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Functional equations and inequalities in paranormed spaces

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Abstract

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of an additive functional equation, a quadratic functional equation, a cubic functional equation and a quartic functional equation in paranormed spaces.

Furthermore, we prove the Hyers-Ulam stability of functional inequalities in paranormed spaces by using the fixed point method and the direct method.

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1 Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently, and since then several generalizations and applications of this notion have been investigated by various authors (see [3–7]). This notion was defined in normed spaces by Kolk [8].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1 [9] Let X be a vector space. A paranorm $P: X \to [0,\infty)$ is a function on X such that

(1) $P(0) = 0$;

(2) $P(-x) = P(x)$;

(3) $P(x + y) \le P(x) + P(y)$ (triangle inequality)

(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [11] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [12] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by

Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [15] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [16], following the same approach as in Rassias [13], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Rassias and Šemrl [17], that one cannot prove a Rassias-type theorem when $p = 1$ (*cf.* the books of Czerwik [18], Hyers, Isac and Rassias [19]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [20] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [21] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [22] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [23–29]).

In [30], Jun and Kim considered the following cubic functional equation:

$$\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) = f(x+y) + f(x-y) + 6f(x). \tag{1.2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation*, and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [31], Lee *et al.* considered the following quartic functional equation:

$$\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) - 3f(y). \tag{1.3}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a *quartic functional equation*, and every solution of the quartic functional equation is said to be a *quartic mapping*.

In [32], Gilányi showed that if f satisfies the functional inequality

$$\left\| 2f(x) + 2f(y) - f\left(xy^{-1}\right) \right\| \leq \left\| f(xy) \right\|, \tag{1.4}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right).$$

See also [33]. Fechner [34] and Gilányi [35] proved the Hyers-Ulam stability of the functional inequality (1.4).

Park, Cho and Han [36] proved the Hyers-Ulam stability of the following functional inequalities:

$$\|f(x)+f(y)+f(z)\| \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \tag{1.5}$$

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|, \tag{1.6}$$

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|. \tag{1.7}$$

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

In this paper, we prove the Hyers-Ulam stability of the Cauchy additive functional equation, the quadratic functional equation (1.2), the cubic functional equation (1.2) and the quartic functional equation (1.3) in paranormed spaces by using the fixed point method and the direct method.

Furthermore, we prove the Hyers-Ulam stability of the functional inequalities (1.5), (1.6) and (1.7) in paranormed spaces by using the fixed point method and the direct method.

2 Hyers-Ulam stability of the Cauchy additive functional equation

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in paranormed spaces.

Let S be a set. A function $m : S \times S \to [0,\infty]$ is called a *generalized metric* on S if m satisfies

(1) $m(x,y) = 0$ if and only if $x = y$;
(2) $m(x,y) = m(y,x)$ for all $x,y \in S$;
(3) $m(x,z) \le m(x,y) + m(y,z)$ for all $x,y,z \in S$.

We recall a fundamental result in fixed point theory.

Theorem 2.1 [37, 38] *Let (S,m) be a complete generalized metric space, and let $J : S \to S$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$. Then, for each given element $x \in S$, either*

$$m(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $m(J^n x, J^{n+1}x) < \infty$, $\forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $W = \{y \in S \mid m(J^{n_0}x, y) < \infty\}$;*
(4) *$m(y,y^*) \le \frac{1}{1-\alpha} m(y,Jy)$ for all $y \in W$.*

In 1996, Isac and Rassias [39] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [40–44]).

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 2.2 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le 2\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.1}$$

for all $x,y \in X$. Let $f : X \to Y$ be a mapping such that

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y) \tag{2.2}$$

for all $x,y \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2-2\alpha}\varphi(x,x) \tag{2.3}$$

for all $x \in X$.

Proof Letting $y = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x,x),$$

and so

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x) \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$m(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x,x),\ \forall x \in X\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,m) is complete (see [45, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $m(g,h) = \varepsilon$. Since

$$\|Jg(x) - Jh(x)\| = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\| \le \alpha\varphi(x,x)$$

for all $x \in X$, $m(g,h) = \varepsilon$ implies that $m(Jg,Jh) \le \alpha\varepsilon$. This means that

$$m(Jg,Jh) \le \alpha m(g,h)$$

for all $g,h \in S$.

It follows from (2.4) that $m(f, Jf) \le \frac{1}{2}$.

By Theorem 2.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A(2x) = 2A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : m(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu\varphi(x, x)$$

for all $x \in X$;

(2) $m(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = A(x)$$

for all $x \in X$;

(3) $m(f, A) \le \frac{1}{1-\alpha} m(f, Jf)$, which implies the inequality

$$m(f, A) \le \frac{1}{2 - 2\alpha}.$$

This implies that the inequality (2.3) holds true.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|A(x + y) - A(x) - A(y)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n(x + y)) - f(2^n x) - f(2^n y)\| \\ &\le \lim_{n\to\infty} \frac{2^n\alpha^n}{2^n} \varphi(x, y) = 0 \end{aligned}$$

for all $x, y \in X$. So, $A(x + y) - A(x) - A(y) = 0$ for all $x, y \in X$. Thus $A : X \to Y$ is an additive mapping, as desired. □

Corollary 2.3 *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be a mapping such that*

$$\|f(x + y) - f(x) - f(y)\| \le P(x)^r + P(y)^r$$

for all $x, y \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2}{2 - 2^r} P(x)^r \tag{2.6}$$

for all $x \in X$.

Proof Taking $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 2.2, we get the desired result. □

Theorem 2.4 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\bigl(2^j x, 2^j y\bigr) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that*

$$\bigl\|f(x) - A(x)\bigr\| \le \frac{1}{2}\Phi(x,x)$$

for all $x \in X$.

Proof The proof is similar to the proof of [46, Theorem 2.2]. □

Remark 2.5 Let $r < 1$. Letting $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ in Theorem 2.4, we obtain the inequality (2.6). The proof is given in [46, Theorem 2.2].

Theorem 2.6 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le \frac{\alpha}{2}\varphi(2x,2y) \tag{2.7}$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping such that

$$P\bigl(f(x+y) - f(x) - f(y)\bigr) \le \varphi(x,y) \tag{2.8}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P\bigl(f(x) - A(x)\bigr) \le \frac{\alpha}{2 - 2\alpha}\varphi(x,x) \tag{2.9}$$

for all $x \in Y$.

Proof Letting $y = x$ in (2.8), we get

$$P\bigl(f(2x) - 2f(x)\bigr) \le \varphi(x,x),$$

and so

$$P\left(f(x) - 2f\left(\frac{x}{2}\right)\right) \le \frac{\alpha}{2}\varphi(x,x) \tag{2.10}$$

for all $x \in Y$.

Consider the set

$$S := \{h : Y \to X\}$$

and introduce the generalized metric on S:

$$m(g,h) = \inf\{\mu \in \mathbb{R}_+ : P(g(x) - h(x)) \leq \mu\varphi(x,x), \ \forall x \in Y\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,m) is complete (see [45, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in Y$.

Let $g, h \in S$ be given such that $m(g,h) = \varepsilon$. Since

$$P(Jg(x) - Jh(x)) = P\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right) \leq \alpha\varphi(x,x)$$

for all $x \in Y$, $m(g,h) = \varepsilon$ implies that $m(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$m(Jg, Jh) \leq \alpha m(g,h)$$

for all $g, h \in S$.

It follows from (2.10) that $m(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 2.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.11}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : m(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.11) such that there exists a $\mu \in (0,\infty)$ satisfying

$$P(f(x) - A(x)) \leq \mu\varphi(x,x)$$

for all $x \in Y$;

(2) $m(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in Y$;

(3) $m(f,A) \leq \frac{1}{1-\alpha} m(f, Jf)$, which implies the inequality

$$m(f,A) \leq \frac{\alpha}{2 - 2\alpha}.$$

This implies that the inequality (2.9) holds true.

It follows from (2.7) and (2.8) that

$$\begin{aligned}
P\big(A(x+y)-A(x)-A(y)\big) &= \lim_{n\to\infty} P\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right)\\
&\le \lim_{n\to\infty} 2^n P\left(\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right)\\
&\le \lim_{n\to\infty} \frac{2^n\alpha^n}{2^n}\varphi(x,y) = 0
\end{aligned}$$

for all $x, y \in Y$. So, $A(x+y)-A(x)-A(y) = 0$ for all $x, y \in Y$. Thus $A : Y \to X$ is an additive mapping, as desired. □

Corollary 2.7 *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be a mapping such that*

$$P\big(f(x+y)-f(x)-f(y)\big) \le \theta\big(\|x\|^r + \|y\|^r\big)$$

for all $x, y \in Y$. Then there exists a unique Cauchy additive mapping $A : Y \to X$ such that

$$P\big(f(x)-A(x)\big) \le \frac{2\theta}{2^r-2}\|x\|^r \tag{2.12}$$

for all $x \in Y$.

Proof Taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ and choosing $\alpha = 2^{1-r}$ in Theorem 2.6, we get the desired result. □

Theorem 2.8 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying (2.8). *Then there exists a unique Cauchy additive mapping $A : Y \to X$ such that*

$$P\big(f(x)-A(x)\big) \le \frac{1}{2}\Phi(x,x)$$

for all $x \in Y$.

Proof The proof is similar to the proof of [46, Theorem 2.1]. □

Remark 2.9 Let $r > 1$. Letting $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 2.8, we obtain the inequality (2.12). The proof is given in [46, Theorem 2.1].

3 Hyers-Ulam stability of the quadratic functional equation (1.1)

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the quadratic functional equation (1.1) in paranormed spaces.

Theorem 3.1 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le 4\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x,y) \tag{3.1}$$

for all $x,y \in X$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\|f(x) - Q_2(x)\| \le \frac{1}{4-4\alpha}\varphi(x,x)$$

for all $x \in X$.

Proof Letting $y = x$ in (3.1), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x),$$

and so

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x,x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2 *Let r be a positive real number with $r < 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le P(x)^r + P(y)^r$$

for all $x,y \in X$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that

$$\|f(x) - Q_2(x)\| \le \frac{2}{4-2^r}P(x)^r \tag{3.2}$$

for all $x \in X$.

Proof Taking $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x,y \in X$ and choosing $\alpha = 2^{r-2}$ in Theorem 3.1, we get the desired result. □

Theorem 3.3 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty}\frac{1}{4^j}\varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x,y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.1). *Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that*

$$\|f(x) - Q_2(x)\| \le \frac{1}{4}\Phi(x,x)$$

for all $x \in X$.

Proof The proof is similar to the proof of [46, Theorem 3.2]. □

Remark 3.4 Let $r < 2$. Letting $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ in Theorem 3.3, we obtain the inequality (3.2). The proof is given in [46, Theorem 3.2].

Theorem 3.5 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le \frac{\alpha}{4}\varphi(2x,2y)$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and

$$P\bigl(f(x+y) + f(x-y) - 2f(x) - 2f(y)\bigr) \le \varphi(x,y) \tag{3.3}$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q_2 : Y \to X$ such that

$$P\bigl(f(x) - Q_2(x)\bigr) \le \frac{\alpha}{4-4\alpha}\varphi(x,x)$$

for all $x \in Y$.

Proof Letting $y = x$ in (3.3), we get

$$P\bigl(f(2x) - 4f(x)\bigr) \le \varphi(x,x),$$

and so

$$P\left(f(x) - 4f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{\alpha}{4}\varphi(x,x)$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.6. □

Corollary 3.6 *Let r, θ be positive real numbers with $r > 2$, and let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and*

$$P\bigl(f(x+y) + f(x-y) - 2f(x) - 2f(y)\bigr) \le \theta\bigl(\|x\|^r + \|y\|^r\bigr)$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q_2 : Y \to X$ such that

$$P\bigl(f(x) - Q_2(x)\bigr) \le \frac{2\theta}{2^r - 4}\|x\|^r \tag{3.4}$$

for all $x \in Y$.

Proof Taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ and choosing $\alpha = 2^{2-r}$ in Theorem 3.5, we get the desired result. □

Theorem 3.7 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and (3.3). *Then there exists a unique quadratic mapping $Q_2 : Y \to X$ such that*

$$P\big(f(x) - Q_2(x)\big) \le \frac{1}{4}\Phi(x,x)$$

for all $x \in Y$.

Proof The proof is similar to the proof of [46, Theorem 3.1]. □

Remark 3.8 Let $r > 2$. Letting $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 3.7, we obtain the inequality (3.4). The proof is given in [46, Theorem 3.1].

4 Hyers-Ulam stability of the cubic functional equation (1.2)

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the cubic functional equation (1.2) in paranormed spaces.

Theorem 4.1 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le 8\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping such that

$$\left\| \frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - f(x+y) - f(x-y) - 6f(x) \right\| \le \varphi(x,y) \tag{4.1}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\big\| f(x) - C(x) \big\| \le \frac{1}{8 - 8\alpha}\varphi(x,0)$$

for all $x \in X$.

Proof Letting $y = 0$ in (4.1), we get

$$\big\| f(2x) - 8f(x) \big\| \le \varphi(x,0),$$

and so

$$\left\| f(x) - \frac{1}{8}f(2x) \right\| \le \frac{1}{8}\varphi(x,0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 4.2 *Let r be a positive real number with $r < 3$, and let $f : Y \to X$ be a mapping such that*

$$\left\| \frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - f(x+y) - f(x-y) - 6f(x) \right\| \le P(x)^r + P(y)^r$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P\big(f(x) - C(x)\big) \leq \frac{1}{8 - 2^r} P(x)^r \tag{4.2}$$

for all $x \in Y$.

Proof Taking $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ and choosing $\alpha = 2^{r-3}$ in Theorem 4.1, we get the desired result. □

Theorem 4.3 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi\big(2^j x, 2^j y\big) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (4.1). *Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\big\|f(x) - C(x)\big\| \leq \frac{1}{8} \Phi(x, 0)$$

for all $x \in X$.

Proof The proof is similar to the proof of [46, Theorem 4.2]. □

Remark 4.4 Let $r < 3$. Letting $\varphi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ in Theorem 4.3, we obtain the inequality (4.2). The proof is given in [46, Theorem 4.2].

Theorem 4.5 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \leq \frac{\alpha}{8} \varphi(2x, 2y)$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping such that

$$P\left(\frac{1}{2} f(2x+y) + \frac{1}{2} f(2x-y) - f(x+y) - f(x-y) - 6f(x)\right) \leq \varphi(x,y) \tag{4.3}$$

for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P\big(f(x) - C(x)\big) \leq \frac{\alpha}{8 - 8\alpha} \varphi(x, 0)$$

for all $x \in Y$.

Proof Letting $y = 0$ in (4.3), we get

$$P\big(f(2x) - 8f(x)\big) \leq \varphi(x, 0),$$

and so

$$P\left(f(x) - 8f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{\alpha}{8} \varphi(x, 0)$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.6. □

Corollary 4.6 *Let r, θ be positive real numbers with $r > 3$, and let $f : Y \to X$ be a mapping such that*

$$P\left(\frac{1}{2}f(2x+y)+\frac{1}{2}f(2x-y)-f(x+y)-f(x-y)-6f(x)\right)\le\theta\left(\|x\|^r+\|y\|^r\right)$$

for all $x, y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P\bigl(f(x)-C(x)\bigr)\le\frac{\theta}{2^r-8}\|x\|^r \tag{4.4}$$

for all $x \in Y$.

Proof Taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ and choosing $\alpha = 2^{3-r}$ in Theorem 4.5, we get the desired result. □

Theorem 4.7 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=1}^{\infty}8^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right)<\infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying (4.3). *Then there exists a unique cubic mapping $C : Y \to X$ such that*

$$P\bigl(f(x)-C(x)\bigr)\le\frac{1}{8}\Phi(x,0)$$

for all $x \in Y$.

Proof The proof is similar to the proof of [46, Theorem 4.1]. □

Remark 4.8 Let $r > 3$. Letting $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 4.7, we obtain the inequality (4.4). The proof is given in [46, Theorem 4.1].

5 Hyers-Ulam stability of the quartic functional equation (1.3)

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the quartic functional equation (1.3) in paranormed spaces.

Theorem 5.1 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y)\le 16\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|\frac{1}{2}f(2x+y)+\frac{1}{2}f(2x-y)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right\|\le\varphi(x,y) \tag{5.1}$$

for all $x, y \in X$. *Then there exists a unique quartic mapping* $Q_4 : X \to Y$ *such that*

$$\|f(x) - Q_4(x)\| \leq \frac{1}{16 - 16\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof Letting $y = 0$ in (5.1), we get

$$\|f(2x) - 16f(x)\| \leq \varphi(x, 0),$$

and so

$$\left\|f(x) - \frac{1}{16}f(2x)\right\| \leq \frac{1}{16}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 5.2 *Let* r *be a positive real number with* $r < 4$, *and let* $f : Y \to X$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\left\|\frac{1}{2}f(2x + y) + \frac{1}{2}f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x) + 3f(y)\right\| \leq P(x)^r + P(y)^r$$

for all $x, y \in X$. *Then there exists a unique quartic mapping* $Q_4 : Y \to X$ *such that*

$$P\big(f(x) - Q_4(x)\big) \leq \frac{1}{16 - 2^r}P(x)^r \tag{5.2}$$

for all $x \in Y$.

Proof Taking $\varphi(x, y) = P(x)^r + P(y)^r$ for all $x, y \in X$ and choosing $\alpha = 2^{r-4}$ in Theorem 5.1, we get the desired result. □

Theorem 5.3 *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{16^j}\varphi\big(2^j x, 2^j y\big) < \infty$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (5.1). *Then there exists a unique quartic mapping* $Q_4 : X \to Y$ *such that*

$$\|f(x) - Q_4(x)\| \leq \frac{1}{16}\Phi(x, 0)$$

for all $x \in X$.

Proof The proof is similar to the proof of [46, Theorem 5.2]. □

Remark 5.4 Let $r < 4$. Letting $\varphi(x, y) = P(x)^r + P(y)^r$ for all $x, y \in X$ in Theorem 5.3, we obtain the inequality (5.2). The proof is given in [46, Theorem 5.2].

Theorem 5.5 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le \frac{\alpha}{16}\varphi(2x,2y)$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and

$$P\left(\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right) \le \varphi(x,y) \tag{5.3}$$

for all $x, y \in Y$. Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that

$$P\big(f(x) - Q_4(x)\big) \le \frac{\alpha}{16 - 16\alpha}\varphi(x,0)$$

for all $x \in Y$.

Proof Letting $y = 0$ in (5.3), we get

$$P\big(f(2x) - 16f(x)\big) \le \varphi(x,0),$$

and so

$$P\left(f(x) - 16f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2},\frac{x}{2}\right) \le \frac{\alpha}{16}\varphi(x,0)$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.6. □

Corollary 5.6 *Let r, θ be positive real numbers with $r > 4$, and let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and*

$$P\left(\frac{1}{2}f(2x+y) + \frac{1}{2}f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right) \le \theta\big(\|x\|^r + \|y\|^r\big)$$

for all $x, y \in Y$. Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that

$$P\big(f(x) - Q_4(x)\big) \le \frac{\theta}{2^r - 16}\|x\|^r \tag{5.4}$$

for all $x \in Y$.

Proof Taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ and choosing $\alpha = 2^{4-r}$ in Theorem 5.5, we get the desired result. □

Theorem 5.7 *Let $\varphi : Y^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 16^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) < \infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and (5.3). *Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that*

$$P\big(f(x) - Q_4(x)\big) \leq \frac{1}{16}\Phi(x, 0)$$

for all $x \in Y$.

Proof The proof is similar to the proof of [46, Theorem 5.1]. □

Remark 5.8 Let $r > 4$. Letting $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$ in Theorem 5.7, we obtain the inequality (5.4). The proof is given in [46, Theorem 5.1].

6 Stability of a functional inequality associated with a three-variable Jensen additive functional equation

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type three-variable Jensen additive functional equation in paranormed spaces.

Proposition 6.1 [36, Proposition 2.1] *Let $f : X \to Y$ be a mapping such that*

$$\big\|f(x) + f(y) + f(z)\big\| \leq \left\|2f\left(\frac{x + y + z}{2}\right)\right\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

Theorem 6.2 *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \leq 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{6.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\big\|f(x) + f(y) + f(z)\big\| \leq \left\|2f\left(\frac{x + y + z}{2}\right)\right\| + \varphi(x, y, z) \tag{6.2}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\big\|f(x) - A(x)\big\| \leq \frac{1}{2 - 2\alpha}\varphi(x, x, -2x) \tag{6.3}$$

for all $x \in X$.

Proof Letting $y = x$ and $z = -2x$ in (6.2), we get

$$\big\|2f(x) - f(2x)\big\| = \big\|2f(x) + f(-2x)\big\| \leq \varphi(x, x, -2x),$$

and so

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x, -2x) \tag{6.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S

$$m(g,h) = \inf\bigl\{\mu \in \mathbb{R}_+ : \bigl\|g(x) - h(x)\bigr\| \le \mu\varphi(x,x,-2x),\ \forall x \in X\bigr\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,m) is complete (see [45, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $m(g,h) = \varepsilon$. Since

$$\bigl\|Jg(x) - Jh(x)\bigr\| = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\| \le \alpha\varphi(x,x,-2x)$$

for all $x \in X$, $m(g,h) = \varepsilon$ implies that $m(Jg,Jh) \le \alpha\varepsilon$. This means that

$$m(Jg,Jh) \le \alpha m(g,h)$$

for all $g,h \in S$.

It follows from (6.4) that $m(f,Jf) \le \frac{1}{2}$.

By Theorem 2.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, *i.e.*,

$$A(2x) = 2A(x) \tag{6.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \bigl\{g \in S : m(f,g) < \infty\bigr\}.$$

This implies that A is a unique mapping satisfying (6.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\bigl\|f(x) - A(x)\bigr\| \le \mu\varphi(x,x,-2x)$$

for all $x \in X$;

(2) $m(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f\bigl(2^n x\bigr) = A(x)$$

for all $x \in X$;

(3) $m(f,A) \leq \frac{1}{1-\alpha} m(f,Jf)$, which implies the inequality

$$m(f,A) \leq \frac{1}{2-2\alpha}.$$

This implies that the inequality (6.3) holds true.

It follows from (6.1) and (6.2) that

$$\begin{aligned}\frac{1}{2^n}\left\|f(2^n x)+f(2^n y)+f(2^n z)\right\| &\leq \frac{1}{2^n}\left\|2f\left(\frac{2^n(x+y+z)}{2}\right)\right\| + \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) \\ &\leq \frac{1}{2^n}\left\|2f\left(\frac{2^n(x+y+z)}{2}\right)\right\| + \frac{2^n\alpha^n}{2^n}\varphi(x,y,z) \qquad (6.6)\end{aligned}$$

for all $x,y,z \in X$. Letting $n \to \infty$ in (6.6), we get

$$\left\|A(x)+A(y)+A(z)\right\| \leq \left\|2A\left(\frac{x+y+z}{2}\right)\right\|$$

for all $x,y,z \in X$. By Proposition 6.1, $A : X \to Y$ is Cauchy additive, as desired. □

Corollary 6.3 [47, Theorem 2.2] *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\left\|f(x)+f(y)+f(z)\right\| \leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\| + P(x)^r + P(y)^r + P(z)^r$$

for all $x,y,z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\left\|f(x)-A(x)\right\| \leq \frac{2+2^r}{2-2^r} P(x)^r \qquad (6.7)$$

for all $x \in X$.

Proof Taking $\varphi(x,y,z) = P(x)^r + P(y)^r + P(z)^r$ for all $x,y,z \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 6.2, we get the desired result. □

Theorem 6.4 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (6.2). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that*

$$\left\|f(x)-A(x)\right\| \leq \frac{1}{2}\Phi(x,x,-2x)$$

for all $x \in X$.

Proof The proof is similar to the proof of [47, Theorem 2.2]. □

Remark 6.5 Let $r < 1$. Letting $\varphi(x,y,z) = P(x)^r + P(y)^r + P(z)^r$ for all $x, y \in X$ in Theorem 6.4, we obtain the inequality (6.7). The proof is given in [47, Theorem 2.2].

7 Stability of a functional inequality associated with a three-variable Cauchy additive functional equation

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type three-variable Cauchy additive functional equation in paranormed spaces.

Proposition 7.1 [36, Proposition 2.2] *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

Theorem 7.2 *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \le 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\| + \varphi(x, y, z) \tag{7.1}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2 - 2\alpha}\varphi(x, x, -2x)$$

for all $x \in X$.

Proof Letting $y = x$ and $z = -2x$ in (7.1), we get

$$\|2f(x) - f(2x)\| = \|2f(x) + f(-2x)\| \le \varphi(x, x, -2x),$$

and so

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x, x, -2x) \tag{7.2}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 6.2. □

Corollary 7.3 [47, Theorem 3.2] *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\| + P(x)^r + P(y)^r + P(z)^r$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2 + 2^r}{2 - 2^r}P(x)^r \tag{7.3}$$

for all $x \in X$.

Proof Taking $\varphi(x,y,z) = P(x)^r + P(y)^r + P(z)^r$ for all $x,y,z \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 7.2, we get the desired result. □

Theorem 7.4 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y, 2^j z\right) < \infty$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (7.1). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that*

$$\left\|f(x) - A(x)\right\| \le \frac{1}{2}\Phi(x,x,-2x)$$

for all $x \in X$.

Proof The proof is similar to the proof of [47, Theorem 3.2]. □

Remark 7.5 Let $r < 1$. Letting $\varphi(x,y,z) = P(x)^r + P(y)^r + P(z)^r$ for all $x,y \in X$ in Theorem 7.4, we obtain the inequality (7.3). The proof is given in [47, Theorem 3.2].

8 Stability of a functional inequality associated with the Cauchy-Jensen functional equation

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type Cauchy-Jensen additive functional equation in paranormed spaces.

Proposition 8.1 [36, Proposition 2.3] *Let $f : X \to Y$ be a mapping such that*

$$\left\|f(x) + f(y) + 2f(z)\right\| \le \left\|2f\left(\frac{x+y}{2} + z\right)\right\|$$

for all $x,y,z \in X$. Then f is Cauchy additive.

Theorem 8.2 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y,z) \le 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x,y,z \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\left\|f(x) + f(y) + 2f(z)\right\| \le \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + \varphi(x,y,z) \tag{8.1}$$

for all $x,y,z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\left\|f(x) - A(x)\right\| \le \frac{1}{2-2\alpha}\varphi(2x,0,-x)$$

for all $x \in X$.

Proof Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (8.1), we get

$$\|2f(x) - f(2x)\| = \|2f(x) + f(-2x)\| \leq \varphi(2x, 0, -x)$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 6.2. □

Corollary 8.3 [47, Theorem 4.2] *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + P(x)^r + P(y)^r + P(z)^r$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1 + 2^r}{2 - 2^r} P(x)^r \tag{8.2}$$

for all $x \in X$.

Proof Taking $\varphi(x, y, z) = P(x)^r + P(y)^r + P(z)^r$ for all $x, y, z \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 8.2, we get the desired result. □

Theorem 8.4 *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, 2^j z\right) < \infty$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (8.1). *Then there exists a unique Cauchy additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2} \Phi(2x, 0, -x) \tag{8.3}$$

for all $x \in X$.

Proof The proof is similar to the proof of [47, Theorem 4.2]. □

Remark 8.5 Let $r < 1$. Letting $\varphi(x, y, z) = P(x)^r + P(y)^r + P(z)^r$ for all $x, y \in X$ in Theorem 8.4, we obtain the inequality (8.3). The proof is given in [47, Theorem 4.2].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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