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Covariant representations on Krein C^*-modules associated to pairs of two maps

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ABSTRACT

In this paper we construct a KSGNS type covariant representation on a Krein C^*-module for a covariant α-completely positive map, and the result is applied to construct a KSGNS type covariant representation associated with a pair of two maps (ρ, Φ) where ρ is a covariant α-completely positive map on a C^*-algebra and Φ is a covariant ρ-map on a Krein C^*-module. The KSGNS type covariant representation for a pair (ρ, Φ) is applied to give a new covariant J-representation of a crossed product of a C^*-algebra and a new covariant map of a crossed product of a Hilbert C^*-module by a discrete group.

1. Introduction

Stinespring's dilation theorem for a completely positive map on a C^*-algebra is one of the fundamental and important results for the study of operator algebras and mathematical physics. In particular, Stinespring's theorem is the basic structure theorem for quantum channels: it states that any quantum channel arises from a unitary evolution on a larger system. By constructing Hilbert C^*-modules, Kasparov [1] gave the Stinespring type representation for a completely positive map between two C^*-algebras, which is called a Kasparov–Stinespring–Gelfand–Naimark–Segal (KSGNS) representation [2]. This construction generalized the Stinespring's construction as well as the classical GNS construction. Recently, Asadi [3] gave a Stinespring type representation for a pair of two unital maps on a C^*-algebra and a Hilbert C^*-module, and then Bhat–Ramesh–Sumesh [4] strengthened Asadi's result by removing a technical condition and unicity on the maps under consideration. Moreover, there are covariant versions of the KSGNS representations for various covariant completely positive maps [5].

A generalization of positive linear functionals to Hermitian linear functionals yielding representations on (indefinite) inner product spaces was studied by Scheibe [6]. Since the positivity is lacking in some local quantum field theories, the GNS-construction on an indefinite inner product is of increasing interest in the general (axiomatic) quantum field theory. In particular, in the gauge quantum field theory, locality is in conflict with positivity and then from the axiomatic point of view, it is better to keep the locality condition and to give up the positivity condition which leads to the modification of the axiom of positivity. For more detailed motivation, we refer to [7]. Motivated by the lack of positivity in some models in

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local quantum field theories, Constantinescu and Gheondea studied Kolmogorov decompositions of Hermitian kernels in [8] and the Stinespring representation theorem and its covariant version for Hermitian maps in [9]. Here the Hermitian kernels and Hermitian maps are associated with the involution with respect to indefinite inner product and satisfy the Schwartz's bounded condition which has been studied with more concrete expressions in [10]. With same motivation, Heo–Hong–Ji [11] introduced a notion of the α-complete positivity as a generalization of complete positivity. Here a positivity is inherent in Hermitian maps in terms of the map α. The α-complete positivity provides a positive definite inner product associated to the indefinite one, and the interplay between these two is indeed the characteristic feature of Krein spaces among all indefinite metric spaces.

Krein spaces arise naturally in situations where the indefinite inner product has an analytically useful property (such as Lorentz invariance) which the Hilbert inner product lacks. It is known that in massless quantum field theory the state space may be a space with an indefinite metric. Motivated by this physical fact, many people extended the GNS construction to Krein spaces. More generally, Heo–Hong–Ji [11] provided such a KSGNS type representation on a Krein C^*-module for an α-completely positive map on a C^*-algebra or a *-algebra. Moreover, Heo–Ji [12] constructed a Stinespring type covariant representation for a pair of a covariant completely positive map ρ and a covariant ρ-map. In this paper, motivated by the results in [3,13,11,12,14], we construct a KSGNS type covariant representation for a pair of a covariant α-completely positive map ρ on a C^*-algebra and a covariant ρ-map on a Krein C^*-module, using the KSGNS type representations on Krein C^*-modules associated to α-completely positive maps. Furthermore, we give a new covariant J-representation of a crossed product of a C^*-algebra by a locally compact group and a new covariant map on the crossed product of a Hilbert C^*-module by a locally compact group. We refer to [13,15] for the crossed products of Hilbert C^*-modules.

This paper is organized as follows. In Section 2, we review some basic notions of a Krein C^*-module and an α-completely positive map on a C^*-algebra. We also recall the KSGNS type construction associated to an α-completely positive map on a C^*-algebra $\mathcal{A}$, which leads to a J-representation of a C^*-algebra on a Krein C^*-module. We give a covariant version of the KSGNS type representation on a Krein C^*-module for a covariant α-completely positive map, which is unique up to unitary equivalence. In Section 3, we give some examples of a pair of two maps on a C^*-algebra and a Hilbert C^*-module and concerned with a pair of a covariant α-completely positive map on a C^*-algebra and a covariant map on a Krein C^*-module. We prove a KSGNS type covariant representation theorem for such a pair. Finally, in Section 4, we give a new covariant J-representation of a crossed product of a C^*-algebra and an associated covariant map of a crossed product of a Hilbert C^*-module using the extension of an α-completely positive map to a C^*-crossed product.

2. KSGNS constructions for covariant α-CP maps

Let $\mathcal{B}$ be a C^*-algebra and let X, Y be Hilbert $\mathcal{B}$-modules. An operator $T : X \to Y$ is *adjointable* if there is an adjoint operator $T^* : Y \to X$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle, \quad (x \in X,\ y \in Y).$$

We note that every adjointable map automatically becomes a continuous $\mathcal{B}$-module map, i.e., $T(xa) = T(x)a$ for all $x \in X$ and $a \in \mathcal{B}$. We denote by $\mathcal{L}(X, Y)$ the set of all adjointable operators from X into Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$ which becomes a C^*-algebra with the operator norm. For detailed information on Hilbert C^*-modules, we refer to [2].

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear map φ from $\mathcal{A}$ into $\mathcal{B}$ is said to be *completely positive* if for any $n \in \mathbb{N}$, the linear map $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\varphi_n\left(\left(a_{ij}\right)_{n\times n}\right) = \left(\varphi(a_{ij})\right)_{n\times n}, \quad (a_{ij} \in \mathcal{A},\ i, j = 1, \dots, n)$$

is positive where $(\cdot)_{n\times n}$ is an $n \times n$ operator matrix. We note that if $\rho : \mathcal{A} \to \mathcal{B}$ is a Hermitian map, i.e., $\rho(a^*) = \rho(a)^*$, then ρ_n is also a Hermitian map.

Definition 2.1 (*[11]*)**.** Let $\mathcal{A}$ be a unital C^*-algebra with a unit $\mathbf{1}$ and let X be a Hilbert $\mathcal{B}$-module. A Hermitian map ρ from $\mathcal{A}$ into $\mathcal{L}(X)$ is called *α-completely positive* (*α-CP*) if there is a bounded Hermitian map $\alpha : \mathcal{A} \to \mathcal{A}$ such that

(ρ1) $\alpha^2 = I$ (the identity mapping on $\mathcal{A}$),
(ρ2) $\alpha(\mathbf{1}) = \mathbf{1}$,
(ρ3) $\rho(ab) = \rho\big(\alpha(a)\alpha(b)\big) = \rho\big(\alpha(ab)\big)$ for any $a, b \in \mathcal{A}$,
(ρ4) $\sum_{i,j=1}^{n} \langle x_i, \rho(\alpha(a_i)^* a_j)x_j\rangle \geq 0$ for any $n \geq 1$, $a_1, \dots, a_n \in \mathcal{A}$ and $x_1, \dots, x_n \in X$,
(ρ5) for each $a \in \mathcal{A}$, there exists a constant $C(a) \geq 0$ such that

$$\big(\rho(\alpha(aa_i)^* aa_j)\big)_{n\times n} \leq C(a)\big(\rho(\alpha(a_i)^* a_j)\big)_{n\times n}$$

for any $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}$.

Remark 2.2. In Definition 2.1, if $\mathcal{A}$ is non-unital, then (ρ2) is replaced by

($\rho 2'$) for any approximate unit $\{f_i\}_{i\in I}$ for $\mathcal{A}$, $\{\alpha(f_i)\}_{i\in I}$ is also an approximate unit.

Let J be a (fundamental) symmetry on a Hilbert $\mathcal{B}$-module X, i.e., $J = J^* = J^{-1}$. Then we define a $\mathcal{B}$-valued indefinite inner product by

$$\langle x, y\rangle_J = \langle x, Jy\rangle, \quad (x, y \in X).$$

In this case, the pair (X, J) is called a *Krein $\mathcal{B}$-module*. For each $T \in \mathcal{L}(X)$, there exists an operator $T^J \in \mathcal{L}(X)$ such that

$$\langle T(x), y\rangle_J = \langle x, T^J(y)\rangle_J, \quad (x, y \in X).$$

The operator T^J is called the *J-adjoint* of T and we can see that $T^J = JT^*J$. For more detailed study for indefinite inner product spaces, we refer to [16].

Let (X, J) be a Krein $\mathcal{B}$-module. An algebra homomorphism $\pi : \mathcal{A} \to \mathcal{L}(X)$ is called a *representation* of $\mathcal{A}$ on X. A *$*$-representation* π of $\mathcal{A}$ on X is a representation of $\mathcal{A}$ on X such that $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. A representation π of $\mathcal{A}$ on X is called a *J-representation* of $\mathcal{A}$ on the Krein C^*-module (X, J) if

$$\pi(a^*) = \pi(a)^J \equiv J\pi(a)^*J \quad \text{for all } a \in \mathcal{A}.$$

Furthermore, if $[\pi(\mathcal{A})X] = X$ where $[\pi(\mathcal{A})X]$ is the closed linear span of the set $\{\pi(a)x : a \in \mathcal{A},\ x \in X\}$, then π is said to be *nondegenerate*.

Theorem 2.3 (*[11]*). *Let $\mathcal{A}$ be a unital C^*-algebra with a unit $\mathbf{1}$ and let X be a Hilbert $\mathcal{B}$-module. If $\rho : \mathcal{A} \to \mathcal{L}(X)$ is an α-CP linear map, then there exist a Krein $\mathcal{B}$-module (Y, J), a J-representation $\pi : \mathcal{A} \to \mathcal{L}(Y)$ and an operator V in $\mathcal{L}(X, Y)$ such that*

(i) *$\rho(a) = V^*\pi(a)V$ and so $\rho(a^*) = V^*\pi(a^*)V = V^*\pi(a)^JV$ for any $a \in \mathcal{A}$,*
(ii) *$\pi(\mathcal{A})[V(X)]$ is dense in Y,*
(iii) *$V^*\pi(a)^*\pi(b)V = V^*\pi(\alpha(a^*)b)V$ for any $a, b \in \mathcal{A}$.*

Moreover, if there are a Krein $\mathcal{B}$-module (Y', J'), a J'-representation $\pi' : \mathcal{A} \to \mathcal{L}(Y')$ and an operator $V' \in \mathcal{L}(X, Y')$ satisfying

(i′) *$\rho(a) = V'^*\pi'(a)V'$ for any $a \in \mathcal{A}$,*
(ii′) *$\pi'(\mathcal{A})[V'(X)]$ is dense in Y',*
(iii′) *$V'^*\pi'(a)^*\pi'(b)V' = V'^*\pi'(\alpha(a^*)b)V'$ for any $a, b \in \mathcal{A}$,*

then there is a unitary operator U in $\mathcal{L}(Y, Y')$ such that

$$V' = UV \quad \text{and} \quad \pi'(a) = U\pi(a)U^*, \quad (a \in \mathcal{A}).$$

For the proof of Theorem 2.4, we note that $Y = \mathcal{A} \otimes_\rho X$ is the completion of the quotient space $\mathcal{A} \otimes_{\text{alg}} X/\mathcal{N}_\rho$, where $\otimes_{\text{alg}}$ is the algebraic tensor product and $\mathcal{N}_\rho$ is the kernel space defined by

$$\mathcal{N}_\rho = \left\{ \sum_i a_i \otimes x_i \in \mathcal{A} \otimes_{\text{alg}} X : \sum_{i,j} \langle x_i, \rho(\alpha(a_i^*)a_j)x_j\rangle = 0 \right\}.$$

The symmetry J is explicitly given by $J = \alpha \otimes I$, and the operator V is given by $V(x) = \mathbf{1} \otimes x$ $(x \in X)$. The KSGNS type representation $((Y, J), \pi, V)$ satisfying (ii) in Theorem 2.3 is said to be *minimal*.

In the remainder of this section, we construct a covariant representation associated to a covariant α-CP map.

Let θ be an action of a locally compact group G on a C^*-algebra $\mathcal{A}$. Here, an action means a group homomorphism $\theta : G \to \text{Aut}(\mathcal{A})$ such that for each $a \in \mathcal{A}$, the map $G \ni s \mapsto \theta_s(a) \in \mathcal{A}$ is continuous with respect to the norm topology on $\mathcal{A}$. Let (Y, J) be a Krein $\mathcal{B}$-module and let v be a J-unitary representation of G into the J-unitary group $\mathcal{U}_J(Y)$ which is the set of all J-unitary operators in $\mathcal{L}(Y)$, i.e., for each $s \in G$, $v_s^Jv_s = v_sv_s^J = I$, which is equivalent to

$$v_s^* = Jv_{s^{-1}}J \quad \text{or} \quad v_s^J = v_{s^{-1}}. \tag{1}$$

A linear map $\rho : \mathcal{A} \to \mathcal{L}(Y)$ is said to be *(θ, v)-covariant* if

$$\rho(\theta_s(a)) = v_s\rho(a)v_s^J, \quad (s \in G,\ a \in \mathcal{A}). \tag{2}$$

A *covariant J-representation* of a C^*-dynamical system $(\mathcal{A}, G, \theta)$ on a Krein $\mathcal{B}$-module (Y, J) is a triple $(\pi, v, (Y, J))$, where π is a J-representation of $\mathcal{A}$ on (Y, J) and v is a J-unitary representation of G into $\mathcal{U}_J(Y)$ such that the (θ, v)-covariance property holds: for any $s \in G$ and $a \in \mathcal{A}$,

$$\pi(\theta_s(a)) = v_s\pi(a)v_s^J.$$

Theorem 2.4. *Let $(\mathcal{A}, G, \theta)$ be a unital C^*-dynamical system and let $u : G \to \mathcal{U}(X)$ be a unitary representation on a Hilbert $\mathcal{B}$-module X. If $\rho : \mathcal{A} \to \mathcal{L}(X)$ is a unital (θ, u)-covariant α-CP map, then there exist a covariant J-representation $(\pi, v, (Y, J))$ of $(\mathcal{A}, G, \theta)$ and an isometry $V \in \mathcal{L}(X, Y)$ such that*

(i) *$\rho(a) = V^*\pi(a)V$ for any $a \in \mathcal{A}$,*
(ii) *$\pi(\theta_s(a)) = v_s\pi(a)v_s^J$ for any $s \in G$ and $a \in \mathcal{A}$,*
(iii) *$Vu_s = v_sV$ for any $s \in G$.*

Proof. By Theorem 2.3, there exist a Krein $\mathcal{B}$-module (Y, J), a J-representation $\pi : \mathcal{A} \to \mathcal{L}(Y)$ and an operator V in $\mathcal{L}(X, Y)$ such that $\rho(a) = V^*\pi(a)V$ for any $a \in \mathcal{A}$. Hence, it is enough to construct a J-unitary representation υ of G on Y satisfying (ii) and (iii). We define $\upsilon : G \to \mathcal{L}(Y)$ by $\upsilon_s = \theta_s \otimes u_s$ on $Y = \mathcal{A} \otimes_\rho X$. For any $a_i, a_j' \in \mathcal{A}$ and $x_i, x_j' \in X$ $(i = 1, \ldots, n, j = 1, \ldots, m)$, we obtain that

$$\begin{aligned}\left\langle \upsilon_s\left(\sum_{i=1}^n a_i \otimes x_i\right), \sum_{j=1}^m a_j' \otimes x_j'\right\rangle &= \sum_{i=1}^n \sum_{j=1}^m \langle \theta_s(a_i) \otimes u_s(x_i), a_j' \otimes x_j'\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \left\langle x_i, \rho\Big(\alpha(a_i^*)\alpha\big(\theta_{s^{-1}}(\alpha(a_j'))\big)\Big)u_s^*(x_j')\right\rangle \\ &= \left\langle \sum_{i=1}^n a_i \otimes x_i, ((\alpha \circ \theta_{s^{-1}} \circ \alpha) \otimes u_s^*)\left(\sum_{j=1}^m a_j' \otimes x_j'\right)\right\rangle,\end{aligned}$$

where the third equality follows from the (θ, u)-covariance property of ρ. Hence we have that $\upsilon_s^* = (\alpha \circ \theta_{s^{-1}} \circ \alpha) \otimes u_s^*$ $(s \in G)$, so that

$$\upsilon_s^J = \theta_{s^{-1}} \otimes u_s^*.$$

This shows that υ is a J-unitary representation of G on Y.

Since $V(y) = 1 \otimes y$ and ρ is unital, we can see that $Vu_s = \upsilon_s V$ for all $s \in G$. Moreover, we obtain that $\pi(\theta_s(a)) = \upsilon_s \pi(a)\upsilon_s^J$ for all $s \in G$ and $a \in \mathcal{A}$. Indeed, for any $a_i \in \mathcal{A}$ and $x_i \in X$ $(i = 1, \ldots, n)$, let $y = \sum_{i=1}^n a_i \otimes x_i$. We have that

$$\pi\big(\theta_s(a)\big)(y) = \sum_{i=1}^n \theta_s\big(a\theta_{s^{-1}}(a_i)\big) \otimes u_s u_{s^{-1}}(x_i) = \upsilon_s \pi(a)\upsilon_s^J(y),$$

which completes the proof. □

3. KSGNS type representations for a pair of covariant maps

In this section, we construct a covariant representation on a Krein C^*-module associated to a pair of two covariant maps on Krein C^*-modules, which may be regarded as a generalization of Theorem 3.2 in [12].

Let Y and Z be Hilbert $\mathcal{B}$-modules. Then $\mathcal{L}(Y, Z)$ can be regarded as a Hilbert $\mathcal{L}(Y)$-module with the following operations:

(i) (*module map*) $\mathcal{L}(Y, Z) \times \mathcal{L}(Y) \ni (T, S) \mapsto TS \in \mathcal{L}(Y, Z)$,
(ii) (*inner product*) $\mathcal{L}(Y, Z) \times \mathcal{L}(Y, Z) \ni (T_1, T_2) \mapsto \langle T_1, T_2\rangle = T_1^*T_2 \in \mathcal{L}(Y)$.

Let X be a Hilbert $\mathcal{A}$-module and let ρ be a linear map from $\mathcal{A}$ into $\mathcal{L}(Y)$. A linear map $\Phi : X \to \mathcal{L}(Y, Z)$ is said to be a *ρ-map* if

$$\langle \Phi(x), \Phi(y)\rangle = \rho(\langle x, y\rangle), \quad (x, y \in X).$$

If (Y, J_Y) is a Krein $\mathcal{B}$-module and ρ is a nondegenerate J_Y-representation, then a ρ-map $\Phi : X \to \mathcal{L}(Y, Z)$ is automatically linear and satisfies the relation

$$\Phi(xa) = \Phi(x)J_Y\rho(a)J_Y \quad \text{for any } x \in X \text{ and } a \in \mathcal{A}.$$

Example 3.1. Let X be a Hilbert $\mathcal{A}$-module.

(1) Consider the right Hilbert $\mathcal{A}$-module $X \oplus \mathcal{A}$ consisting of columns (x, a) and equipped with an inner product $\langle (x_1, a_1), (x_2, a_2)\rangle = \langle x_1, x_2\rangle + a_1^*a_2$. We identify each $x \in X$ with the corresponding adjointable operator ϕ_x from $\mathcal{A}$ to X defined by $a \mapsto xa$ of which the adjoint operator is given by $\phi_x^*(y) = \langle x, y\rangle$, $(y \in X)$. The C^*-algebra $\mathcal{K}(X \oplus \mathcal{A})$ consisting of compact operators is called the *linking algebra* of X. It is known that $\mathcal{K}(\mathcal{A}, X) = X$ and $\mathcal{K}(X, \mathcal{A}) = X^*$. If $\pi : \mathcal{K}(X \oplus \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ is a representation on a Hilbert space $\mathcal{H}$, then by restriction π defines two maps $\phi = \pi|_{\mathcal{A}}$ and $\Phi = \pi|_X$, which together constitute a representation of X. It is proved in [17] that every Hilbert C^*-module has such a representation.
(2) Let I be an ideal of $\mathcal{A}$ and let X_I be the closed linear span of the products xa for $x \in X$ and $a \in I$. The inner product on X modulo I gives an $\mathcal{A}/I$-valued inner product on the quotient space X/X_I and the quotient map $\Phi : X \to X/X_I$ is a φ-morphism, where $\varphi : \mathcal{A} \to \mathcal{A}/I = \mathcal{B}$ is the quotient map. For the exact sequence:

$$I \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/I$$

we have the associated sequence of Hilbert C^*-modules $X_I \xrightarrow{\jmath} X \xrightarrow{\Pi} X/X_I$ where $\jmath$ is the inclusion map and Π is the canonical quotient map. It is not difficult to see that $\jmath$ is an ι-representation and Π is a π-representation. Bakić and Guljaš [18] gave a correspondence between the class of representations of Hilbert C^*-modules and the class of morphisms of the corresponding linking algebras.

For a covariant representation theorem for an α-CP map, we introduce a compatible action and the covariance property of a map on a Krein C^*-module. Let $(\mathcal{A}, G, \theta)$ be a C^*-dynamical system and let (X, J_X) be a Krein $\mathcal{A}$-module. A group homomorphism τ from G into the J_X-unitary group $\mathcal{U}_{J_X}(X)$ such that for any $s \in G$, $a \in \mathcal{A}$ and $x, x' \in X$,

(i) $\tau_s(xa) = \tau_s(x)\theta_s(a)$,
(ii) $\langle \tau_s(x), \tau_s(x')\rangle_{J_X} = \theta_s(\langle x, x'\rangle_{J_X})$,

is called a *θ-compatible action* of G on (X, J_X). Let (Y, J_Y) and (Z, J_Z) be Krein $\mathcal{B}$-modules. For a θ-compatible action τ of G on (X, J_X) and a map $\Phi : X \to \mathcal{L}(Y, Z)$, if there are a J_Y-unitary representation $\upsilon : G \to \mathcal{U}_{J_Y}(Y)$ and a J_Z-unitary representation $\sigma : G \to \mathcal{U}_{J_Z}(Z)$ such that

$$\Phi(\tau_s(x)) = \sigma_s \Phi(x) \upsilon_s^{J_Y} \quad \text{for any } x \in X \text{ and } s \in G,$$

then Φ is said to be *(τ, σ, υ)-covariant*.

In the following theorem, we construct a covariant representation associated to a pair of two covariant maps under a technical assumption which can be replaced by the existence of some element in X (see Remark 3.3).

Theorem 3.2. *Let X be a Hilbert $\mathcal{A}$-module and let Y, Z be Hilbert $\mathcal{B}$-modules. If $\rho : \mathcal{A} \to \mathcal{L}(Y)$ is a unital (θ, u)-covariant α-CP linear map and if $\Phi : X \to \mathcal{L}(Y, Z)$ is a (τ, σ, u)-covariant ρ-map such that*

(P) *the closure $[\Phi(X)Y]$ is orthogonal complemented in Z,*

then there exists a pair $\big((\pi, V, (E, J)), (\Pi, W, F)\big)$ such that

(i) *(E, J) is a Krein $\mathcal{B}$-module and F is a Hilbert $\mathcal{B}$-module,*
(ii) *$\pi : \mathcal{A} \to \mathcal{L}(E)$ is a J-representation,*
(iii) *$\Pi : X \to \mathcal{L}(E, F)$ is a $J \circ \pi$-map,*
(iv) *$V \in \mathcal{L}(Y, E)$ is an isometry and $W \in \mathcal{L}(Z, F)$ is a projection*

satisfying the conditions (i)–(iii) in Theorem 2.3 and $\Phi(x) = W^*\Pi(x)V$ for all $x \in X$. Moreover, there exist a J-unitary representation υ and a map $\sigma' : G \to \mathcal{U}(F)$ such that

(1) *$(\pi, \upsilon, (E, J))$ is a covariant J-representation of $(\mathcal{A}, G, \theta)$,*
(2) *Π is $(\tau, \sigma', \upsilon)$-covariant.*

Remark 3.3. In Theorem 3.2, if $\mathcal{B} = \mathbb{C}$ and Y, Z are Hilbert spaces, then the hypothesis (P) is redundant (see [4]). Moreover, we see that the hypothesis (P) in Theorem 3.2 can be replaced by the existence of $x_0 \in X$ with $\Phi(x_0)\Phi(x_0)^* = 1_{\mathcal{L}(Z)}$ as in [3]. Indeed, we consider the algebraic tensor product $X \otimes Z$ equipped with

$$\left\langle \sum_i x_i \otimes z_i, \sum_j x'_j \otimes z'_j \right\rangle = \sum_{i,j} \langle z_i, \Phi(x'_j)\Phi(x_i)^* z'_j\rangle.$$

The Hilbert $\mathcal{B}$-module F is obtained by the completion of the quotient space $X \otimes Z/\mathcal{N}$, where $\mathcal{N}$ is the kernel space of $\langle \cdot, \cdot \rangle$. The map $\Pi : X \to \mathcal{L}(E, F)$ defined by

$$\Pi(x)\left(\sum_i \pi(a_i)Ve_i + \mathcal{N}\right) = \sum_i x_0 \otimes \Phi(xa_i) + \mathcal{N}$$

becomes a π-map and the map $W : Z \to F$ given by $W(z) = x_0 \otimes z + N$ satisfies the relation $W^*\Pi(x)V = \Phi(x)$ $(x \in X)$. However, this is not a minimal representation in the following sense.

Proof. By Theorem 2.3, there exist a Krein $\mathcal{B}$-module $(E = \mathcal{A} \otimes_\rho Y, J)$, a J-representation $\pi : \mathcal{A} \to \mathcal{L}(E)$ and an operator $V \in \mathcal{L}(Y, E)$ such that the conditions (i)–(iii) in Theorem 2.3 hold. Let F be the closed linear span $[\Phi(X)Y]$ of the set $\Phi(X)Y = \left\{\sum_i \Phi(x_i)y_i : x_i \in X, y_i \in Y\right\}$ in Z. It is clear that the set F becomes a Hilbert $\mathcal{B}$-module. For each $x \in X$, we define a map $\Pi(x) : \pi(\mathcal{A})V(Y) \to F$ by

$$\Pi(x)\left(\sum_{i=1}^n \pi(a_i)Vy_i\right) = \sum_{i=1}^n \Phi(xa_i)y_i.$$

Let $a_i \in \mathcal{A}$ and $y_i \in Y$ for $i = 1, \ldots, n$. Since Φ is a ρ-map, we have that

$$\left\| \Pi(x)\left(\sum_{i=1}^n \pi(a_i)Vy_i\right)\right\|^2 = \left\| \sum_{i,j=1}^n \langle y_i, \rho(a_i^* \langle x, x\rangle a_j) y_j\rangle \right\|.$$

It follows from $(\rho 3)$ in Definition 2.1 and (i), (iii) in Theorem 2.3 that

$$\begin{aligned}\left\| \Pi(x)\left(\sum_{i=1}^{n}\pi(a_i)Vy_i\right)\right\|^2 &= \left\|\sum_{i,j=1}^{n}\langle y_i, V^*\pi(\alpha(a_i)^*\alpha(\langle x,x\rangle a_j))Vy_j\rangle\right\| \\ &\leq \|J\pi(\langle x,x\rangle)\|\left\|\sum_{i=1}^{n}\pi(a_i)Vy_i\right\|^2.\end{aligned}$$

Thus, $\Pi(x)$ is bounded on $\pi(\mathcal{A})V(Y)$ which is dense in E. Hence for each $x\in X$, $\Pi(x)$ can be extended to E and the extension is still denoted by the same symbol.

By similar arguments, we also have that

$$\left\langle \Pi(x)\left(\sum_{i=1}^{n}\pi(a_i)Vy_i\right), \sum_{j=1}^{m}\Phi(x_j)y'_j\right\rangle = \left\langle \sum_{i=1}^{n}\pi(a_i)Vy_i, \sum_{j=1}^{m}J\pi\left(\langle x,x_j\rangle\right)Vy'_j\right\rangle,$$

which implies that

$$\Pi(x)^*\left(\sum_{j=1}^{m}\Phi(x_j)y'_j\right) = \sum_{j=1}^{m}J\pi\left(\langle x,x_j\rangle\right)Vy'_j.$$

Since the set $\Phi(X)Y$ is dense in the Hilbert $\mathcal{B}$-module F, the operator $\Pi(x)$ is adjointable. Hence, Π is a map from X into $\mathcal{L}(E,F)$. Let $a_i,\ a'_j\in\mathcal{A}$ and $y_i,\ y'_j\in Y$ $(i=1,\ldots,n, j=1,\ldots,m)$. It follows from (i) and (iii) in Theorem 2.3 that

$$\left\langle \Pi(x)^*\Pi(x')\left(\sum_{i=1}^{n}\pi(a_i)Vy_i\right), \sum_{j=1}^{m}\pi(a'_j)Vy'_j\right\rangle = \left\langle J\circ\pi(\langle x,x'\rangle)\sum_{i=1}^{n}\pi(a_i)Vy_i, \sum_{j=1}^{m}\pi(a'_j)Vy'_j\right\rangle.$$

This equality implies that

$$\Pi(x)^*\Pi(x') = (J\circ\pi)(\langle x,x'\rangle) \tag{3}$$

on the set $\pi(\mathcal{A})V(Y)$ for any $x, x'\in X$, so that the map Π is a $J\circ\pi$-map.

On the other hand, by the hypothesis (P), the space $F=[\Phi(X)Y]$ is orthogonal complemented in Z, so that there exists an orthogonal projection W from Z onto F as an adjointable map. In fact, the adjoint operator $W^*: F\hookrightarrow Z$ is the inclusion map. For any $x\in X$ and $y\in Y$, we have that

$$W^*(\Pi(x)Vy) = W^*(\Pi(x)(\pi(1)Vy)) = W^*(\Phi(x)y) = \Phi(x)y. \tag{4}$$

From Theorem 2.4, there exists a covariant J-representation $(\pi, \upsilon, (E,J))$ of $(\mathcal{A}, G, \theta)$. For any $x\in X$ and $s\in G$, we obtain that

$$\begin{aligned}\Pi(\tau_s(x))\left(\sum_{i=1}^{n}\pi(a_i)Vy_i\right) &= \sum_{i=1}^{n}\Phi\left(\tau_s(x\theta_{s^{-1}}(a_i))\right)y_i \\ &= W\sigma_s W^*\Pi(x)\upsilon_s^J\left(\sum_{i=1}^{n}\pi(a_i)Vy_i\right).\end{aligned}$$

Put $\sigma'_s = W\sigma_s W^*$ for each $s\in G$. Since Φ is (τ,σ,u)-covariant, the set $\Phi(X)Y$ is a invariant subspace under the set $\{\sigma_s : s\in G\}$. Thus, we have that σ'_s is the restriction of σ_s to $[\Phi(X)Y]$ and that σ' is a unitary representation on F. Since F is the closed linear span of the set $\Phi(X)Y$, we obtain that

$$\Pi(\tau_s(x)) = \sigma'_s\Pi(x)\upsilon_s^J \quad \text{for any } x\in X \text{ and } s\in G.$$

This implies that Π is (τ,σ',υ)-covariant. □

Remark 3.4. By the definition of Π and its continuity, it is easy to see that

$$\Pi(xa) = \Pi(x)\pi(a) \quad (x\in X,\ a\in\mathcal{A}). \tag{5}$$

A pair $\left((\pi,V,(E,J)),(\Pi,W,F)\right)$ satisfying the conditions (i) and (iii) in Theorem 2.3 and $\Phi(x)=W^*\Pi(x)V$ $(x\in X)$ is called a *KSGNS type representation* for a pair (ρ,Φ). Such a representation is said to be *minimal* if

$$E=[\pi(\mathcal{A})V(Y)] \quad \text{and} \quad F=[\Phi(X)(Y)].$$

Hence the pair $\left((\pi,V,(E,J)),(\Pi,W,F)\right)$ constructed in Theorem 3.2 is minimal.

The following theorem says that the representations constructed in Theorem 3.2 are unique up to unitary equivalence. The proof is routine, but we give a proof for the reader's convenience.

Theorem 3.5. *Let $\big((\pi_1, V_1, (E_1, J_1)), (\Pi_1, W_1, F_1)\big)$ be another minimal KSGNS representation for (ρ, Φ) which is given as in Theorem 3.2. Then there exist two unitary operators $U : E \to E_1$ and $U_1 : F \to F_1$ such that*

(1) $UV = V_1, U\pi(a) = \pi_1(a)U$ *for any* $a \in \mathcal{A}$,
(2) $W_1^* U_1 = W^*, U_1\Pi(x) = \Pi_1(x)U$ *for any* $x \in X$.

Proof. From Theorem 2.3, there is a unitary operator U in $\mathcal{L}(E, E_1)$ such that $\pi_1(a) = U\pi(a)U^*$ and $V_1 = UV$ where π and $V \in \mathcal{L}(Y, E)$ are given as in Theorem 2.3. In fact, we have that

$$U\left(\sum_{i=1}^{n} \pi(a_i)Vy_i\right) = \sum_{i=1}^{n} \pi_1(a_i)V_1y_i.$$

We similarly define $U_1 : F \to F_1$ by $U_1(\sum_{i=1}^{n} \Pi(x_i)Vy_i) = \sum_{i=1}^{n} \Pi_1(x_i)V_1y_i$ for any $x_i \in X$ and $y_i \in Y, i = 1, \dots, n$ for $n \geq 1$. Then we obtain that

$$\left\|\sum_{i=1}^{n} \Pi_1(x_i)V_1y_i\right\|^2 = \left\|\sum_{i,j=1}^{n} \langle y_i, V_1^*\pi_1\big(\alpha(\langle x_i, x_j\rangle)\big)V_1y_j\rangle\right\| = \left\|\sum_{i=1}^{n} \Pi(x_i)Vy_i\right\|^2.$$

Hence, U_1 is an isometry and it can be extended to the whole space F as a unitary. From the minimality of two representations for (ρ, Φ), we have that

$$\Phi(x) = W^*\Pi(x)V = W_1^*\Pi_1(x)V_1 = W_1^*U_1\Pi(x)V \quad \text{for any } x \in X.$$

Hence we have that $(W^* - W_1^*U_1)\Pi(x)V = 0$, i.e., $(W^* - W_1^*U_1)\Pi(x)Vy = 0$ for all $y \in Y$. Since $\Pi(x)VY$ is dense in F, we get $W^* = W_1^*U_1$. Moreover, we have that

$$U_1\Pi(x)\left(\sum_{i=1}^{n} \pi(a_i)Vy_i\right) = U_1\left(\sum_{i=1}^{n} \Pi(xa_i)Vy_i\right) = \Pi_1(x)U_1\left(\sum_{i=1}^{n} \pi(a_i)Vy_i\right),$$

which completes the proof. □

4. Representations of crossed products of Hilbert C^*-modules

Let G be a locally compact group with left Haar measure dt. By uniqueness of left Haar measure, there exists a function $\Delta : G \to (0, \infty)$ such that $d(ts) = \Delta(s)dt$ and $d(t^{-1}) = \Delta(t)^{-1}dt$. If $\Delta \equiv 1$, then G is said to be *unimodular*. It is known that G is unimodular if and only if a left Haar measure is also a right Haar measure.

Let $(\mathcal{A}, G, \theta)$ be a unital C^*-dynamical system and let $C_c(G, \mathcal{A})$ be the set of all continuous functions from G into $\mathcal{A}$ with compact support. The set $C_c(G, \mathcal{A})$ is a linear space with the multiplication, involution and norm of $C_c(G, \mathcal{A})$ as follows:

$$(f * g)(s) = \int_G f(t)\theta_t(g(t^{-1}s))dt, \qquad f^*(s) = \Delta(s)^{-1}[\theta_s(f(s^{-1}))]^*,$$
$$\|f\|_1 = \int_G \|f(t)\|dt.$$

The completion of $C_c(G, \mathcal{A})$ with respect to $\|\cdot\|_1$ becomes a Banach $*$-algebra for which we denote by $L^1(G, \mathcal{A})$. Now, we define a new norm on $L^1(G, \mathcal{A})$ by

$$\|f\| = \sup_{\pi} \|\pi(f)\|,$$

where π ranges over all Hilbert space representations of $L^1(G, \mathcal{A})$. Then the norm $\|\cdot\|$ becomes a C^*-norm. The completion of $L^1(G, \mathcal{A})$ with respect to this norm is called the *crossed product of* $\mathcal{A}$ *by* G, and denoted by $\mathcal{A} \rtimes_\theta G$.

Let X be a Hilbert $\mathcal{A}$-module. The linear space $C_c(G, X)$ is a pre-Hilbert $\mathcal{A} \rtimes_\theta G$-module with the action of $\mathcal{A} \rtimes_\theta G$ on $C_c(G, X)$ and the inner product given by

$$(\xi \cdot f)(s) = \int_G \xi(t)\theta_t\left(f(t^{-1}s)\right)dt, \quad (\xi \in C_c(G, X),\ f \in C_c(G, \mathcal{A})), \tag{6}$$

$$\langle \xi, \xi'\rangle(s) = \int_G \theta_{t^{-1}}\big(\langle \xi(t), \xi'(ts)\rangle\big)dt, \quad (\xi, \xi' \in C_c(G, X)), \tag{7}$$

respectively. The *crossed product* $X \rtimes_\tau G$ *of* X *by* G is defined by the completion of $C_c(G, X)$ with respect to the inner product. Then $X \rtimes_\tau G$ becomes a Hilbert $\mathcal{A} \rtimes_\theta G$-module (see [13, Proposition 3.5]).

In this section, we give a covariant J-representation of a crossed product of a C^*-algebra and an associated covariant map of a crossed product of a Hilbert C^*-module by a discrete group. To do this, we first extend an α-CP map to a C^*-crossed product. However, it is not easy in general, to show that a map on a C^*-algebra is α-CP.

From now on, let X be a Hilbert $\mathcal{A}$-module and let Y, Z be Hilbert $\mathcal{B}$-modules. Suppose that $\rho : \mathcal{A} \to \mathcal{L}(Y)$ is a unital (θ, u)-covariant α-CP linear map and that $\Phi : X \to \mathcal{L}(Y, Z)$ is a (τ, σ, u)-covariant ρ-map satisfying the condition (P) in Theorem 3.2. By Theorem 3.2, there exists a KSGNS type representation $\big((\pi, V, (E, J)), (\Pi, W, F)\big)$ for a pair (ρ, Φ) such that $(\pi, v, (E, J))$ is a covariant J-representation of $(\mathcal{A}, G, \theta)$ and Π is (τ, σ', v)-covariant. The bounded maps $(\pi \times v) : C_c(G, \mathcal{A}) \to \mathcal{L}(E)$ and $\Pi \times v : C_c(G, X) \to \mathcal{L}(E, F)$ defined by

$$(\pi \times v)(f) = \int_G \pi(f(s))v_s ds, \quad f \in C_c(G, \mathcal{A}), \tag{8}$$

$$(\Pi \times v)(\xi) = \int_G \Pi\big(\xi(s)\big)v_s ds, \quad (\xi \in C_c(G, X)), \tag{9}$$

can be extended to $\mathcal{A} \rtimes_\theta G$ and $X \rtimes_\tau G$, respectively, and the extensions are denoted by the same symbols. In fact, for each $f \in C_c(G, \mathcal{A})$ and $x \in X$, we have that

$$(\Pi \times v)(\xi) = \int_G \Pi(xf(s))v_s ds = \Pi(x)(\pi \times v)(f), \quad \xi = xf \in C_c(G, X),$$

and that $\big((\Pi \times v)(\xi)\big)^* = \big((\pi \times v)(f)\big)^*\big(\Pi(x)\big)^*$. We also obtain that the closed linear span $[(\Pi \times v)(X \rtimes_\tau G)E]$ is equal to the Hilbert $\mathcal{B}$-module F, which means that $\Pi \times v$ is nondegenerate.

Proposition 4.1. *Let (E, J) be a Krein $\mathcal{B}$-module constructed in Theorem 3.2.*

(1) *$(\pi \times v)$ is a J-representation of $\mathcal{A} \rtimes_\theta G$ on E,*
(2) *$(\Pi \times v)$ is a $J \circ (\pi \times v)$-map and*

$$(\Pi \times v)(\xi \cdot f) = (\Pi \times v)(\xi) \cdot (\pi \times v)(f) \tag{10}$$

for $\xi \in C_c(G, X)$ and $f \in C_c(G, \mathcal{A})$.

Proof. (1) Since π is (θ, v)-covariant, for any $f, g \in C_c(G, \mathcal{A})$,

$$(\pi \times v)(f * g) = (\pi \times v)(f) \cdot (\pi \times v)(g), \tag{11}$$

$$(\pi \times v)(f^*) = (\pi \times v)(f)^J. \tag{12}$$

(2) Let $\xi, \xi' \in C_c(G, X)$. Since π is (θ, v)-covariant, it follows from (7) and (3) that

$$\begin{aligned}(\pi \times v)(\langle \xi, \xi' \rangle) &= \int_G \int_G J v_t^* \big(\Pi(\xi(t))\big)^* \Pi(\xi'(ts))v_{ts} ds dt \\ &= J\big((\Pi \times v)(\xi)\big)^*\big((\Pi \times v)(\xi')\big),\end{aligned}$$

which implies that $(\Pi \times v)$ is a $J \circ (\pi \times v)$-map. We can also get (10) from (5). □

In the remainder of this section, we assume that G is a discrete group and $\mathcal{A}$ is a unital C^*-algebra. Then the crossed product $\mathcal{A} \rtimes_\theta G$ has a unit element δ_e. We define maps $\widetilde{\alpha} : \mathcal{A} \rtimes_\theta G \to \mathcal{A} \rtimes_\theta G$ and $\widetilde{\rho} : \mathcal{A} \rtimes_\theta G \to \mathcal{L}(Y)$ by

$$\widetilde{\alpha}(f)(s) = \alpha(f(s)), \qquad \widetilde{\rho}(f) = V^*(\pi \times v)(f)V \quad (f \in C_c(G, \mathcal{A}),\ s \in G).$$

We also define a group action $\widetilde{\theta}$ of G on $\mathcal{A} \rtimes_\theta G$ by

$$(\widetilde{\theta}_t(f))(s) = \theta_t\left(f(t^{-1}st)\right), \quad f \in C_c(G, \mathcal{A}), \tag{13}$$

where π and V are given as in Theorem 2.3 and v is given as in Theorem 2.4.

Theorem 4.2. *If α and θ are equivariant, i.e. $\alpha \circ \theta_s = \theta_s \circ \alpha$ $(s \in G)$ and $\alpha(0) = 0$, then*

(1) *$\widetilde{\alpha}$ is a bounded Hermitian map with $\widetilde{\alpha}^2 = I$ and $\widetilde{\alpha}(\delta_e) = \delta_e$,*
(2) *$\widetilde{\rho}$ is a $(\widetilde{\theta}, u)$-covariant $\widetilde{\alpha}$-completely positive map such that*

$$\widetilde{\rho}(f) = \int_G \rho(f(s))u_s ds, \quad f \in C_c(G, \mathcal{A}), \tag{14}$$

(3) *$\big(\mathcal{A} \rtimes_\theta G, G, \widetilde{\theta}\big)$ is a C^*-dynamical system,*
(4) *the triple $(\pi \times v, v, (E, J))$ is a covariant J-representation of $\big(\mathcal{A} \rtimes_\theta G, G, \widetilde{\theta}\big)$.*

Proof. (1) The proof is straightforward.

(2) By (iii) and (i) in Theorem 2.4, we have that for any $f \in C_c(G, \mathcal{A})$

$$\widetilde{\rho}(f) = V^*(\pi \times v)(f)V = \int_G V^*\pi(f(s))v_s V ds = \int_G \rho(f(s))u_s ds,$$

which gives the proof of (14). The conditions (i) and (ii) in Definition 2.1 are satisfied by the above argument. Since α and θ are equivariant, it follows from (14) and (ρ3) in Definition 2.1 that for any $f, g \in C_c(G, \mathcal{A})$,

$$\widetilde{\rho}\big(\widetilde{\alpha}(f) * \widetilde{\alpha}(g)\big) = \widetilde{\rho}(f * g).$$

For any $n \in \mathbb{N}$, let $y_i \in Y$ and $f_i \in C_c(G, \mathcal{A})$ $(i = 1, \ldots, n)$. By (11) and (12), we have that

$$\sum_{i,j=1}^{n} \langle y_i, \widetilde{\rho}(\widetilde{\alpha}(f_i)^* * f_j) y_j \rangle = \sum_{i,j=1}^{n} \langle V y_i, \big((\pi \times v)(f_i)\big)^* (\pi \times v)(f_j) V y_j \rangle \geq 0.$$

Moreover, we see that

$$\begin{aligned} \big(\widetilde{\rho}(\widetilde{\alpha}(f * f_i)^* * (f * f_j))\big)_{n \times n} &\leq \|(\pi \times v)(f)\|^2 \left(V^* \big((\pi \times v)(f_i)\big)^* (\pi \times v)(f_j) V\right) \\ &= \|(\pi \times v)(f)\|^2 \left(\widetilde{\rho}(\widetilde{\alpha}(f_i)^* * f_j)\right). \end{aligned}$$

Hence, the map $\widetilde{\rho}$ is $\widetilde{\alpha}$-completely positive. On the other hand, for any $f \in C_c(G, \mathcal{A})$ and $t \in G$, by (14), (13) and (θ, u)-covariant property of ρ, we have that

$$\widetilde{\rho}\left(\widetilde{\theta}_t(f)\right) = \int_G u_t \rho\left(f(t^{-1}st)\right) u_{t^{-1}st} u_t^* ds = u_t \widetilde{\rho}(f) u_t^*.$$

It follows from the boundedness that the map $\widetilde{\rho}$ is $(\widetilde{\theta}, u)$-covariant.

(3) It is enough to see that for each $t \in G$, $\widetilde{\theta}_t$ is an automorphism on $\mathcal{A} \rtimes_\theta G$. Since each θ_t is bijective, we see that $\widetilde{\theta}_t$ is also bijective. For any $s, t \in G$ and $f, g \in C_c(G, \mathcal{A})$, we have that

$$\begin{aligned} \left(\widetilde{\theta}_t(f * g)\right)(s) &= \int_G \theta_t\big(f(k)\big)\theta_{tk}\left(g(k^{-1}t^{-1}st)\right) dk \\ &= \int_G \theta_t\left(f(t^{-1}kt)\right)\theta_{kt}\left(g(t^{-1}k^{-1}st)\right) dk \\ &= \left(\widetilde{\theta}_t(f)\right) * \left(\widetilde{\theta}_t(g)\right)(s), \end{aligned}$$

and that

$$\left(\widetilde{\theta}_t(f^*)\right)(s) = \theta_t\left((f^*)(t^{-1}st)\right) = \theta_s\left(\theta_t\left(f(t^{-1}s^{-1}t)\right)\right)^* = \left(\widetilde{\theta}_t(f)\right)^*(s).$$

By continuity of $\widetilde{\theta}_t$, we see that each $\widetilde{\theta}_t$ is a automorphism on $\mathcal{A} \rtimes_\theta G$.

(4) Let $t \in G$ and $f \in C_c(G, \mathcal{A})$. Since π is (θ, v)-covariant, we have that

$$(\pi \times v)\left(\widetilde{\theta}_t(f)\right) = \int_G v_t \pi(f(s)) v_s v_t^J ds = v_t (\pi \times v)(f) v_t^J,$$

where the first equality follows from the change of variables. □

For a given θ-compatible action τ of G on X, we define a group action $\widetilde{\tau}$ of G on the crossed product $X \rtimes_\theta G$ by

$$\big(\widetilde{\tau}_t(\xi)\big)(s) = \tau_t\big(\xi(t^{-1}st)\big), \quad \xi \in C_c(G, X)$$

and, by its boundedness, it can be extended to the crossed product $X \rtimes_\theta G$. Also, we define a map $\widetilde{\Phi} : X \rtimes_\theta G \to \mathcal{L}(Y, Z)$ by

$$\widetilde{\Phi}(\xi) = W^*(\Pi \times v)(\xi) V \quad (\xi \in X \rtimes_\theta G).$$

Let $t \in G, f \in C_c(G, \mathcal{A})$ and $\xi \in C_c(G, X)$. Since Φ is (τ, σ, u)-covariant, we have that

$$\widetilde{\Phi}(\widetilde{\tau}_t(\xi)) = \int_G \sigma_t \Phi\left(\xi(t^{-1}st)\right) u_t^* u_s ds = \sigma_t \widetilde{\Phi}(\xi) u_t^*,$$

which shows that $\widetilde{\Phi}$ is $(\widetilde{\tau}, \sigma, u)$-covariant.

Corollary 4.3. *Let $\widetilde{\rho}$ be given as in (14).*

(1) *The action $\widetilde{\tau} : G \to \mathcal{U}(X \rtimes_\theta G)$ is $\widetilde{\theta}$-compatible.*
(2) *The above map $\widetilde{\Phi}$ is a $\widetilde{\rho}$-map satisfying*

$$\widetilde{\Phi}(\xi) = \int_G \Phi(\xi(s)) u_s ds \quad \text{for any } \xi \in C_c(G, X). \tag{15}$$

(3) *The map $\Pi \times v$ defined as in (9) is $(\widetilde{\tau}, \sigma', v)$-covariant.*

Proof. (1) Indeed, we note that each $\widetilde{\tau}_t$ is a unitary operator on $X \rtimes_\theta G$. Let $s, t \in G, f \in C_c(G, \mathcal{A})$ and $\xi, \xi' \in C_c(G, X)$. Since τ is θ-compatible, it follows from (6) and similar arguments used in the proof (3) in Theorem 4.2 that

$$\begin{aligned}\widetilde{\tau}_t(\xi \cdot f)(s) &= \int_G \tau_t\big(\xi(k)\big)\theta_{tk}\left(f(k^{-1}t^{-1}st)\right) dk \\ &= \int_G \tau_t\left(\xi(t^{-1}kt)\right)\theta_{kt}\left(f(t^{-1}k^{-1}st)\right) dk = \left(\widetilde{\tau}_t(\xi)\cdot\widetilde{\theta}_t(f)\right)(s).\end{aligned}$$

By the definition (7) of the inner product, we have that

$$\begin{aligned}\langle\widetilde{\tau}_t(\xi), \widetilde{\tau}_t(\xi')\rangle(s) &= \int_G \theta_{k^{-1}t}\left(\langle\xi(t^{-1}kt), \xi'(t^{-1}kst)\rangle\right) dk \\ &= \int_G \theta_{tk^{-1}}\left(\langle\xi(k), \xi'(kt^{-1}st)\rangle\right) dk = \widetilde{\theta}_t\big(\langle\xi, \xi'\rangle\big)(s),\end{aligned}$$

which implies that $\widetilde{\tau}$ is $\widetilde{\theta}$-compatible.

(2) For any $\xi \in C_c(G, X)$, we have that

$$\widetilde{\Phi}(\xi) = \int_G W^*\Pi(\xi(s))v_sVds = \int_G W^*\Pi(\xi(s))Vu_sds = \int_G \Phi(\xi(s))u_sds,$$

which gives the proof of (15). Let $\xi, \xi' \in C_c(G, X)$ be any elements. Since $(\Pi \times v)$ is a $J \circ (\pi \times v)$-map in Proposition 4.1, it follows from (i) in Theorem 2.4 that

$$\begin{aligned}\left\langle\widetilde{\Phi}(\xi), \widetilde{\Phi}(\xi')\right\rangle &= V^*(\Pi \times v)(\xi)^*(\Pi \times v)(\xi')V \\ &= \int_G \rho\big(\alpha(\langle\xi, \xi'\rangle(s))\big)u_sds = \widetilde{\rho}\big(\langle\xi, \xi'\rangle\big).\end{aligned}$$

Since $\widetilde{\Phi}$ is bounded and $C_c(G, X)$ is dense in $X \rtimes_\theta G$, $\widetilde{\Phi}$ is a $\widetilde{\rho}$-map of $X \rtimes_\theta G$.

(3) Since Π is a (τ, σ', v)-covariant map constructed in Theorem 3.2, we have that for any $t \in G$ and $\xi \in C_c(G, X)$,

$$(\Pi \times v)\big(\widetilde{\tau}_t(\xi)\big) = \int_G \sigma'_t\Pi\left(\xi(t^{-1}st)\right)v_t^Jv_sds = \sigma'_t(\Pi \times v)(\xi)v_t^J,$$

which implies that $\Pi \times v$ is $(\widetilde{\tau}, \sigma', v)$-covariant. □

Remark 4.4. (1) The pair $\big((\pi \times v, V, (E, J)), (\Pi \times v, W, F)\big)$ is a minimal KSGNS type representation of the pair $(\widetilde{\rho}, \widetilde{\Phi})$, where $\widetilde{\rho}$ and $\widetilde{\Phi}$ are given as in (14) and (15), respectively.

(2) For any $s \in G$, $a \in \mathcal{A}$ and $x \in X$, we denote (a, s) and (x, s) elements of $\mathcal{A} \rtimes_\theta G$ and $X \rtimes_\theta G$, respectively. We can consider inclusion maps $\mathcal{A} \ni a \hookrightarrow (a, e) \in \mathcal{A} \rtimes_\theta G$ and $X \ni x \hookrightarrow (x, e) \in X \rtimes_\tau G$ with the unit e in G. Then we see that

$$\widetilde{\rho}(a, e) = \rho(a) \quad \text{and} \quad \widetilde{\Phi}(x, e) = \Phi(x).$$

Moreover, we have that $\widetilde{\alpha}(a, e) = \alpha(a)$, $\widetilde{\theta}_t(a, e) = \theta_t(a)$ and $\widetilde{\tau}_t(x, e) = \tau_t(x)$ for any $t \in G$, $a \in \mathcal{A}$ and $x \in X$.

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References

[1] G.G. Kasparov, Hilbert C^*-modules: theorems of Stinespring and Voiculescu, J. Oper. Theory 4 (1980) 133–150.
[2] E.C. Lance, Hilbert C^*-Modules. A Toolkit for Operator Algebraists, in: London Mathematical Society Lecture Series, vol. 210, Cambridge University Press, Cambridge, 1995.
[3] M.B. Asadi, Stinespring's theorem for Hilbert C^*-modules, J. Oper. Theory 62 (2008) 235–238.
[4] B.V.R. Bhat, G. Ramesh, K. Sumesh, Stinespring's theorem for maps on Hilbert C^*-modules, 2010. Preprint.
[5] J. Heo, Completely multi-positive linear maps and representations on Hilbert C^*-modules, J. Oper. Theory 41 (1999) 3–22.
[6] E. Scheibe, Über Feldtheorien in Zustandsräumen mit indefiniter Metrik, Mimeographed notes of the Max–Planck Institut für Physik und Astrophysik in München, München, 1960.
[7] J. Heo, U.C. Ji, Radon–Nikodým type theorem for α-completely positive maps, J. Math. Phys. 51 (2010) 103505.
[8] T. Constantinescu, A. Gheondea, Representations of Hermitian kernels by means of Kreĭn spaces, Publ. RIMS, Kyoto Univ. 33 (1997) 917–951.

[9] T. Constantinescu, A. Gheondea, Representations of Hermitian kernels by means of Kreĭn spaces II. Invariant kernels, Comm. Math. Phys. 216 (2001) 409–430.
[10] T. Constantinescu, A. Gheondea, On L. Schwartz's boundedness condition for kernels, Positivity 10 (2006) 65–86.
[11] J. Heo, J.P. Hong, U.C. Ji, On KSGNS representations on Krein C^*-modules, J. Math. Phys. 51 (2010) 053504.
[12] J. Heo, U.C. Ji, Quantum stochastic processes for pairs of maps on Hilbert C^*-modules, J. Math. Phys. 52 (2011) 053501.
[13] S. Echterhoff, S. Kaliszewski, J. Quigg, I. Raeburn, Naturality and induced representations, Bull. Aust. Math. Soc. 61 (2000) 415–438.
[14] M. Joita, Covariant version of the Stinespring type theorem for Hilbert C^*-modules. Preprint.
[15] M. Kusuda, Duality for crossed products of Hilbert C^*-modules, J. Oper. Theory 60 (2008) 85–112.
[16] J. Bogner, Indefinite Inner Product Spaces, Springer-Verlag, Berlin, 1974.
[17] G. Murphy, Positive definite kernels and Hilbert C^*-modules, Proc. Edinburgh Math. Soc. 40 (1997) 367–374.
[18] D. Bakić, B. Guljaš, On a class of module maps of Hilbert C^*-modules, Math. Commun. 7 (2002) 177–192.