Journal of Combinatorial Theory, Series A 120 (2013) 1087–1115

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Zigzag strip bundles and crystals

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ARTICLE INFO

Article history:
Received 27 October 2012

Keywords:
Zigzag strip bundles
Nakajima monomials
Kashiwara embeddings
Crystals

ABSTRACT

We introduce new combinatorial models, called *zigzag strip bundles*, over quantum affine algebras $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$, and show that the sets of all zigzag strip bundles for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$ realize the crystal bases $B(\infty)$ of $U_q^-(B_n^{(1)})$, $U_q^-(D_n^{(1)})$ and $U_q^-(D_{n+1}^{(2)})$, respectively. Further, we discuss the connection between zigzag strip bundle realization, Nakajima monomial realization, and polyhedral realization of the crystal $B(\infty)$.

0. Introduction

Since Kashiwara introduced the *crystal basis* theory in [11,12], it has played an important role in answering problems in many areas of mathematics and mathematical physics. In particular, crystal basis theory is a powerful combinatorial tool to investigate the structures of integrable modules over quantum groups and quantum groups themselves. Hence one of the most important problems in the crystal basis theory is to realize crystal bases explicitly using several combinatorial objects. In many articles, one can find several kinds of realizations of crystal bases of integrable highest weight modules over quantum groups (for example, [2,3,5,7,8,14–16,23–25]).

For the crystal bases $B(\infty)$ of the negative parts $U_q^-(\mathfrak{g})$ of quantum groups $U_q(\mathfrak{g})$, in [4], Kang, Kashiwara and Misra gave the *path realizations* of the crystals $B(\infty)$ for quantum affine algebras using the *perfect crystal* theory developed in [5,6].

In [13], Kashiwara introduced an embedding of crystals $\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow B(\mathbf{i})$, where $\mathbf{i}$ is some infinite sequence from the index set of simple roots. But, in general, it is not easy to find the image $\operatorname{Im}\Psi_{\mathbf{i}}$. In [1], Cliff described the image $\operatorname{Im}\Psi_{\mathbf{i}}$ of the Kashiwara embedding for classical Lie algebras. For

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1 This research was supported by NRF Grant #2010-0006797.
2 This research was supported by KOSEF Grant #2009-0068820.

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2013.02.007

more general types, Zelevinsky and Nakashima obtained the image of the Kashiwara embedding by a unified method, called the *polyhedral realization* [26]. However, their descriptions of the images of the Kashiwara embeddings contain many redundant inequalities, and except for classical quantum finite algebras and quantum affine algebra of type A, the minimal set of inequalities is unknown.

In [9], Kang and the authors modified the notion of Nakajima monomials, a combinatorial model realizing the highest weight crystal, by adding a new variable $\mathbf{1}$, and defined a crystal structure on the set of modified Nakajima monomials. Moreover, they showed that the connected component containing $\mathbf{1}$ is isomorphic to the crystal basis $B(\infty)$ of $U_q^-(\mathfrak{g})$. Even though the characterization of the connected component containing $\mathbf{1}$ for classical quantum finite algebras and the quantum affine algebra of type A have been obtained in [9,17], it is still unknown for general quantum affine algebras.

In [18], the authors gave a new realization of the crystal $B(\infty)$ for the quantum affine algebra $U_q(A_n^{(1)})$ using so-called *generalized Young walls*, and they gave a 1-1 correspondence between the set of generalized Young walls and the set of paths given in [4]. However, it is not easy to extend the theory of generalized Young walls to other quantum affine algebras.

The main goal of this paper is to give a new realization of the crystal $B(\infty)$ over the quantum affine algebra $U_q(\mathfrak{g})$, where $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$ or $D_{n+1}^{(2)}$. More precisely, we introduce new combinatorial objects *zigzag strip bundles* for $U_q(\mathfrak{g})$, and give the rules and patterns for building zigzag strip bundles and the action of Kashiwara operators explicitly in terms of zigzag strip bundles. Then we show that the set $\mathcal{S}(\infty)$ of all zigzag strip bundles for $U_q(\mathfrak{g})$ is isomorphic to the crystal $B(\infty)$ by giving a $U_q(\mathfrak{g})$-crystal isomorphism between $\mathcal{S}(\infty)$ and the connected component $\mathcal{M}(\infty)$ containing $\mathbf{1}$ in the set of all Nakajima monomials. Moreover, using the crystal isomorphism between $\mathcal{M}(\infty)$ and the image of the Kashiwara embedding $\Psi_{\mathbf{i}}$ given in Section 2.3, we give a crystal isomorphism from $\mathcal{S}(\infty)$ to the image of $\Psi_{\mathbf{i}}$. It means that one can understand the description of the connected component $\mathcal{M}(\infty)$ and the image of $\Psi_{\mathbf{i}}$ using the combinatorics of zigzag strip bundles.

We expect that we can extend the theory of zigzag strip bundles to other quantum affine algebras [20]. Also, we believe that the combinatorics of zigzag strip bundles can be applied to the construction of highest weight crystals for quantum affine algebras [21]. In [19], motivated by the zigzag strip bundle realizations of the crystals $B(\infty)$ over quantum affine algebras, the author introduced strip bundles and realized the crystals $B(\infty)$ over $U_q(B_\infty)$, $U_q(C_\infty)$ and $U_q(D_\infty)$ using strip bundles. Recently, Lee and Salisbury gave a combinatorial description of the Gindikin–Karpelevich formula in type A using Young tableaux [22], and Kang, Lee, Ryu, and Salisbury extend their work to the affine type $A_n^{(1)}$ using generalized Young walls [10]. We think that zigzag strip bundles can be used to give a combinatorial description of the affine Gindikin–Karpelevich formula in type $B_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$.

1. Crystals

1.1. Quantum affine algebras

Let $I = \{0, 1, \dots, n\}$ be an index set and let $A = (a_{ij})_{i,j\in I}$ be a *Cartan matrix* of affine type. Let

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$$

be the *dual weight lattice* and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ be the *Cartan subalgebra*. We define the linear functionals α_i and Λ_i $(i \in I)$ on $\mathfrak{h}$ by

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d) = \delta_{0,i}, \qquad \Lambda_i(h_j) = \delta_{i,j}, \qquad \Lambda_i(d) = 0 \quad (i, j \in I).$$

The α_i and h_i $(i \in I)$ are called the *simple roots* and *simple coroots*, respectively, and the Λ_i $(i \in I)$ are called the *fundamental weights*. We denote by

$$\Pi = \{\alpha_i \mid i \in I\} \quad \text{and} \quad \Pi^\vee = \{h_i \mid i \in I\}$$

the set of simple roots and simple coroots, respectively. The *affine weight lattice* is defined to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}.$$

The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called an *affine Cartan datum*. To each affine Cartan datum, we can associate infinite dimensional Lie algebras $\mathfrak{g}$ and $U_q(\mathfrak{g})$ called the *affine Kac–Moody algebra* and the *quantum affine algebra*, respectively. Let us denote by $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ the subalgebras of $U_q(\mathfrak{g})$ generated by the e_i's and the f_i's, respectively. We also denote by $P^+ = \{\lambda \in P \mid \lambda(h_i) \geqslant 0 \text{ for all } i \in I\}$ the set of *dominant integral weights*.

1.2. Abstract crystals

An *abstract crystal* for $U_q(\mathfrak{g})$ or a $U_q(\mathfrak{g})$-*crystal* is a set B together with the maps

$$\mathrm{wt} : B \to P, \qquad \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}, \qquad \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \quad (i \in I)$$

such that for all $i \in I$ and $b \in B$,

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(ii) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq 0$,
(iii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq 0$,
(iv) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \neq 0$,
(v) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \neq 0$,
(vi) $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $b, b' \in B$,
(vii) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

For the crystals B_1 and B_2, a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ sending 0 to 0 is called a *crystal morphism* from B_1 to B_2 if the following conditions are satisfied: for all $b, b' \in B_1$, $i \in I$

(i) if $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$,
(ii) if $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$ and $\psi(b) = \tilde{e}_i \psi(b')$.

Example 1.1. (a) The crystal basis $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ with $\lambda \in P^+$ is a $U_q(\mathfrak{g})$-crystal, where the maps ε_i, φ_i $(i \in I)$ are given by

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \qquad \varphi_i(b) = \max\{k \geqslant 0 \mid \tilde{f}_i^k b \neq 0\}.$$

(b) The crystal basis $B(\infty)$ of the negative part $U_q^-(\mathfrak{g})$ of a quantum group $U_q(\mathfrak{g})$ is a $U_q(\mathfrak{g})$-crystal, where

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \qquad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle.$$

(c) For $\lambda \in P$, the singleton $T_\lambda = \{t_\lambda\}$ is a crystal with the maps defined by

$$\mathrm{wt}(t_\lambda) = \lambda, \qquad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \qquad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for all } i \in I.$$

(d) For each $i \in I$, let $B_i = \{b_i(n) \mid n \in \mathbb{Z}\}$. Then B_i is a crystal with the maps defined by

$$\begin{aligned}
&\mathrm{wt}(b_i(n)) = n\alpha_i, \\
&\varepsilon_i(b_i(n)) = -n, \qquad \varphi_i(b_i(n)) = n, \qquad \tilde{e}_i b_i(n) = b_i(n+1), \qquad \tilde{f}_i b_i(n) = b_i(n-1), \\
&\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty, \qquad \tilde{e}_j b_i(n) = \tilde{f}_j b_i(n) = 0 \quad \text{if } j \neq i.
\end{aligned}$$

The crystal B_i is called an *elementary crystal*.

1.3. Tensor product of crystals

Let B and B' be crystals. Then their tensor product

$$B \otimes B' = \{b \otimes b' \mid b \in B,\ b' \in B'\}$$

is also a crystal with the maps wt, ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ given by

$$\mathrm{wt}\big(b\otimes b'\big)=\mathrm{wt}(b)+\mathrm{wt}\big(b'\big),$$
$$\varepsilon_i\big(b\otimes b'\big)=\max\big\{\varepsilon_i(b),\varepsilon_i\big(b'\big)-\big\langle h_i,\mathrm{wt}(b)\big\rangle\big\},$$
$$\varphi_i\big(b\otimes b'\big)=\max\big\{\varphi_i\big(b'\big),\varphi_i(b)+\big\langle h_i,\mathrm{wt}\big(b'\big)\big\rangle\big\},$$
$$\tilde{e}_i\big(b\otimes b'\big)=\begin{cases}\tilde{e}_ib\otimes b' & \text{if } \varphi_i(b)\geqslant\varepsilon_i(b'),\\ b\otimes\tilde{e}_ib' & \text{if } \varphi_i(b)<\varepsilon_i(b'),\end{cases}$$
$$\tilde{f}_i\big(b\otimes b'\big)=\begin{cases}\tilde{f}_ib\otimes b' & \text{if } \varphi_i(b)>\varepsilon_i(b'),\\ b\otimes\tilde{f}_ib' & \text{if } \varphi_i(b)\leqslant\varepsilon_i(b').\end{cases}$$

2. Kashiwara embeddings and Nakajima monomials

2.1. Kashiwara embeddings

Let $\mathbf{i}=(i_1,i_2,\ldots)$ be an infinite sequence of indices in I such that every $i\in I$ appears infinitely many times, and let

$$B(\mathbf{i})=\big\{\big(b_{i_k}(-x_k)\big)_{k=1}^{\infty}\in\cdots\otimes B_{i_k}\otimes\cdots\otimes B_{i_1}\;\big|\;x_k\in\mathbb{Z}_{\geqslant0},\ x_k=0\text{ for all }k\gg0\big\}.$$

Fix $j\in I$. For $b=\cdots\otimes b_{i_k}(-x_k)\otimes\cdots\otimes b_{i_1}(-x_1)\in B(\mathbf{i})$, choose a sufficiently large $N>0$ such that $x_k=0$ for all $k>N$ and $j=i_k$ for some $k=1,\ldots,N$. Set

$$b'=\cdots\otimes b_{i_{N+2}}(0)\otimes b_{i_{N+1}}(0),\qquad b''=b_{i_N}(-x_N)\otimes\cdots\otimes b_{i_1}(-x_1).$$

We define

$$\varepsilon_j\big(b'\big)=\varphi_j\big(b'\big)=0,$$
$$\varepsilon_j\big(b''\big)=\max\left\{\varepsilon_j\big(b_{i_k}(-x_k)\big)+\sum_{l=k+1}^{N}\langle h_j,\alpha_{i_l}\rangle x_{i_l}\;\middle|\;1\leqslant k\leqslant N\right\},$$
$$\varphi_j\big(b''\big)=\max\left\{\varphi_j\big(b_{i_k}(-x_k)\big)-\sum_{l=1}^{k-1}\langle h_j,\alpha_{i_l}\rangle x_{i_l}\;\middle|\;1\leqslant k\leqslant N\right\},$$
$$\tilde{e}_jb'=0,$$
$$\tilde{f}_jb'=\cdots\otimes\tilde{f}_jb_{i_k}(0)\otimes\cdots\otimes b_{i_{N+1}}(0)\quad\big(k=\min\{t\mid t>N,\ i_t=j\}\big),$$

and

$$\varepsilon_j(b)=\max\big\{0,\varepsilon_j\big(b''\big)\big\},\qquad\varphi_j(b)=\max\big\{\varphi_j\big(b''\big),\big\langle h_j,\mathrm{wt}\big(b''\big)\big\rangle\big\},$$
$$\tilde{e}_jb=\begin{cases}0 & \text{if } \varepsilon_j(b'')\leqslant0,\\ b'\otimes\tilde{e}_jb'' & \text{if } \varepsilon_j(b'')>0,\end{cases}\qquad\tilde{f}_jb=\begin{cases}\tilde{f}_jb'\otimes b'' & \text{if } \varepsilon_j(b'')<0,\\ b'\otimes\tilde{f}_jb'' & \text{if } \varepsilon_j(b'')\geqslant0.\end{cases}$$

Then $B(\mathbf{i})$ is a $U_q(\mathfrak{g})$-crystal. Moreover, we have

Theorem 2.1. *(See [13].) There exists a unique strict embedding of crystals*

$$\Psi_{\mathbf{i}}:B(\infty)\hookrightarrow B(\mathbf{i})\quad\text{such that } u_\infty\mapsto\cdots\otimes b_{i_{N+1}}(0)\otimes b_{i_N}(0)\otimes\cdots\otimes b_{i_1}(0),$$

where u_∞ is the highest weight vector in $B(\infty)$.

2.2. Nakajima monomials

Let $Y_i(n)$ $(i \in I, n \in \mathbb{Z})$ and $\mathbf{1}$ be commuting variables and let $\mathcal{M}$ be the set of all monomials of the form

$$M = \mathbf{1} \cdot \prod_{i \in I,\, n \geqslant 0} Y_i(n)^{y_i(n)}$$

such that $y_i(n) \in \mathbb{Z}$ and $y_i(n) = 0$ for all but finitely many n's. We call the monomials in $\mathcal{M}$ the *Nakajima monomials*.

For a Nakajima monomial $M \in \mathcal{M}$, we define

$$\begin{aligned}
&\mathrm{wt}(M) = \sum_{i \in I} \Bigl(\sum_{n \geqslant 0} y_i(n) \Bigr) \Lambda_i, \\
&\varphi_i(M) = \max\Bigl\{ \sum_{0 \leqslant k \leqslant n} y_i(k) \Bigm| n \geqslant 0 \Bigr\}, \\
&\varepsilon_i(M) = \varphi_i(M) - \langle h_i, \mathrm{wt}(M) \rangle, \\
&n_f = n_f(M) = \min\Bigl\{ n \geqslant 0 \Bigm| \varphi_i(M) = \sum_{0 \leqslant k \leqslant n} y_i(k) \Bigr\}, \\
&n_e = n_e(M) = \max\Bigl\{ n \geqslant 0 \Bigm| \varphi_i(M) = \sum_{0 \leqslant k \leqslant n} y_i(k) \Bigr\}.
\end{aligned} \tag{2.1}$$

Choose a set $C = (c_{ij})_{i \neq j}$ of nonnegative integers such that $c_{ij} + c_{ji} = 1$, and for each $i \in I$, $n \in \mathbb{Z}_{\geqslant 0}$, define

$$A_i(n) = Y_i(n) Y_i(n+1) \prod_{j \neq i} Y_j(n + c_{ji})^{a_{ji}}.$$

We define the *Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ on $\mathcal{M}$ as follows.

$$\begin{aligned}
&\tilde{e}_i M = \begin{cases} 0 & \text{if } \varepsilon_i(M) = 0, \\ A_i(n_e) M & \text{if } \varepsilon_i(M) > 0, \end{cases} \\
&\tilde{f}_i M = A_i(n_f)^{-1} M.
\end{aligned} \tag{2.2}$$

Then $\mathcal{M}$ becomes a $U_q(\mathfrak{g})$-crystal with the maps wt, ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ defined in (2.1) and (2.2). Moreover, we have

Theorem 2.2. *(See [9].) For any maximal vector $M \in \mathcal{M}$, i.e., $\tilde{e}_i M = 0$ for all $i \in I$, the connected component $C(M)$ of $\mathcal{M}$ containing M is isomorphic to the $U_q(\mathfrak{g})$-crystal $B(\infty) \otimes T_{\mathrm{wt}(M)}$. In particular, we have $C(\mathbf{1}) \xrightarrow{\sim} B(\infty)$.*

2.3. Kashiwara embeddings and Nakajima monomials

Let $\mathcal{M}(\infty)$ be the connected component containing $\mathbf{1}$ in $\mathcal{M}$ when we choose the set $C = (c_{ij})_{i \neq j}$ of nonnegative integers as follows.

$$c_{ij} = 0 \quad \text{if } i > j, \quad \text{and} \quad c_{ij} = 1 \quad \text{if } i < j.$$

Let $\mathbf{i} = (0, 1 \ldots, n, 0, 1, \ldots, n, 0, \ldots)$ be an infinite sequence of indices in I, and let $\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow B(\mathbf{i})$ be the crystal embedding given in Theorem 2.1.

For a monomial

$$M = \mathbf{1} \cdot \prod_{i \in I,\, k \geqslant 0} A_i(k)^{-a_i(k)} \in \mathcal{M}(\infty),$$

we now define a map $\phi : \mathcal{M}(\infty) \to \operatorname{Im} \Psi_{\mathbf{i}}$ by

$$\phi(M) = \cdots \otimes b_0\big(-a_0(2)\big) \otimes b_n\big(-a_n(1)\big) \cdots \otimes b_1\big(-a_1(1)\big) \otimes b_0\big(-a_0(1)\big) \otimes b_n\big(-a_n(0)\big) \cdots \otimes b_1\big(-a_1(0)\big) \otimes b_0\big(-a_0(0)\big).$$

Then we have

Theorem 2.3. *(See [17].) The map $\phi : \mathcal{M}(\infty) \to \operatorname{Im} \Psi_{\mathbf{i}}$ is a $U_q(\mathfrak{g})$-crystal isomorphism.*

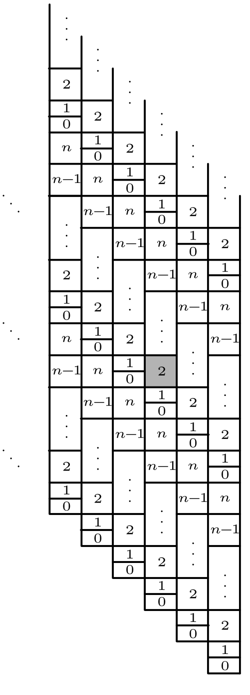
3. Zigzag strip bundles

Young walls, combinatorial models realizing affine highest crystals introduced by Kang in [3], were devised from the perfect crystal theory. In [18], motivated by the combinatorics of Young walls, the authors introduced new combinatorial objects called *generalized Young walls* and they gave a realization of the crystal $B(\infty)$ over $U_q(A_n^{(1)})$ using generalized Young walls. In this section, from the theories of Nakajima monomials and crystal embeddings, we introduce the notion of *zigzag strip bundles* for the quantum affine Lie algebras $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$. Moreover, in the next section, we show that the set of all zigzag strip bundles for $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$ or $D_{n+1}^{(2)}$) realizes the crystal $B(\infty)$ over $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$ or $D_{n+1}^{(2)}$).

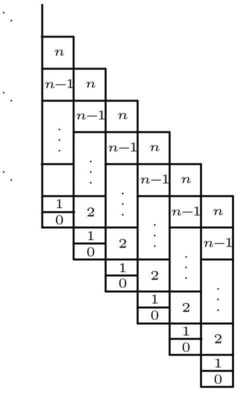
3.1. Boards and blocks for zigzag strip bundles

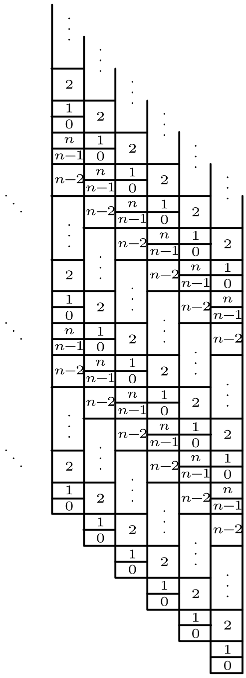
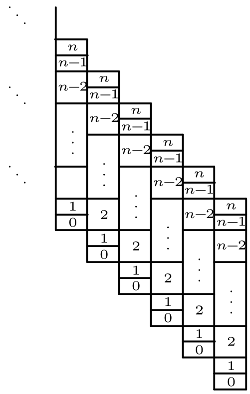
For each quantum affine algebra $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$), we fix boards $\mathcal{B}$ and $\overline{\mathcal{B}}$ with coloring as follows:

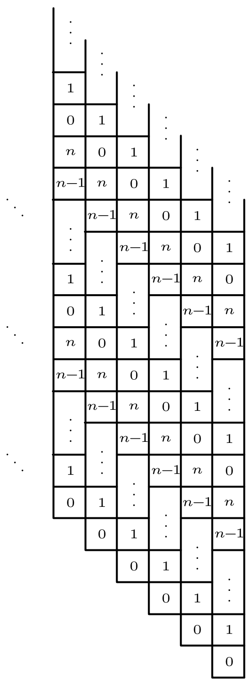
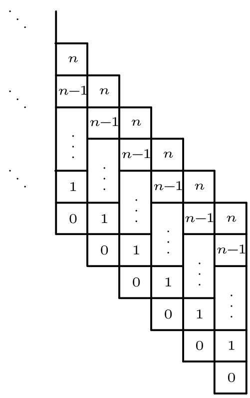
(i) $\mathfrak{g} = B_n^{(1)}$ ($n \geqslant 3$); $\mathcal{B} =$



$\overline{\mathcal{B}} =$



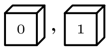
(ii) $\mathfrak{g} = D_n^{(1)}$ $(n \geqslant 4)$; $\mathcal{B} =$  $\overline{\mathcal{B}} =$ 

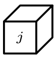
(iii) $\mathfrak{g} = D_{n+1}^{(2)}$ $(n \geqslant 2)$; $\mathcal{B} =$  $\overline{\mathcal{B}} =$ 

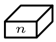
Then we see that the $\mathcal{B}$ can be obtained by adding up the same subboard as $\overline{\mathcal{B}}$ on the top of each $\overline{\mathcal{B}}$ continuously. We denote by $\mathcal{B}_t$ $(t \geqslant 1)$ the tth subboard $\overline{\mathcal{B}}$ appearing in $\mathcal{B}$ from the bottom. We also see that the board $\mathcal{B}$ consists of squares with colorings, which we call *sites* of $\mathcal{B}$, and we regard the board $\mathcal{B}$ as the set $\mathbb{N} \times \mathbb{N}$ by identifying the jth site from the bottom of the ith column from the right with $(i, j) \in \mathbb{N} \times \mathbb{N}$. For instance, the site of coloring 2, the shaded site, appearing in the second from the bottom of the third column in $\mathcal{B}$ for $U_q(B_n^{(1)})$ corresponds to $(3, n+2)$.

The zigzag strip bundles for $U_q(\mathfrak{g})$ ($\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$ or $D_{n+1}^{(2)}$) are built of colored blocks of three different shapes on the board $\mathcal{B}$ according to the colors of sites of $\mathcal{B}$:

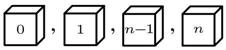
(i) $\mathfrak{g} = B_n^{(1)}$;

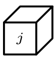
 : unit width, unit height, half-unit thickness,

 $(j = 2, \ldots, n-1)$: unit width, unit height, unit thickness,

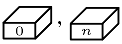
 : unit width, half-unit height, unit thickness.

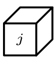
(ii) $\mathfrak{g} = D_n^{(1)}$;

 : unit width, unit height, half-unit thickness,

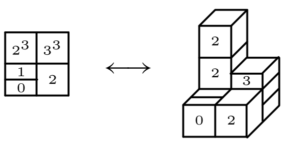
 $(j = 2, \ldots, n-2)$: unit width, unit height, unit thickness.

(iii) $\mathfrak{g} = D_{n+1}^{(2)}$;

 : unit width, half-unit height, unit thickness,

 $(j = 1, \ldots, n-1)$: unit width, unit height, unit thickness.

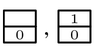
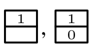
With these colored blocks, we will pile up them on the board $\mathcal{B}$ according to some rules. For convenience, we will use the notation $\boxed{j^k}$ for the stack of k-many j-blocks ($j = 0, 1, \ldots, n$). For instance, for $U_q(B_3^{(1)})$,

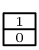


3.2. *Zigzag strip bundles for $U_q(B_n^{(1)})$*

Before we introduce the zigzag strip bundles for $U_q(B_n^{(1)})$, we define zigzag strips as follows.

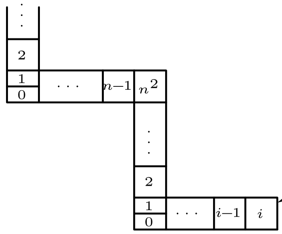
Definition 3.1. (a) We define a *zigzag* 0 (resp. 1)*-strip* for $U_q(B_n^{(1)})$ by a pile of colored blocks stacked on a site with colorings 0 and 1 of the rightmost column of $\mathcal{B}$ as follows

 (resp. ).

Note that the pile  can be regarded as both 0-strip and 1-strip.

(b) For each $i \in I - \{0, 1, n\}$, we define a *zigzag i-strip* for $U_q(B_n^{(1)})$ by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

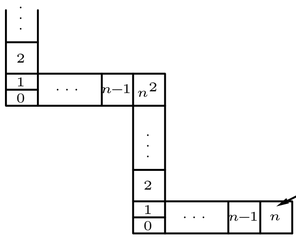
(i) It is obtained by stacking colored blocks starting from an i-colored site of the first column from the right in the pattern given below.



(ii) The volume of stacked blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

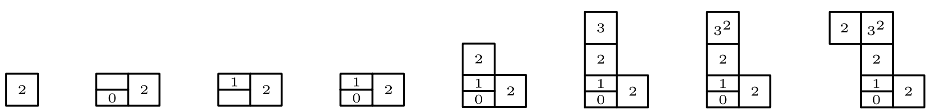
(c) For each $k \in \mathbb{Z}_{>0}$, we define a kth *zigzag n-strip* for $U_q(B_n^{(1)})$ by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

(i) It is obtained by stacking colored blocks starting from an n-colored site of the kth column from the right in the pattern given below.



(ii) Except for the rightmost site of the pile, the volume of blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

Example 3.2. The following are zigzag 2-strips in $B_3^{(1)}$.



On the other hand, the following are not zigzag 2-strips.

Indeed, in the second column of the left pile, the volume of 2-colored blocks is bigger than the volume of 1-colored blocks, and in the right pile, the volume of 2-colored blocks on the leftmost site is bigger than the volume of 3-colored blocks.

Definition 3.3. A pile S of stacked blocks on the board $\mathcal{B}$ is called a *zigzag strip bundle* for $U_q(B_n^{(1)})$ if there is a decomposition of S into zigzag strips satisfying the following conditions:

(i) For each $t, k \geqslant 1$ and $i \in I - \{n\}$, there exist at most one zigzag i-strip and one kth zigzag n-strip in $\mathcal{B}_t$.

(ii) For each $t, k \geqslant 1$ and $i \in I - \{n\}$, if we let S_i^t, $S_{n,k}^t$ be the zigzag i-strip and the kth zigzag n-strip in $\mathcal{B}_t$, respectively, then

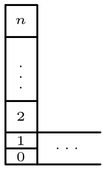
$$S_i^t \supset S_i^{t+1}, \qquad S_{n,k}^t \supset S_{n,k}^{t+1} \quad \text{for all } k, t \geqslant 1. \tag{3.1}$$

Here, equalities hold only if zigzag strips consist of blocks of full height (unit height).

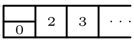
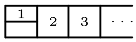
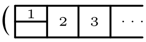
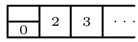
(iii) For each $k, t \geqslant 1$, if we let $|S_{n,k}^t|$ be the number of blocks in the kth zigzag n-strip $S_{n,k}^t$, then

$$\left|S_{n,k}^t\right| \geqslant \left|S_{n,k+1}^t\right|.$$

Here, $|S_{n,k}^t| = |S_{n,k+1}^t|$ if and only if

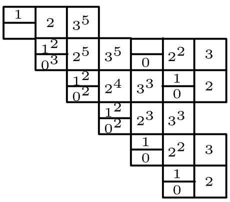
$$S_{n,k}^t = S_{n,k+1}^t = \emptyset \quad \text{or} \quad S_{n,k}^t = S_{n,k+1}^t =$$ 

or

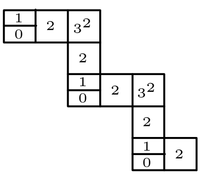
$$\left(S_{n,k}^t, S_{n,k+1}^t\right) = \Big($$  ,  $\Big)$ or $\Big($  ,  $\Big)$.

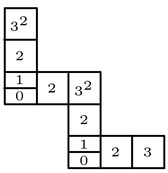
Remark 3.4. In (3.1), we regard S_i^t and $S_{n,k}^t$ as only piles of stacked blocks without considering positions on the board $\mathcal{B}$.

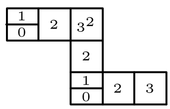
Example 3.5. Let

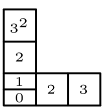
$S =$ 

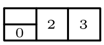
be a pile for $U_q(B_3^{(1)})$. Then it is decomposed into the following zigzag strips:

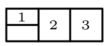
$S_2^1 =$ 

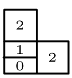
$S_{3,1}^1 =$ 

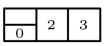
$S_{3,2}^1 =$ 

$S_{3,3}^1 =$ 

$S_{3,4}^1 =$ 

$S_{3,5}^1 =$ 

$S_2^2 =$ 

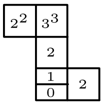
$S_{3,1}^2 =$ 

We can easily check that

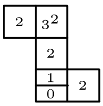
(i) $S_2^1 \supset S_2^2$, $S_{3,1}^1 \supset S_{3,1}^2$,
(ii) $|S_{3,1}^1| > |S_{3,2}^1| > |S_{3,3}^1| > |S_{3,4}^1| = |S_{3,5}^1|$.

Therefore, S is a zigzag strip bundle for $U_q(B_3^{(1)})$.

(b) Now, consider the following pile S for $U_q(B_3^{(1)})$.



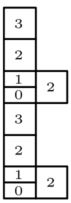
In any decomposition of S into zigzag strips, we have $S_{3,1}^1 = \emptyset$ and $S_{3,2}^1 \neq \emptyset$, and so it is not a zigzag strip bundle. Indeed, according to the condition (i) in Definition 3.3, the only possible decomposition of S into zigzag strips is as follows.

$S_2^1 =$ 

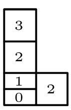
and $S_{3,2}^1 = \boxed{2}\boxed{3}$.

Since $|S_{3,1}^1| = 0 < 2 = |S_{3,2}^1|$, it is not a zigzag strip bundle.

Example 3.6. The following pile for $U_q(B_3^{(1)})$



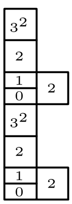
is decomposed as follows:

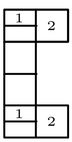
$S_2^1 = S_2^2 =$ 

.

But, the 3-block of S_2^1 (and S_2^2) is a block of half height, it does not satisfy the condition (ii) of Definition 3.3.

On the other hand, the following piles are zigzag strip bundles for $U_q(B_3^{(1)})$.





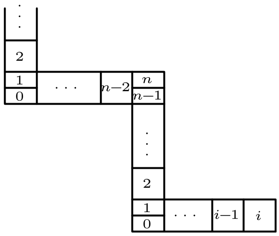
3.3. $D_n^{(1)}$-type

As in the case $U_q(B_n^{(1)})$, we first define zigzag strips for $U_q(D_n^{(1)})$ as follows.

Definition 3.7. (a) The *zigzag* 0 (resp. 1)-*strips* for $U_q(D_n^{(1)})$ are the same as those for $U_q(B_n^{(1)})$.

(b) For each $i \in I - \{0, 1, n-1, n\}$, we define a *zigzag i-strip* by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

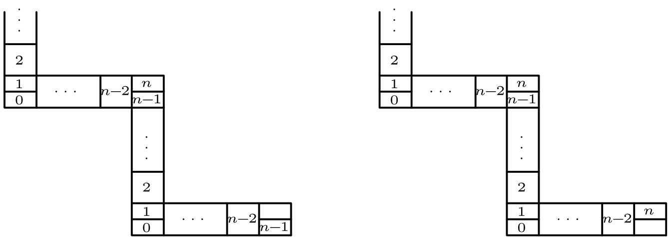
(i) It is obtained by stacking colored blocks starting from an i-colored site of the first column from the right in the pattern given below.



(ii) The volume of stacked blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

(c) For each $k \in \mathbb{Z}_{>0}$, we define a kth zigzag $(n-1)$-strip and a kth zigzag n-strip by piles of colored blocks on the board $\mathcal{B}$ satisfying the following conditions:

(i) They are obtained by stacking colored blocks starting from an $(n-1)$ or n-colored site of the kth column in the pattern given below.



(ii) Except for the rightmost site of the pile, the volume of blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

Definition 3.8. A pile S of stacked blocks on the board $\mathcal{B}$ is called a *zigzag strip bundle* for $U_q(D_n^{(1)})$ if there is a decomposition of S into zigzag strips satisfying the following conditions:

(i) For each $t, k \geqslant 1$ and $i \in I - \{n-1, n\}$, there are at most one zigzag i-strip, one kth zigzag $(n-1)$-strip and one kth zigzag n-strip in $\mathcal{B}_t$.

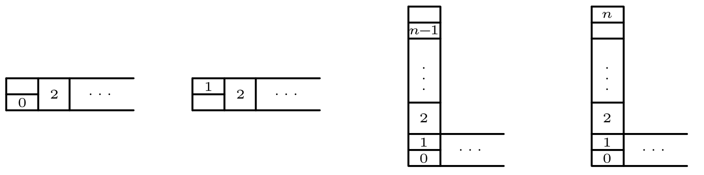
(ii) For each $t, k \geqslant 1$ and $i \in I - \{n-1, n\}$, if we let S_i^t, $S_{n-1,k}^t$ and $S_{n,k}^t$ be the zigzag i-strip, the kth zigzag $(n-1)$-strip and the kth zigzag n-strip in $\mathcal{B}_t$, respectively, then

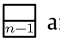
$$S_i^t \supset S_i^{t+1}, \qquad S_{n-1,k}^t \supset S_{n-1,k}^{t+1}, \qquad S_{n,k}^t \supset S_{n,k}^{t+1}. \tag{3.2}$$

(iii) For each $k, t \geqslant 1$, if we let $|S_{n-1,k}^t|$ and $|S_{n,k}^t|$ be the numbers of blocks in the kth zigzag $(n-1)$-strip $S_{n-1,k}^t$ and the kth zigzag n-strip $S_{n,k}^t$, respectively, then

$$\left|S_{n-1,k}^t\right| \geqslant \left|S_{n,k+1}^t\right| \quad \text{and} \quad \left|S_{n,k}^t\right| \geqslant \left|S_{n-1,k+1}^t\right|.$$

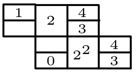
Here, $|S_{n-1,k}^t| = |S_{n,k+1}^t|$ (resp. $|S_{n,k}^t| = |S_{n-1,k+1}^t|$) if and only if $S_{n-1,k}^t$ and $S_{n,k+1}^t$ (resp. $S_{n,k}^t$ and $S_{n-1,k+1}^t$) are emptysets, or they are one of the following forms and the last blocks, the blocks stacked on top of the leftmost column, in $S^t_{n-1,k}$ and $S^t_{n,k+1}$ (resp. $S^t_{n,k}$ and $S^t_{n-1,k+1}$) are different:



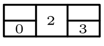
Here, in right two zigzag strips, the numbers of blocks are strictly bigger than 1. That is,  and  are not permitted.

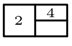
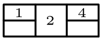
Remark 3.9. In (3.2), we regard S^t_i, $S^t_{n-1,k}$ and $S^t_{n,k}$ as only piles of stacked blocks without considering positions on the board $\mathcal{B}$.

Example 3.10. Let

$S =$ 

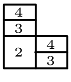
be a pile for $U_q(D_4^{(1)})$. Then it is decomposed into the following zigzag strips:

$S^1_{3,1} =$  $S^1_{3,2} =$ 

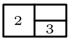
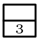
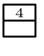
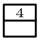
$S^1_{4,1} =$  $S^1_{4,2} =$ 

Therefore, it is clear that S is a zigzag strip bundle. Indeed, we can see that $|S^1_{4,1}| = 2 > 1 = |S^1_{3,2}|$, and $|S^1_{3,1}| = |S^1_{4,2}| = 3$. Moreover, we find that $S^1_{3,1}$ and $S^1_{4,2}$ have the forms given in (iii) of Definition 3.8.

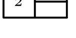
On the other hand, the following pile for $U_q(D_4^{(1)})$



is decomposed into zigzag strips as follows:

$S^1_{3,1} =$  $S^1_{3,2} =$  $S^1_{4,1} =$  $S^1_{4,2} =$ 

or

$S^1_{3,1} =$  $S^1_{3,2} =$  $S^1_{4,1} =$  $S^1_{4,2} =$ 

In any case, they do not have the forms given in (iii) of Definition 3.8. Therefore, it is not a zigzag strip bundle for $U_q(D_4^{(1)})$.

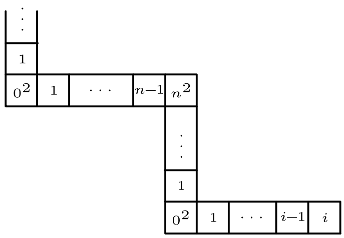
3.4. $D^{(2)}_{n+1}$-type

As in the case $U_q(B_n^{(1)})$ and $U_q(D_n^{(1)})$, we first define zigzag strips for $U_q(D^{(2)}_{n+1})$ as follows.

Definition 3.11. (a) We define a *zigzag* 0*-strip* for $U_q(D_{n+1}^{(2)})$ by a pile consisting of only one 0-colored block stacked on a 0-colored site of the rightmost column of $\mathcal{B}$.

(b) For each $i \in I - \{0, n\}$, we define a *zigzag* i*-strip* for $U_q(D_{n+1}^{(2)})$ by a pile of colored blocks stacked on the board $\mathcal{B}$ satisfying the following conditions:

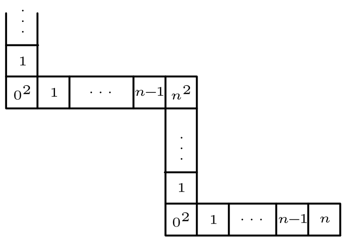
(i) It is obtained by stacking colored blocks starting from an i-colored site of the first column from the right in the pattern given below.



(ii) The volume of stacked blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

(c) For each $k \in \mathbb{Z}_{>0}$, we define a kth *zigzag* n*-strip* for $U_q(D_{n+1}^{(2)})$ by a pile of colored blocks on the board $\mathcal{B}$ satisfying the following conditions:

(i) It is obtained by stacking colored blocks starting from an n-colored site of the kth column from the right in the pattern given below.



(ii) Except the rightmost site of the pile, the volume of blocks on each site of the pile is weakly decreasing from right to left and from bottom to top.

Definition 3.12. A pile S of stacked blocks on the board $\mathcal{B}$ is called a *zigzag strip bundle* for $U_q(D_{n+1}^{(2)})$ if there is a decomposition of S into zigzag strips satisfying the following conditions are satisfied:

(i) For each $t, k \geqslant 1$ and $i \in I - \{n\}$, there are at most one zigzag i-strip and one kth zigzag n-strip in $\mathcal{B}_t$.

(ii) For each $t, k \geqslant 1$ and $i \in I - \{n\}$, if we let S_i^t, $S_{n,k}^t$ be the zigzag i-strip and the kth zigzag n-strip in $\mathcal{B}_t$, respectively, then

$$S_i^t \supset S_i^{t+1}, \qquad S_{n,k}^t \supset S_{n,k}^{t+1} \quad \text{for all } k, t \geqslant 1. \tag{3.3}$$

Here, equalities hold only if zigzag strips consist of blocks of full height (unit height).

(iii) For each $k, t \geqslant 1$, if we let $|S_{n,k}^t|$ be the number of blocks in the kth zigzag n-strip $S_{n,k}^t$ in $\mathcal{B}_t$, then

$$\left|S_{n,k}^t\right| \geqslant \left|S_{n,k+1}^t\right|.$$

Here, the equality holds if and only if

$$S^t_{n,k} = S^t_{n,k+1} = \emptyset, \quad \begin{array}{|c|c|c|c} \hline 0 & 1 & 2 & \cdots \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c} \hline n & \\ \cline{1-1} \vdots & \\ \cline{1-1} 1 & \\ \hline 0^2 & \cdots \\ \hline \end{array}$$

Remark 3.13. In (3.3), we regard S^t_i and $S^t_{n,k}$ as only piles of stacked blocks without considering positions on the board $\mathcal{B}$.

Example 3.14. (a) Let

$$S = \begin{array}{ccccc} 3 & & & & \\ 2 & 3 & & & \\ 1 & 2 & & & \\ 0^2 & 1^2 & 2 & 3^2 & \\ & 0^2 & 1 & 2^2 & 3 \\ & & & 1 & \\ & & & 0^2 & 1 \end{array}$$

be a pile for $U_q(D_4^{(2)})$. Then it is decomposed into the following zigzag strips:

$$S^1_1 = \begin{array}{cc} 3 & \\ 2 & \\ 1 & \\ 0^2 & 1 \end{array} \qquad S^1_{3,1} = S^1_{3,2} = \begin{array}{cccc} 3 & & & \\ 2 & & & \\ 1 & & & \\ 0^2 & 1 & 2 & 3 \end{array}$$

Therefore, S is a zigzag strip bundle for $U_q(D_4^{(2)})$.

(b) Let

$$S = \begin{array}{cc} 0^2 & 1 \\ \vdots & \vdots \\ 0^2 & 1 \end{array} \quad \text{and} \quad S' = \begin{array}{cc} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{array}$$

be piles for $U_q(D_4^{(2)})$. Then by condition (ii) of Definition 3.12, S is a zigzag strip, but S' is not.

4. Realization of the crystal $B(\infty)$

4.1. Crystal structure on the set of zigzag strip bundles

In this subsection, we give a crystal structure on the set of all zigzag strip bundles.

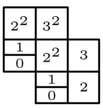
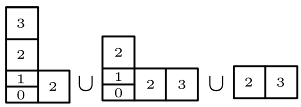
Definition 4.1. Let S be a zigzag strip bundle for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$, and for each $t \geqslant 1$, let $\overline{S}_t$ be the zigzag strip bundle in $\mathcal{B}_t$ consisting of blocks in $\mathcal{B}_t$ of S. Now, let $\overline{S}_t$ be decomposed into zigzag strips $\{S_j \mid j = 1, \ldots, N\}$. From now on, for simplicity, we denote $\{S_j \mid j = 1, \ldots, N\}$ by $\{S_j\}$.

(a) An i-colored block in a zigzag strip S_j $(j = 1, \ldots, N)$ of $\bar{S}_t$ is called a *removable i-block* with respect to the zigzag strips $\{S_j\}$ if one can have another zigzag strip bundle in $\mathcal{B}_t$ from $\{S_j\}$ by removing that block, that is, the zigzag strips obtained from $\{S_j\}$ by removing that block satisfy the conditions (i) and (iii) given in Definition 3.3, Definition 3.8 and Definition 3.12.

(b) An i-colored site in $\mathcal{B}_t$ is called an *i-admissible slot* with respect to the zigzag strips $\{S_j\}$ if one can have another zigzag strip bundle in $\mathcal{B}_t$ from $\{S_j\}$ by stacking an i-block on that site, that is, the zigzag strips obtained from $\{S_j\}$ by stacking an i-block on that site satisfy the conditions (i) and (iii) given in Definition 3.3, Definition 3.8 and Definition 3.12.

(c) A site P of the board $\mathcal{B}_t$ is said to be *k-times i-removable* and *l-times i-admissible* with respect to the zigzag strips $\{S_j\}$ if there are k removable i-blocks on P and one can have another zigzag strip bundle in $\mathcal{B}_t$ from $\{S_j\}$ by stacking l i-blocks on P.

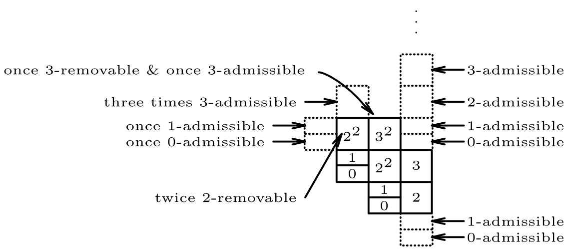
Example 4.2. Let

$S =$  $= S^1_2 \cup S^1_{3,1} \cup S^1_{3,2} =$ 

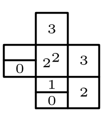
be a zigzag strip bundle lying on $\mathcal{B}_1$ for $U_q(B^{(1)}_3)$. Then it is easy to see that

(i) the site $(2,3)$ is once 3-removable and once 3-admissible from the strip S^1_2,
(ii) the sites $(3,2)$ and $(3,3)$ are once 2-removable and twice 3-admissible from the strip $S^1_{3,1}$, respectively,
(iii) the sites $(4,1)$ and $(3,2)$ are once 0-admissible and once 1-admissible, and once 2-removable from the strip $S^1_{3,2}$, respectively,
(iv) the site $(3,3)$ is once 3-admissible because if we stack a 3-block on $(3,3)$, then we have $S^1_{3,3} = \boxed{3}$, and $S^1_{3,2}$ and $S^1_{3,3}$ satisfy the condition (iii) of Definition 3.3,
(v) the sites $(1,1)$ is once 0-admissible and once 1-admissible, and all sites in the rightmost column of $\mathcal{B}_t$ $(t \geqslant 2)$ are admissible.

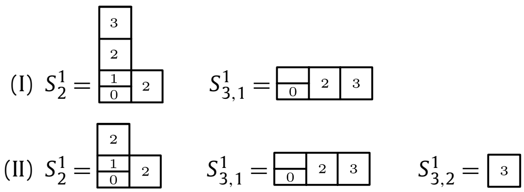
The following describes all the removable and admissible sites with respect to the above zigzag strips $\{S^1_2, S^1_{3,1}, S^1_{3,2}\}$.



Remark 4.3. Note that the concepts of removable blocks, admissible slots, removable sites, and admissible sites for zigzag strip bundles totally depend on the decomposition into zigzag strips. For instance, for a zigzag strip bundle

$S =$ 

over $U_q(B^{(1)}_3)$, we can decompose into the following two different bundles of zigzag strips:



Then in case (I), there is no removable 2-block and 2-admissible slot in $\mathcal{B}_1$, but in case (II), the site $(2,2)$ is once 2-removable from S_2^1 and the site $(3,2)$ is once 2-admissible from $S_{3,2}^1$.

Fix $i \in I$ and let a given zigzag strip bundle S be decomposed into zigzag strips $\{S_j \mid j = 1, \ldots, N\}$. Let $P_1, P_2, \ldots, P_t$ be the all i-removable or i-admissible sites of $\mathcal{B}$ with respect to $\{S_j\}$ from west to east and from south to north. To each site P_α $(\alpha = 1, \ldots, t)$, we assign its i-signature $\mathrm{sgn}_i(P_\alpha)$ as

$$\underbrace{- - - \cdots -}_{p\text{-times}} \underbrace{+ + + \cdots +}_{q\text{-times}}$$

if P_α is p-times i-removable and q-times i-admissible. From the sequence

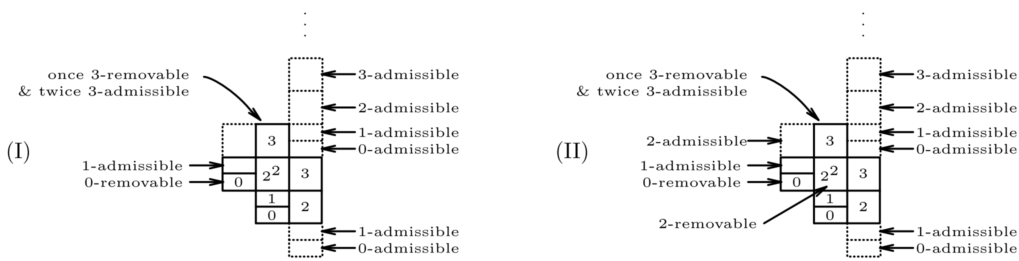
$$\big(\mathrm{sgn}_i(P_1), \mathrm{sgn}_i(P_2), \ldots, \mathrm{sgn}_i(P_t)\big)$$

of $+$'s and $-$'s, cancel out every $(+,-)$-pair to obtain a sequence of $-$'s followed by $+$'s, reading from left to right. This sequence is called the *i-signature* of S with respect to $\{S_j\}$.

Remark 4.4. (a) Note that the *i-signature* of S with respect to $\{S_j\}$ contains the information of i-removable or i-admissible sites.

(b) We can redefine a removable i-block and an i-admissible slot as follows. For each j, an i-block of S_j is a removable i-block if one can have another zigzag strip from S_j by removing that block, and an i-colored site is an i-admissible slot if one can have another zigzag strip from S_j by stacking an i-block on that site. Also, when $\mathfrak{g} = B_n^{(1)}$ or $D_{n+1}^{(2)}$ (resp. $D_n^{(1)}$), if there exists an $(n-1)$-block (resp. $(n-2)$-block) in the last zigzag n-strip (resp. $(n-1)$ or n-strip), then above n-site (resp. n or $(n-1)$-site) of $(n-1)$-block (resp. $(n-2)$-block) is an n-admissible (resp. n or $(n-1)$-admissible) slot. Then we can easily check that the i-signature of S with respect to $\{S_j\}$ according to Definition 4.1 and the i-signature of S according to these exposition are the same.

Example 4.5. (a) Let S be a zigzag strip bundle for $U_q(B_3^{(1)})$ given in Remark 4.3. Then it is easy to see that all the removable and admissible sites of S with respect to two types (I) and (II) of decompositions into zigzag strips are given below.

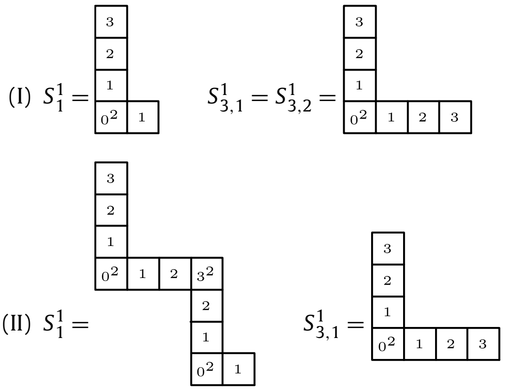


Therefore, we can easily see that the i-signatures $(i = 0, 1, 2, 3)$ of S with respect to (I) and (II) are the same. Indeed, the 2-signatures of S with respect to (I) and (II) are

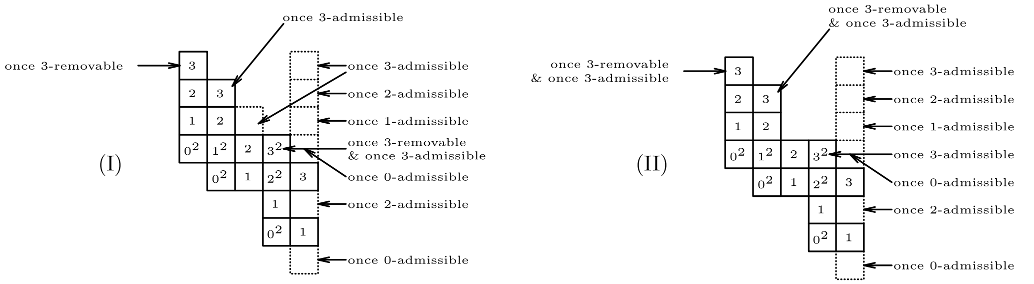
$$(+ + \cdots) \quad \text{and} \quad (+, -, + + \cdots) = (+ + \cdots),$$

respectively.

(b) Let S be the pile for $U_q(D_4^{(2)})$ given in Example 3.14. Then as one can see in Example 3.14 and Example A.5, we have the following two different decomposition into zigzag strips of S:



Then it is easy to see that all the removable and admissible sites of S with respect to two types (I) and (II) of decompositions into zigzag strips are given below.



Therefore, it is easy to see that the i-signatures ($i = 0, 1, 2, 3$) of S with respect to (I) and (II) are the same.

In Example 4.5, we see that the i-signatures ($i = 0, 1, 2, 3$) of S with respect to (I) and (II) are the same. Indeed, it is not strange as one can see in below.

Proposition 4.6. *Let S be a zigzag strip bundle for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$, and let $\{S_j \mid j = 1, \dots, M\}$ and $\{T_k \mid k = 1, \dots, N\}$ be any two decompositions into zigzag strips satisfying the conditions of Definition 3.3, Definition 3.8 or Definition 3.12. Then the i-signature of S with respect to $\{S_j \mid j = 1, \dots, M\}$ and the i-signature of S with respect to $\{T_k \mid k = 1, \dots, N\}$ are the same. In other words, the i-signature of S is independent of decompositions into zigzag strips of S.*

Remark 4.7. In Proposition 4.6, the statement that *the i-signature of S with respect to $\{S_j \mid j = 1, \dots, M\}$ and the i-signature of S with respect to $\{T_k \mid k = 1, \dots, N\}$ are the same* means that they are the same sequences, and the sites corresponding to $-$ and $+$ of the i-signatures are the same.

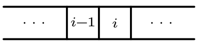
Proof of Proposition 4.6. Since the proofs are the similar, we only treat $U_q(B_n^{(1)})$-type. Now, for the sites of board $\mathcal{B}$, we adopt new exposition of admissibility and removability different from those given in Definition 4.1 as follows.

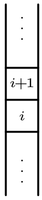
Another exposition of admissibility and removability: Consider an i-colored site P on which there are k i-blocks. Then if $i \neq n$, the below and right sites are $(i-1)$ and $(i+1)$-colored sites, respectively. On the other hand, if P is an n-colored site, the below site is an $(n-1)$-colored site and there is no right site of P. For the numbers $\#(i-1)$, $\#(i+1)$ and $\#(n-1)$ of $(i-1)$, $(i+1)$ and $(n-1)$-blocks on these $(i-1)$, $(i+1)$ and $(n-1)$-sites, respectively, we set

$$l = \begin{cases} \#(i-1) + \#(i+1) - k & \text{if } i \neq n, \\ 2 \cdot \#(n-1) - k & \text{if } i = n. \end{cases}$$

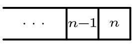
Then the site P is said to be k-times i-removable and l-times i-admissible.

In Definition 4.1, removable blocks of a zigzag strip are the leftmost top blocks, and left or top site of the leftmost top block is an admissible slot, and so we can easily check that i-signatures of a zigzag strip according to Definition 4.1 and above exposition are the same. Indeed, for the following cases

(i) 

(ii) 

there is no i-admissible slot and removable i-block according to Definition 4.1, but if we adopt above exposition, in (i), above site of $(i-1)$-block is once i-admissible and the site of i-block is once i-removable, and in (ii), left site of $(i+1)$-block is once i-admissible and the site of i-block is once i-removable. Thus, the i-signatures of a zigzag strip according to Definition 4.1 and above exposition are the same. Also, consider the zigzag strip

$S =$ 

Then the above site of $(n-1)$-block is once n-admissible according to Definition 4.1, but in the above exposition, the above site of $(n-1)$-block is twice n-admissible and the site of n-block is once n-removable. Thus, the n-signatures of S according to Definition 4.1 and above exposition are the same.

Now, as one can see in Remark 4.4, the i-signatures of a zigzag strip bundle S are determined by the i-admissible slots and removable i-blocks of decomposed zigzag strips S_j. Thus, the i-signatures of zigzag strip bundle according to Definition 4.1 and above exposition are the same. Therefore, the i-signatures of a zigzag strip bundle are also totally determined by the number of stacked blocks on the sites, and so are independent of the decomposition into zigzag strips. $\square$

According to Proposition 4.6, we may say the i-signature of S for a zigzag strip bundle S. Now, let $\mathcal{S}(\infty)$ be the set of all zigzag strip bundles, and let $S \in \mathcal{S}(\infty)$ be a zigzag strip bundle. We define $\tilde{f}_i S$ to be the zigzag strip bundle obtained from S by stacking an i-colored block on the i-admissible site corresponding to the leftmost $+$ in the i-signature of S. We define $\tilde{e}_i S$ to be the zigzag strip bundle obtained from S by eliminating the removable i-block corresponding to the rightmost $-$ in the i-signature of S. If there is no $-$ in the i-signature of S, we define $\tilde{e}_i S = 0$.

We also define the maps

$$\mathrm{wt} : \mathcal{S}(\infty) \to P, \qquad \varepsilon_i : \mathcal{S}(\infty) \to \mathbb{Z}, \qquad \varphi_i : \mathcal{S}(\infty) \to \mathbb{Z}$$

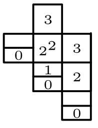
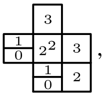
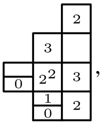
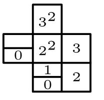
by

$$\mathrm{wt}(S) = -\sum_{i \in I} k_i \alpha_i,$$

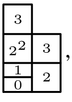
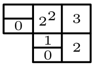
$$\varepsilon_i(S) = \text{the number of } -\text{'s in the } i\text{-signature of } S,$$
$$\varphi_i(S) = \varepsilon_i(S) + \langle h_i, \mathrm{wt}(S) \rangle,$$

where k_i is the number of i-colored blocks in S. Then it is straightforward to verify that $(\mathcal{S}(\infty), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is an affine crystal.

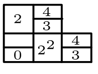
Example 4.8. Let S be a zigzag strip bundle for $U_q(B_3^{(1)})$ given in Remark 4.3. Then from the removable and admissible sites of S given in Example 4.5(a), we have

$\tilde{f}_0 S =$ , $\tilde{f}_1 S =$ , $\tilde{f}_2 S =$ , $\tilde{f}_3 S =$ 

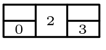
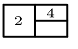
and

$\tilde{e}_0 S =$ , $\tilde{e}_1 S = \tilde{e}_2 S = 0,$ $\tilde{e}_3 S =$ 

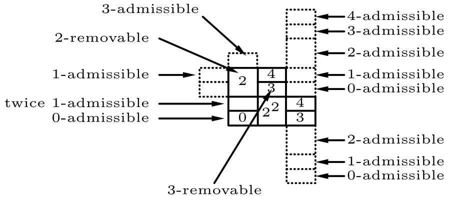
Example 4.9. Let

$S =$ 

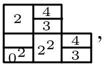
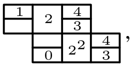
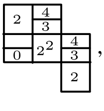
be a zigzag strip bundle for $U_q(D_4^{(1)})$. Then it is decomposed into the zigzag strips as follows.

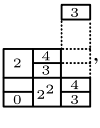
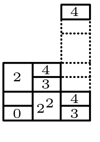
$S^1_{3,1} =$  $S^1_{3,2} =$  $S^1_{4,1} = S^1_{4,2} =$ 

Hence all the removable and admissible sites are given below.

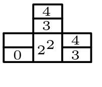


Therefore, we have

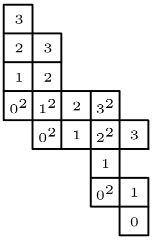
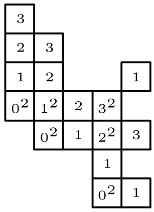
$\tilde{f}_0 S =$ , $\tilde{f}_1 S =$ , $\tilde{f}_2 S =$ ,

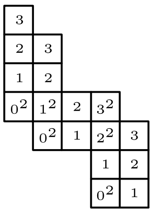
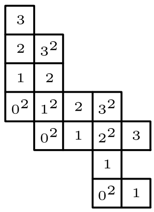
$\tilde{f}_3 S =$ , $\tilde{f}_4 S =$ 

and

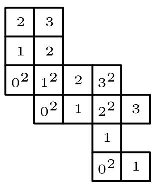
$\tilde{e}_0 S = \tilde{e}_1 S = \tilde{e}_3 S = \tilde{e}_4 S = 0,$ $\tilde{e}_2 S =$ 

Example 4.10. Let S be the pile for $U_q(D_4^{(2)})$ given in Example 3.14. Then from the removable and admissible sites of S with respect to two types (I) or (II) of decompositions into zigzag strips are given in Example 4.5(b), we have

$\tilde{f}_0 S =$  , $\tilde{f}_1 S =$  ,

$\tilde{f}_2 S =$  , $\tilde{f}_3 S =$ 

and

$\tilde{e}_0 S = \tilde{e}_1 S = \tilde{e}_2 S = 0,$ $\tilde{e}_3 S =$ 

4.2. Realization of the crystal $B(\infty)$

Let $\mathcal{S}(\infty)$ be the set of all zigzag strip bundles for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$.

Proposition 4.11. *Every zigzag strip bundle $S \in S(\infty)$ for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$ is connected with the empty zigzag strip bundle $\emptyset$. That is,*

$$\text{if } \tilde{e}_i S = 0 \text{ for all } i \in I, \text{ then } S = \emptyset.$$

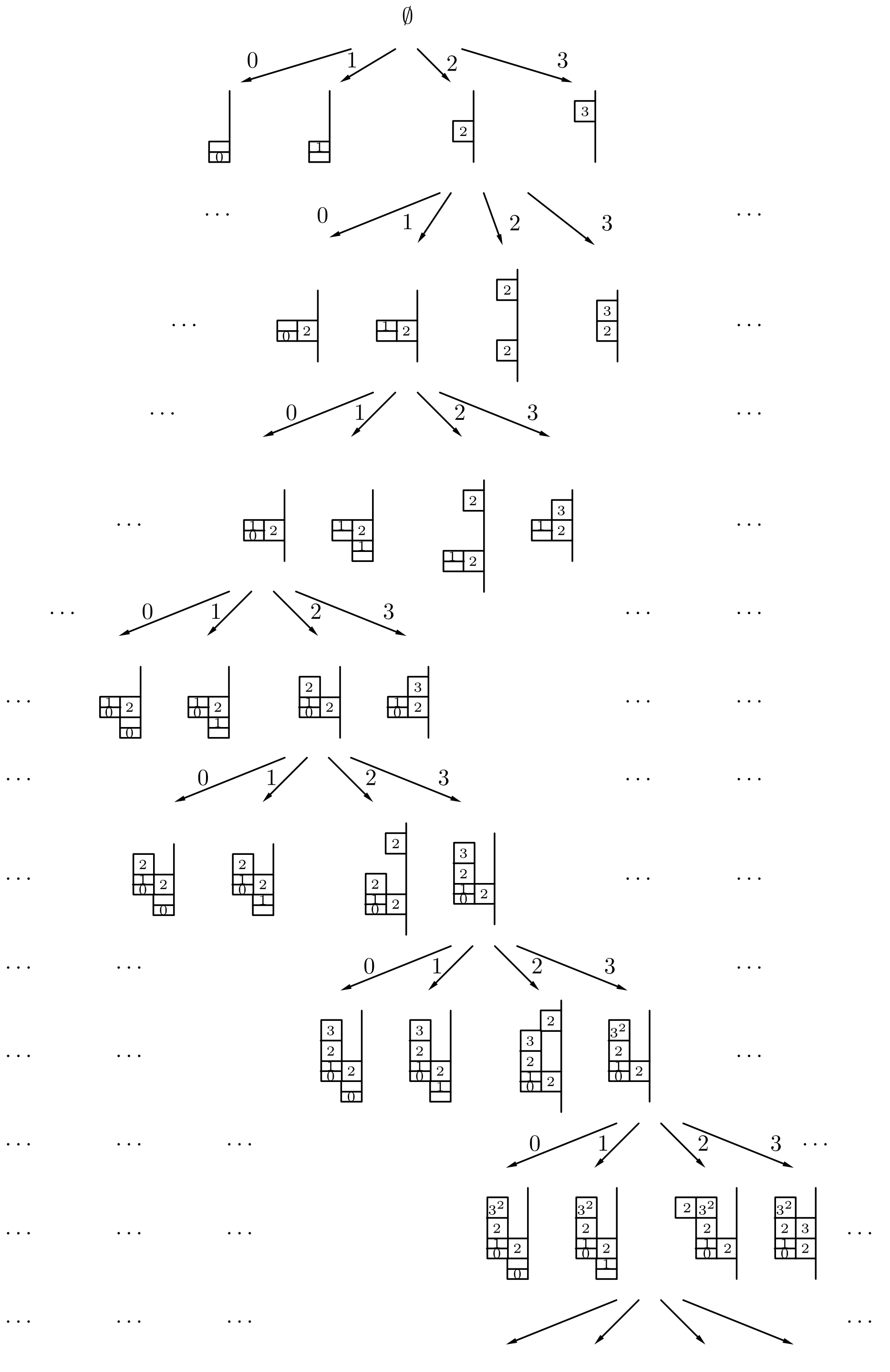
Proof. Suppose that $S \neq \emptyset$. Let B be the top block stacked on the top site of the leftmost column of S, and assume that its color is α. If above α-block B does not belong to a zigzag n-strip, it is clear that B is a removable block, and by the definition of zigzag strip bundle and the definition of i-signature, we have

$$\mathrm{sgn}_\alpha(S) = (-, \ldots),$$

where the leftmost $-$ corresponds to the removable block B. Thus $\tilde{e}_\alpha S \neq 0$.

Also, if B belongs to the kth zigzag n-strip $S_{n,k}^t$ on $\mathcal{B}_t$ and $S_{n,k+1}^t = \emptyset$, then clearly $\tilde{e}_\alpha S \neq 0$. Now, consider the case that B belongs to kth zigzag n-strip $S_{n,k}^t$ on $\mathcal{B}_t$ and $S_{n,k+1}^t \neq \emptyset$. In this case, note that $|S_{n,k}^t| \geqslant |S_{n,k+1}^t| + 2$ because of the location of B and the definition of zigzag strip bundle, and so B is a removable block. Thus we also have $\tilde{e}_\alpha S \neq 0$. $\square$

Example 4.12. The following is the top part of $\mathcal{S}(\infty)$ over $U_q(B_3^{(1)})$.



Let $\mathcal{M}(\infty)$ be the connected component containing $\mathbf{1}$ given in Section 2.3. Then we have

Theorem 4.13. *For $\mathfrak{g} = B_n^{(1)}$, $D_n^{(1)}$ or $D_{n+1}^{(2)}$, there is a $U_q(\mathfrak{g})$-crystal isomorphism*

$$\Phi : \mathcal{S}(\infty) \xrightarrow{\sim} \mathcal{M}(\infty) \cong B(\infty)$$

sending $\emptyset$ to $\mathbf{1}$.

Remark 4.14. For quantum affine algebras, it is not easy to find the explicit description of monomials in the connected component containing a maximal vector. Theorem 4.13 provides a characterization of $\mathcal{M}(\infty)$ using the combinatorics of zigzag strip bundles.

Proof of Theorem 4.13. Since the proofs are the similar, we only treat $U_q(B_n^{(1)})$-type. Let S be a zigzag strip bundle in $\mathcal{S}(\infty)$. We define $\Phi(S)$ to be the Nakajima monomial

$$\Phi(S) = \mathbf{1} \cdot \prod_{i \in I, k \geqslant 0} A_i(k)^{-s_i(k+1)},$$

where $s_i(k+1)$ $(k \geqslant 0)$ is the number of i-blocks in the $(k+1)$st column of S. Then clearly $\Phi(\emptyset) = \mathbf{1}$. But, we cannot convince that $\Phi(S)$ belongs to $\mathcal{M}(\infty)$. Nevertheless, since $S(\infty)$ and $\mathcal{M}(\infty)$ are the connected components containing $\emptyset$ and $\mathbf{1}$, respectively, in order to show that Φ is a crystal isomorphism, it suffices to show that Φ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ and it is 1-1.

For this, we use induction on the number of zigzag strips of zigzag strip bundles. First, consider the case when the zigzag strip bundle S consists of only one zigzag strip, and assume that S is a zigzag α-strip $(\alpha \in I)$. If $\alpha = 0$ or 1, then S is one of the zigzag strips $\begin{array}{|c|}\hline \\ \hline 0 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}$, and

$$\Phi(S) = \begin{cases} \mathbf{1} \cdot Y_0(0)^{-1} Y_0(1)^{-1} Y_2(0) & \text{for } S = \begin{array}{|c|}\hline \\ \hline 0 \\ \hline\end{array}, \\ \mathbf{1} \cdot Y_1(0)^{-1} Y_1(1)^{-1} Y_2(0) & \text{for } S = \begin{array}{|c|}\hline 1 \\ \hline \\ \hline\end{array}, \\ \mathbf{1} \cdot Y_0(0)^{-1} Y_0(1)^{-1} Y_1(0)^{-1} Y_1(1)^{-1} Y_2(0)^2 & \text{for } S = \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array} \end{cases}$$

Thus, $\Phi(\tilde{e}_i S) = \tilde{e}_i \Phi(S)$ and $\Phi(\tilde{f}_i S) = \tilde{f}_i \Phi(S)$ for all $i \in I$. Consider the case that $\alpha = 2, \dots, n-2$. Suppose that S has a u-admissible slot and a removable v-block at the pth and qth columns respectively, then we have two cases

(i) $p = q$,
(ii) $p = q + 1$.

Moreover, when $p = q$, we have the following three cases:

(i-1) $2 \leqslant v \leqslant n-1$, $u = v + 1$, i.e.,

$$S = \begin{array}{c} u \\ \begin{array}{|c|}\hline v \\ \hline \vdots \\ \end{array} \end{array}$$

(i-2) $v = 0$ and $u = 1$, or $v = 1$ and $u = 0$, i.e.,

$$S = \begin{array}{|c|c}\hline & \cdots \\ \hline 0 & \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c}\hline 1 & \cdots \\ \hline & \\ \hline\end{array}$$

(i-3) $u = v = n$, i.e.,

$$S = \begin{array}{|c|}\hline n \\ \hline \vdots \\ \end{array}$$

For these cases (i-1), (i-2) and (i-3), we have $\Phi(S)$ as follows.

(i-1) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_{u-1}(p)^{-1}Y_u(p-1)$;
(i-2) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_0(p)^{-1}Y_1(p-1)$ or $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_0(p-1)Y_1(p)^{-1}$;
(i-3) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_n(p-1)Y_n(p)^{-1}$.

Thus, it is clear that $\Phi(\tilde{e}_iS)=\tilde{e}_i\Phi(S)$ and $\Phi(\tilde{f}_iS)=\tilde{f}_i\Phi(S)$ for all $i\in I$. Also, when $p=q+1$, there are three possibilities as follows:

(ii-1) $3\leqslant v\leqslant n-1$, $u=v-1$, i.e., $S={}_{u}\,\boxed{v}\boxed{\ \cdots\ }$;
(ii-2) $v=2$ and $u=1$, or $v=2$ and $u=0$, i.e., $S={}^{1}_{0}\,\boxed{2}\boxed{\ \cdots\ }$;
(ii-3) $u=n-1$ and $v=n$, i.e.,

$$S=\begin{array}{|c|}\hline n\,2\\ \hline \vdots\\ \end{array}$$

In each case, $\Phi(S)$ is given below.

(ii-1) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_u(q)Y_{u+1}(q)^{-1}$;
(ii-2) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_0(q)Y_1(q)Y_2(q)^{-1}$;
(ii-3) $\mathbf{1}\cdot Y_\alpha(0)^{-1}Y_{\alpha+1}(0)Y_{n-1}(q)Y_n(q)^{-2}$.

Therefore, $\Phi(\tilde{e}_iS)=\tilde{e}_i\Phi(S)$ and $\Phi(\tilde{f}_iS)=\tilde{f}_i\Phi(S)$ for all $i\in I$. Similarly, for the cases $\alpha=n-1$ and $\alpha=n$, we can show that $\Phi(\tilde{e}_iS)=\tilde{e}_i\Phi(S)$ and $\Phi(\tilde{f}_iS)=\tilde{f}_i\Phi(S)$ for all $i\in I$. Indeed, the only difference between the case $\alpha=2,\dots,n-2$ and the cases $\alpha=n-1$ and $\alpha=n$ is that $Y_\alpha(0)^{-1}Y_{\alpha+1}(0)$ appearing in $\Phi(S)$ for $\alpha=2,\dots,n-2$ is replaced with $Y_{n-1}(0)^{-1}Y_n(0)^2$ and $Y_n(0)^{-1}$ in $\Phi(S)$ for $\alpha=n-1$ and $\alpha=n$, respectively.

Now, let $S\in S(\infty)$ be decomposed into the zigzag strips $\{S_1,\dots,S_N\}$. Then there is a zigzag strip S_j such that $\{S_1,\dots,S_N\}-\{S_j\}$ is a decomposition of zigzag strip bundle S'. By the induction hypothesis, we have $\tilde{f}_i\Phi(S')=\Phi(\tilde{f}_iS')$ and $\tilde{e}_i\Phi(S')=\Phi(\tilde{e}_iS')$ for all $i\in I$. Also, by the definition of Φ, we have $\Phi(S)=\Phi(S')\Phi(S_j)$. As we can see in the case when the zigzag strip bundle consists of only one zigzag strip, if S_j has a u-admissible slot and a removable v-block at the pth and qth columns respectively, then

$$\Phi(S_j)=\mathbf{1}\cdot Y_u(p-1)Y_v(q)^{-1}\cdot R(S_j), \tag{4.1}$$

where $R(S_j)$ is the product $Y_i(m)$'s $(i\neq u,v)$ in $\Phi(S_j)$. Note that $\tilde{e}_i(\mathbf{1}\cdot R(S_j))=0$ and $\tilde{f}_i(\mathbf{1}\cdot R(S_j))=\mathbf{1}\cdot R(S_j)\cdot A_i(0)^{-1}$ for all $i\in I$. Thus, for $i\neq u,v$, clearly $\tilde{f}_iS$ (resp. $\tilde{e}_iS$) is the union of $\tilde{f}_iS'$ ($\tilde{e}_iS'$) and S_j, and $\tilde{f}_i\Phi(S)=\tilde{f}_i\Phi(S')\cdot\Phi(S_j)$ (resp. $\tilde{e}_i\Phi(S)=\tilde{e}_i\Phi(S')\cdot\Phi(S_j)$), and so $\tilde{f}_i$ (resp. $\tilde{e}_i$) commutes with Φ. Hence it suffices to show that $\tilde{f}_u$, $\tilde{e}_u$, $\tilde{f}_v$, and $\tilde{e}_v$ commute with Φ. Since the arguments are the similar, we only show that $\Phi(\tilde{f}_uS)$ and $\tilde{f}_u\Phi(S)$ are the same. First, consider the case when $\tilde{f}_uS$ is the union of $\tilde{f}_uS'$ and S_j. In this case, if $\tilde{f}_uS'$ is obtained by stacking u-block on the lth $(l\geqslant p)$ column, then $n_f(\Phi(S'))=l-1$, and so by (4.1), $n_f(\Phi(S))$ is also $l-1$. Also, if $\tilde{f}_uS'$ is obtained by stacking u-block on the lth $(l<p)$ column, there should exist a removable u-block between lth column and pth column. By (4.1), it means that $\varphi_u(S)=\varphi_u(S')+1$, but $n_f(\Phi(S))=n_f(\Phi(S'))$. Second, if $\tilde{f}_uS$ is the union of S' and $\tilde{f}_uS_j$, and if $\tilde{f}_uS'$ is obtained by stacking u-block on the lth column, then $p\geqslant l$, and so $n_f(\Phi(S))=p-1$. Therefore, $\Phi(\tilde{f}_uS)$ and $\tilde{f}_u\Phi(S)$ are the same.

Moreover, given the set $(s_i(k)\colon\ i\in I,\ k\in\mathbb{Z}_{>0})$ of nonnegative integers, by the condition (ii) of definition of zigzag strip bundle, there exists at most one zigzag strip bundle consisting of $s_i(k)$ i-blocks in the kth column of $\mathcal{B}$. Thus, Φ is 1-1, which completes the proof. $\square$

Let $\mathbf{i} = (0, 1 \ldots, n, 0, 1, \ldots, n, 0, \ldots)$ be an infinite sequence of indices in I. Let $\Psi_i : B(\infty) \hookrightarrow B(\mathbf{i})$ be the crystal embedding given in Theorem 2.1, and let $\phi : \mathcal{M}(\infty) \to \operatorname{Im} \Psi_i$ be the crystal isomorphism given in Theorem 2.3.

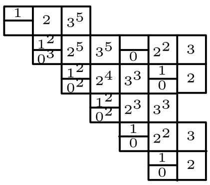
Corollary 4.15. *For $\mathfrak{g} = B_n^{(1)}$, $D_n^{(1)}$ or $D_{n+1}^{(2)}$, there is a $U_q(\mathfrak{g})$-crystal isomorphism*

$$\Psi : \mathcal{S}(\infty) \xrightarrow{\Phi} \mathcal{M}(\infty) \xrightarrow{\phi} \operatorname{Im} \Psi_i \cong B(\infty)$$

sending $\emptyset$ to $\cdots \otimes b_0(0) \otimes b_n(0) \cdots \otimes b_1(0) \otimes b_0(0)$.

Remark 4.16. In [26], Zelevinsky and Nakashima obtained the images of the Kashiwara embeddings by a unified method, called the *polyhedral realization*. However, their descriptions of the images of the Kashiwara embeddings contain many redundant (linear) inequalities. According to Corollary 4.15, one can understand the description of the connected component $\mathcal{M}(\infty)$ and the image of $\Psi_{\mathbf{i}}$ using the combinatorics of zigzag strip bundles.

Example 4.17. Let

$S =$ 

be a zigzag strip bundle for $U_q(B_3^{(1)})$ given in Example 3.5(a). Then

$$\begin{aligned}\Phi(S) = \mathbf{1} &\cdot \big(A_2(0)^{-2} A_3(0)^{-2}\big) \cdot \big(A_0(1)^{-2} A_1(1)^{-2} A_2(1)^{-4} A_3(1)^{-3}\big) \\ &\cdot \big(A_0(2)^{-2} A_1(2)^{-1} A_2(2)^{-3} A_3(2)^{-3}\big) \cdot \big(A_0(3)^{-2} A_1(3)^{-2} A_2(3)^{-4} A_3(3)^{-5}\big) \\ &\cdot \big(A_0(4)^{-2} A_1(4)^{-2} A_2(4)^{-5} A_3(4)^{-5}\big) \cdot \big(A_0(5)^{-3} A_1(5)^{-2} A_2(5)^{-1}\big) \cdot A_1(6)^{-1}.\end{aligned}$$

Moreover, for the sequence $\mathbf{i} = (0, 1, 2, 3, 0, 1, 2, 3, 0, \ldots)$,

$$\begin{aligned}\psi\big(\Phi(S)\big) = \cdots &\otimes b_3(0) \otimes b_2(0) \otimes b_1(0) \otimes b_0(0) \otimes b_3(0) \otimes b_2(0) \otimes b_1(-1) \otimes b_0(0) \\ &\otimes b_3(0) \otimes b_2(-1) \otimes b_1(-2) \otimes b_0(-3) \otimes b_3(-5) \otimes b_2(-5) \otimes b_1(-2) \otimes b_0(-2) \\ &\otimes b_3(-5) \otimes b_2(-4) \otimes b_1(-2) \otimes b_0(-2) \otimes b_3(-3) \otimes b_2(-3) \otimes b_1(-1) \otimes b_0(-2) \\ &\otimes b_3(-3) \otimes b_2(-4) \otimes b_1(-2) \otimes b_0(-2) \otimes b_3(-2) \otimes b_2(-2) \otimes b_1(0) \otimes b_0(0).\end{aligned}$$

Acknowledgments

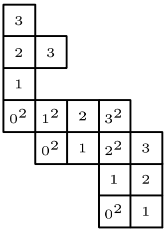
Part of this work was done during the authors visited KIAS, in the winter of 2008. They would like to express their sincere gratitude to the staff of KIAS for their hospitality and support.

Appendix A. An algorithm to decompose zigzag strip bundles

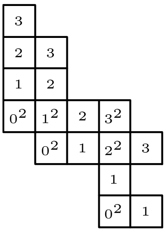
In this section, we give an algorithm to decompose zigzag strip bundles for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$ into zigzag strips. Before we describe the algorithm, we give the notion of *connectedness* of a pile as follows.

Definition A.1. A pile of colored blocks stacked on $\mathcal{B}_t$ for some $t \geqslant 1$ is called *connected* if the existence of blocks on any site except for the sites of the rightmost column in $\mathcal{B}_t$ implies the existence of blocks on below or right site of that site. Also, the pile is called *disconnected* if it is not connected.

Example A.2. Let

$S =$ 

be a pile for $U_q(D_4^{(2)})$. Then it is disconnected because there is no block below 3-block of the fourth column from the right. However, the following pile

$S' =$ 

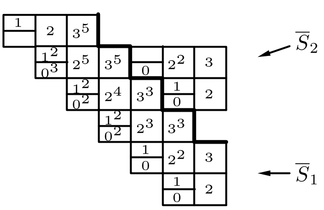
is connected.

Let S be a zigzag strip bundle for $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ or $U_q(D_{n+1}^{(2)})$, and let $\overline{S}_t$ $(t \geqslant 1)$ be the pile consisting of blocks on $\mathcal{B}_t$ of S. Then it is enough to describe an algorithm to decompose $\overline{S}_t$ into zigzag strips.

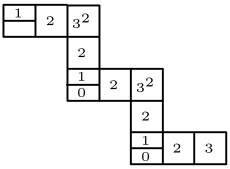
(i) $B_n^{(1)}$ **or** $D_{n+1}^{(2)}$**-type.** We first pick out the zigzag n-strip $S_{n,1}^t$ *as long as possible* so that the remaining pile is connected. Second, if exists, pick out the 2nd zigzag n-strip $S_{n,2}^t$ as long as possible so that $S_{n,2}^t \subset S_{n,1}^t$ and the remaining pile is connected. Continue this process until the last zigzag n-strip, say the kth zigzag n-strip, which means the $(k+1)$st zigzag n-strip does not exist. After that, if exists, pick out the zigzag i-strip S_i^t from 0 to $n-1$ *as long as possible* so that the remaining pile is connected.

(ii) $D_n^{(1)}$**-type.** First, we pick out zigzag $(n-1)$-strip $S_{n-1,1}^t$ and zigzag n-strip $S_{n,1}^t$ *as long as possible* so that the remaining pile is connected. Second, if exists, we pick out the 2nd zigzag n-strip $S_{n,2}^t$ as long as possible so that $S_{n,2}^t \subset S_{n-1,1}^t$ and the remaining pile is connected, and then pick out the 2nd zigzag $(n-1)$-strip $S_{n-1,2}^t$ as long as possible so that $S_{n-1,2}^t \subset S_{n,1}^t$ the remaining pile is connected. Continue this process until the last zigzag $(n-1)$-strip and n-strip. After that, as $B_n^{(1)}$-case, we pick out zigzag i-strips S_i^t $(i = 0, 1, \ldots, n-2)$ from $\overline{S}_t$.

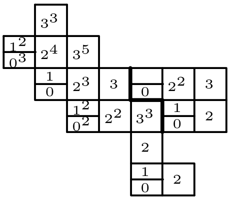
Example A.3. Let

$S =$ 

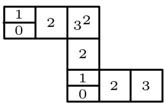
be a pile for $U_q(B_3^{(1)})$ given in Example 3.5(a). Then the following zigzag 3-strip



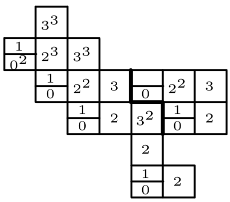
is the longest the one which we can pick out from $\bar{S}_1$ of S so that the remaining pile is connected, we say $S_{3,1}^1$, and we have

$S\backslash S_{3,1}^1 =$ 

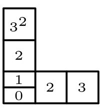
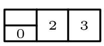
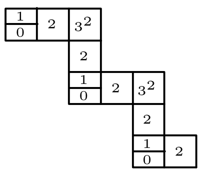
Similarly, we can pick out the following zigzag 3-strip from $S\backslash S_{3,1}^1$



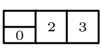
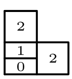
which we call $S_{3,2}^1$. Also, we have

$S\backslash\{S_{3,1}^1, S_{3,2}^1\} =$ 

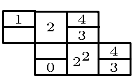
Continuing this process, we have

$S_{3,3}^1 =$  $S_{3,4}^1 =$  $S_{3,5}^1 =$  $S_2^1 =$ 

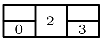
Also, we have

$S_{3,1}^2 =$  and $S_2^2 =$ 

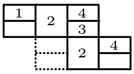
Example A.4. Let

$S =$ 

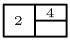
be a pile for $U_q(D_4^{(1)})$ given in Example 3.10. Then the following zigzag first 3-strip is the longest the one which we can pick out from $\bar{S}_1$ of S so that the remaining pile is connected, we say $S_{3,1}^1$,



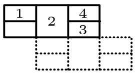
and we have

$S \backslash S_{3,1}^1 =$ 

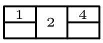
Similarly, we can pick out the following first zigzag 4-strip from $S \backslash S_{3,1}^1$



which we call $S_{4,1}^1$, and we have

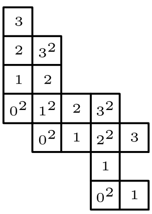
$S \backslash \{S_{3,1}^1, S_{4,1}^1\} =$ 

Also, we can pick out the following second zigzag 4-strip $S_{4,2}^1$ from $S \backslash \{S_{3,1}^1, S_{4,1}^1\}$

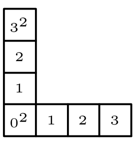


and the remaining pile  is the second zigzag 3-strip.

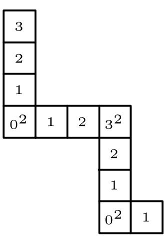
Example A.5. Let

$S =$ 

be a pile for $U_q(D_4^{(2)})$ given in Example 3.14. Then the following zigzag 3-strip



is the longest the one which we can pick out from $\bar{S}_1$ of S so that the remaining pile is connected, we say $S_{3,1}^1$, and we have

$S \backslash S_{3,1}^1 =$  $= S_1^1.$

Remark A.6. Note that the decompositions of the piles given in Example A.3 and Example A.5 are different from those given Example 3.5(a) and Example 3.14.

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