Omega})} = O\left(\frac{1}{d(y, \partial\Omega)}\right) \tag{A.2}$$

*uniformly in $\Omega$. Moreover, we have*

$$H(x, y) = 8\pi H_D(x, y) + \nabla \log a(y) \cdot \nabla(|x - y|^2 \log |x - y|) + H_1(x, y), \tag{A.3}$$

*where $y \mapsto H_1(\cdot, y)$ is a continuous map from $\Omega$ into $C^{1,\gamma}(\bar{\Omega})$. The map $(x, y) \mapsto H_1(x, y)$ is in $C^1(\Omega \times \Omega)$ and, in particular, $x \mapsto H(x, x) \in C^1(\Omega)$.*

Lemma A.3. *Let $H_R$ denote the Robin function $x \to H(x, x)$. Then*

$$H_R(x) = 4 \log(2d(x, \partial\Omega)) + O(d(x, \partial\Omega)), \tag{A.4}$$

$$\nabla H_R(x) = O\left(\frac{1}{d(x, \partial\Omega)}\right) \tag{A.5}$$

*uniformly in $\Omega$.*

*Proof of Lemma* A.1–A.3. Lemma A.1 was proved in [**32**, Lemma 2.1], while Lemma A.2 was proved in [**31**, Lemma 2.1] (see also [**32**, Lemma 2.2] and the paragraph following it) except the $L^\infty$-estimate of $H(\cdot, y)$, which can be obtained from (A.3) and the fact that $H_D(x, y) = O(|\log d(y, \partial\Omega)|)$. Finally, in considering Lemma A.3, the estimate (A.5) was obtained in [**32**, Lemma 2.3], and (A.4) comes from (A.3) since

$$\nabla \log a(y) \cdot \nabla(|x-y|^2 \log|x-y|)|_{y=x} = 0, \quad H_1(x,x) \longrightarrow 0 \text{ as } d(x,\partial\Omega) \longrightarrow 0$$

and

$$H_D(x,x) = \frac{1}{2\pi}\log(2d(x,\partial\Omega)) + O(d(x,\partial\Omega))$$

(see [**4**, Subsection 2.1]). □

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## *References*

**1.** N. Ackermann, M. Clápp and A. Pistoia, 'Boundary clustered layers for some supercritical problems', arXiv:1211.2364 [math.AP].
**2.** Adimurthi and M. Grossi, 'Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity', *Proc. Amer. Math. Soc.* 132 (2004) 1013–1019.
**3.** A. Bahri, Y.-Y. Li and O. Rey, 'On a variational problem with lack of compactness: the topological effect of the critical points at infinity', *Calc. Var. Partial Differential Equations* 3 (1995) 67–93.
**4.** C. Bandle and M. Flucher, 'Harmonic radius and concentration of energy hyperbolic radius and Liouville's equations $\Delta = e^U$ and $\Delta U = U^{(n+2)/(n-2)}$', *SIAM Rev.* 38 (1996) 191–238.
**5.** S. Baraket and F. Pacard, 'Construction of singular limits for a semilinear elliptic equation in dimension 2', *Calc. Var. Partial Differential Equations* 6 (1998) 1–38.
**6.** T. Bartsch, A. M. Micheletti and A. Pistoia, 'On the existence and the profile of nodal solutions of elliptic equations involving critical growth', *Calc. Var. Partial Differential Equations* 26 (2006) 265–282.
**7.** T. Bartsch, A. Pistoia and T. Weth, 'N-vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane–Emden–Fowler equations', *Comm. Math. Phys.* 297 (2010) 653–686.
**8.** M. Ben Ayed, K. El Mehdi, M. Grossi and O. Rey, 'A nonexistence result of single peaked solutions to a supercritical nonlinear problem', *Commun. Contemp. Math.* 5 (2003) 179–195.
**9.** D. Chae and O. Imanuvilov, 'The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory', *Comm. Math. Phys.* 215 (2000) 119–142.
**10.** W. Chen and C. Li, 'Classification of solutions of some nonlinear elliptic equations', *Duke Math. J.* 63 (1991) 615–623.
**11.** K. El Mehdi and M. Grossi, 'Asymptotic estimates and qualitative properties of an elliptic problem in dimension two', *Adv. Nonlinear Stud.* 4 (2004) 15–36.
**12.** P. Esposito, M. Musso and A. Pistoia, 'Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent', *J. Differential Equations* 227 (2006) 29–68.
**13.** P. Esposito, M. Musso and A. Pistoia, 'On the existence and profile of nodal solutions for a two-dimensional elliptic problem with large exponent in nonlinearity', *Proc. London Math. Soc.* 94 (2007) 497–519.
**14.** Y. Ge, M. Musso and A. Pistoia, 'Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains', *Comm. Partial Differential Equations* 35 (2010) 1419–1457.
**15.** J. Kazdan and F. Warner, 'Remarks on some quasilinear elliptic equations', *Comm. Pure Appl. Math.* 28 (1975) 567–597.
**16.** S. Kim and A. Pistoia, 'Towers of sign changing boundary layers for some supercritical problems', arXiv:1302.1217 [math.AP].
**17.** M. Musso and A. Pistoia, 'Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains', *J. Math. Pures Appl.* 86 (2006) 510–528.
**18.** M. Musso and A. Pistoia, 'Tower of bubbles for almost critical problems in general domains', *J. Math. Pures Appl.* 93 (2010) 1–40.
**19.** D. Passaseo, 'Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains', *J. Funct. Anal.* 114 (1993) 97–105.
**20.** D. Passaseo, 'New nonexistence results for elliptic equations with supercritical nonlinearity', *Differential Integral Equations* 8 (1995) 577–586.

**21.** M. del Pino, P. Felmer and M. Musso, 'Two-bubble solutions in the super-critical Bahri–Coron's problem', *Calc. Var. Partial Differential Equations* 16 (2003) 113–145.
**22.** M. del Pino, M. Musso and F. Pacard, 'Bubbling along boundary geodesics near the second critical exponent', *J. Eur. Math. Soc.* 12 (2010) 1553–1605.
**23.** A. Pistoia and O. Rey, 'Multiplicity of solutions to the supercritical Bahri–Coron's problem in pierced domains', *Adv. Differential Equations* 11 (2006) 647–666.
**24.** A. Pistoia and T. Weth, 'Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem', *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007) 325–340.
**25.** S. I. Pohožaev, 'Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$', *Soviet Math. Dokl.* 6 (1965) 1408–1411.
**26.** X. Ren and J. Wei, 'On a two-dimensional elliptic problem with large exponent in nonlinearity', *Trans. Amer. Math. Soc.* 4 (1994) 749–763.
**27.** X. Ren and J. Wei, 'Single point condensation and least energy solutions', *Proc. Amer. Math. Soc.* 124 (1996) 111–120.
**28.** O. Rey, 'The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent', *J. Funct. Anal.* 89 (1990) 1–52.
**29.** L. Wang and D. Ye, 'Concentrating solutions for an anisotropic planar elliptic problem with large exponent', Preprint, 2011.
**30.** L. Wei, 'Changing-sign bubble solutions for an anisotropic sinh-Poisson equation', *Nonlinear Differential Equations Appl.* 18 (2011) 685–706.
**31.** J. Wei, D. Ye and F. Zhou, 'Bubbling solutions for an anisotropic Emden–Fowler equation', *Calc. Var. Partial Differential Equations* 28 (2007) 217–247.
**32.** J. Wei, D. Ye and F. Zhou, 'Analysis of boundary bubbling solutions for an anisotropic Emden–Fowler equation', *Ann. Inst. H. Poincaré, Anal. Non Linéaire* 25 (2008) 425–447.

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