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Hyers-Ulam stability of a generalized additive set-valued functional equation

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Abstract

In this paper, we define a generalized additive set-valued functional equation, which is related to the following generalized additive functional equation:

$$f(x_1 + \cdots + x_l) = (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) + f(x_l)$$

for a fixed integer l with $l > 1$, and prove the Hyers-Ulam stability of the generalized additive set-valued functional equation.

MSC: Primary 39B52; 54C60; 91B44

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1 Introduction and preliminaries

The theory of set-valued functions has been much related to the control theory and the mathematical economics. After the pioneering papers written by Aumann [1] and Debreu [2], set-valued functions in Banach spaces have been developed in the last decades. We can refer to the papers by Arrow and Debreu [3], McKenzie [4], the monographs by Hindenbrand [5], Aubin and Frankowska [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9].

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [11] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [12] for additive mappings and by Th.M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference with a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [15–17]).

Let Y be a Banach space. We define the following:

2^Y: the set of all subsets of Y;

$C_b(Y)$: the set of all closed bounded subsets of Y;

$C_c(Y)$: the set of all closed convex subsets of Y;

$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$d(x, C) = \inf\{\|x - y\| : y \in C\}, \quad x \in Y,$$
$$s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}, \quad x^* \in Y^*.$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.1 *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*:

(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;

(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [7]). Debreu [2] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.2 [2] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties*:

(a) $j(A \oplus B) = j(A) + j(B)$;

(b) $j(\lambda A) = \lambda j(A)$;

(c) $h(A, B) = \|j(A) - j(B)\|_u$;

(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$

for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [18]). In this case, the Debreu integral of f in Ω is the unique element $(D)\int_\Omega f\, d\nu \in C_{cb}(Y)$ such that $j((D)\int_\Omega f\, d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [19–27]).

In this paper, we define a generalized additive set-valued functional equation and prove the Hyers-Ulam stability of the generalized additive set-valued functional equation.

Throughout this paper, let X be a real vector space and Y be a Banach space.

2 Stability of a generalized additive set-valued functional equation

Definition 2.1 Let $f : X \to C_{cb}(Y)$. The generalized additive set-valued functional equation is defined by

$$f(x_1 + \cdots + x_l) = (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l) \tag{1}$$

for all $x_1, \ldots, x_l \in X$. Every solution of the generalized additive set-valued functional equation is called a *generalized additive set-valued mapping.*

Note that there are some examples in [28].

Theorem 2.2 *Let $\varphi : X^l \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x_1, \ldots, x_l) := \sum_{j=0}^{\infty} \frac{1}{l^j}\varphi\left(l^j x_1, \ldots, l^j x_l\right) < \infty \tag{2}$$

for all $x_1, \ldots, x_l \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h\left(f(x_1 + \cdots + x_l), (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l)\right) \le \varphi(x_1, \ldots, x_l) \tag{3}$$

for all $x_1, \ldots, x_l \in X$. Then there exists a unique generalized additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h\big(f(x), A(x)\big) \le \frac{1}{l}\widetilde{\varphi}(x, \ldots, x) \tag{4}$$

for all $x \in X$.

Proof Let $x_1 = \cdots = x_l = x$ in (3). Since $f(x)$ is convex, we get

$$h\big(f(lx), lf(x)\big) \le \varphi(x, \ldots, x), \tag{5}$$

and if we replace x by $l^n x$, $n \in \mathbb{N}$ in (5), then we obtain

$$h\big(f\left(l^{n+1}x\right), lf\left(l^n x\right)\big) \le \varphi\left(l^n x, \ldots, l^n x\right)$$

and

$$h\left(\frac{f(l^{n+1}x)}{l^{n+1}}, \frac{f(l^n x)}{l^n}\right) \le \frac{1}{l^{n+1}}\varphi\left(l^n x, \ldots, l^n x\right).$$

So,

$$h\left(\frac{f(l^n x)}{l^n}, \frac{f(l^m x)}{l^m}\right) \le \frac{1}{l}\sum_{j=m}^{n-1} \frac{1}{l^j}\varphi\left(l^j x, \ldots, l^j x\right) \tag{6}$$

for all integers n, m with $n \ge m$. It follows from (2) and (6) that $\{\frac{f(l^n x)}{l^n}\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$.

Let $A(x) = \lim_{n\to\infty} \frac{f(l^n x)}{l^n}$ for each $x \in X$. Then we claim that A is a generalized additive set-valued mapping. Note that

$$h\left(\frac{f(l^n(x_1 + \cdots + x_l))}{l^n}, (l-1)f\left(\frac{l^n(x_1 + \cdots + x_{l-1})}{l^n(l-1)}\right) \oplus \frac{f(l^n x_l)}{l^n}\right) \le \frac{1}{l^n}\varphi\left(l^n x_1, \ldots, l^n x_l\right).$$

Since $h(A \oplus B, C \oplus D) \le h(A, C) + h(B, D)$, we have

$$\begin{aligned}
&h\left(A(x_1 + \cdots + x_l), (l-1)A\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus A(x_l)\right) \\
&\quad \le h\left(A(x_1 + \cdots + x_l), \frac{f(l^n(x_1 + \cdots + x_l))}{l^n}\right) \\
&\qquad + h\left(\frac{f(l^n(x_1 + \cdots + x_l))}{l^n}, (l-1)f\left(\frac{l^n(x_1 + \cdots + x_{l-1})}{l^n(l-1)}\right) \oplus \frac{f(l^n x_l)}{l^n}\right) \\
&\qquad + h\left((l-1)f\left(\frac{l^n(x_1 + \cdots + x_{l-1})}{l^n(l-1)}\right) \oplus \frac{f(l^n x_l)}{l^n}, (l-1)A\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus A(x_l)\right),
\end{aligned}$$

which tends to zero as $n \to \infty$. So, A is a generalized additive set-valued mapping. Letting $m = 0$ and passing the limit $m \to \infty$ in (6), we get the inequality (4).

Now, let $T : X \to (C_{cb}(Y), h)$ be another generalized additive set-valued mapping satisfying (1) and (4). So,

$$\begin{aligned}
h\left(A(x), T(x)\right) &= \frac{1}{l^n} h\left(A\left(l^n x\right), T\left(l^n x\right)\right) \\
&\le \frac{1}{l^n} h\left(A\left(l^n x\right), f\left(l^n x\right)\right) + \frac{1}{l^n} h\left(T\left(l^n x\right), f\left(l^n x\right)\right) \\
&\le \frac{2}{l^{n+1}} \widetilde{\varphi}\left(l^n x, \ldots, l^n x\right),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $A(x) = T(x)$ for all $x \in X$, which proves the uniqueness of A, as desired. □

Corollary 2.3 *Let $1 > p > 0$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h\left(f(x_1 + \cdots + x_l), (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l)\right) \le \theta \sum_{j=1}^{l} \|x_j\|^p \tag{7}$$

for all $x_1, \ldots, x_l \in X$. Then there exists a unique generalized additive set-valued mapping $A : X \to Y$ satisfying

$$h\left(f(x), A(x)\right) \le \frac{l\theta}{l - l^p} \|x\|^p$$

for all $x \in X$.

Proof The proof follows from Theorem 2.2 by taking

$$\varphi(x_1,\dots,x_l) := \theta\sum_{j=1}^{l}\|x_j\|^p$$

for all $x_1,\dots,x_l \in X$. □

Theorem 2.4 *Let $\varphi : X^l \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x_1,\dots,x_l) := \sum_{j=1}^{\infty} l^j\varphi\left(\frac{x_1}{l^j},\dots,\frac{x_l}{2^j}\right) < \infty$$

for all $x_1,\dots,x_l \in X$. Suppose that $f : X \to (C_{cb}(Y),h)$ is a mapping satisfying (3). *Then there exists a unique generalized additive set-valued mapping $A : X \to (C_{cb}(Y),h)$ such that*

$$h\bigl(f(x),A(x)\bigr) \le \frac{1}{l}\widetilde{\varphi}(x,\dots,x)$$

for all $x \in X$.

Proof It follows from (5) that

$$h\left(f(x),lf\left(\frac{x}{l}\right)\right) \le \varphi\left(\frac{x}{l},\dots,\frac{x}{l}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5 *Let $p > 1$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y),h)$ is a mapping satisfying* (7). *Then there exists a unique generalized additive set-valued mapping $A : X \to Y$ satisfying*

$$h\bigl(f(x),A(x)\bigr) \le \frac{l\theta}{l^p - l}\|x\|^p$$

for all $x \in X$.

Proof The proof follows from Theorem 2.4 by taking

$$\varphi(x_1,\dots,x_l) := \theta\sum_{j=1}^{l}\|x_j\|^p$$

for all $x_1,\dots,x_l \in X$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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